3. 24.11.05

We now aim at developing tools to be able to prove the following.

Theorem 3.1 (Luzin)**.** *Assuming CH, there exists a* Luzin set*, that is, an uncountable set $L \subseteq \mathbb{R}$ such that for any meager set $M \subseteq \mathbb{R}$: $|L \cap M| \leq \aleph_0$.*

Definition 3.2. Suppose X is a set. For an ideal $I \subseteq \mathcal{P}(X)$. Put:

- $\mathsf{add}(I) := \min\{|\mathcal{A}| \mid \mathcal{A} \subseteq I(\bigcup \mathcal{A} \notin I)\}$.
- $\mathsf{cov}(I) := \min\{|\mathcal{A}| \mid \mathcal{A} \subseteq I(\bigcup \mathcal{A} = X)\}$.
- $\mathsf{cof}(I) := \min\{|\mathcal{A}| \mid \mathcal{A} \subseteq I \text{ and } \forall B \in I \exists C \in \mathcal{A}(B \subseteq C)\}$.

If $\mathcal{I}$ is a proper ideal, we may also define:

- $\mathsf{non}(I) := \min\{|A| \mid A \subseteq X \text{ and } A \notin I\}$.

Since an ideal is closed under finite unions, always $\mathsf{add}(I) \geq \aleph_0$. If I is a proper ideal, then also $\mathsf{add}(I) \leq \mathsf{cov}(I)$. If I is non-trivial, then also $\mathsf{cov}(I) \leq \mathsf{cof}(I)$.

Intuitively, an ideal is a collection of negligible sets. Two important examples are:

Definition 3.3. Let $\mathcal{M} := \{A \subseteq \mathbb{R} \mid A \text{ is meager }\}$ and $\mathcal{N} := \{A \subseteq \mathbb{R} \mid A \text{ is a null set }\}$.

We also consider $\mathcal{M}_{[0,1]} := \mathcal{M} \cap \mathcal{P}([0,1])$ and $\mathcal{N}_{[0,1]} := \mathcal{N} \cap \mathcal{P}([0,1])$.

Evidently, $\mathcal{M}, \mathcal{N}$ are non-trivial ideals and $\mathsf{add}(\mathcal{M}), \mathsf{add}(\mathcal{N}) \geq \aleph_1$. $|\mathcal{M}| = |\mathcal{N}| = 2^{\mathfrak{c}}$, since the cantor set $C \in \mathcal{M} \cap \mathcal{N}$ is of size $\mathfrak{c}$ and then $\mathcal{P}(C) \subseteq \mathcal{M} \cap \mathcal{N}$. However:

Lemma 3.4. $\mathsf{cof}(\mathcal{M}) \leq \mathfrak{c}$ *and* $\mathsf{cof}(\mathcal{N}) \leq \mathfrak{c}$.

Proof. As mentioned before, any meager set is contained in some F_σ meager set, and there are only $\mathfrak{c}$ many F_σ sets, hence, $\mathsf{cof}(\mathcal{M}) \leq \mathfrak{c}$.

If $A \in \mathcal{N}$, then for all $n \in \mathbb{N}$, there exists some open G_n containing A and of measure $< \frac{1}{n+1}$. It follows that any null set is contained in some G_δ null set, thus, $\mathsf{cof}(\mathcal{N}) \leq \mathfrak{c}$. □

Lemma 3.5. *Assume $\mathcal{I}$ is an ideal over some infinite set X, then* $\mathrm{cf}(\mathsf{add}(\mathcal{I})) = \mathsf{add}(\mathcal{I})$.

If $\mathsf{non}(\mathcal{I})$ *is defined, then* $\mathsf{add}(\mathcal{I}) \leq \mathrm{cf}(\mathsf{non}(\mathcal{I}))$.

If $\mathsf{cof}(\mathcal{I})$ *is infinite, then* $\mathsf{add}(\mathcal{I}) \leq \mathrm{cf}(\mathsf{cof}(\mathcal{I}))$.

Proof. Put $\lambda := \mathsf{add}(\mathcal{I})$, $\kappa := \mathrm{cf}(\lambda)$ and pick a family $\{\lambda_i \in \lambda \mid i < \kappa\}$ with $\sup_{i<\kappa} \lambda_i = \lambda$. Let $\{A_\alpha \in \mathcal{I} \mid \alpha < \lambda\}$ witness $\mathsf{add}(\mathcal{I}) = \lambda$. By the definition of $\mathsf{add}(\mathcal{I})$, for all $i < \kappa$, $B_i := \bigcup_{\alpha<\lambda_i} A_\alpha$ is in $\mathcal{I}$. Now if λ was a singular cardinal, i.e., if $\kappa < \mathsf{add}(\mathcal{I})$, then $\bigcup_{\alpha<\lambda} A_\alpha = \bigcup_{i<\kappa} B_i \in \mathcal{I}$. A Contradiction.

Put $\theta := \mathsf{cof}(\mathcal{I})$ and pick a witness $\mathcal{C} := \{C_\alpha \in \mathcal{I} \mid \alpha < \theta\}$. Also, find $\{\theta_i < \theta \mid i < \tau\}$ witnessing $\tau := \mathrm{cf}(\theta)$. By thinning-out if needed, we may assume non-redundancy of $\mathcal{C}$, i.e.:

$$(\star) \quad \alpha < \beta < \theta \rightarrow C_\beta \nsubseteq C_\alpha.$$

Put $\mathcal{C}' := \{C_{\theta_i} \mid i < \tau\}$. Now, if $\tau < \mathsf{add}(\mathcal{I})$, then $\bigcup \mathcal{C}' \in \mathcal{I}$, and there must exist some $\alpha < \theta$ with $\bigcup \mathcal{C}' \subseteq C_\alpha$. Find $i < \tau$ with $\alpha < \theta_i$, then in particular $C_{\theta_i} \subseteq \bigcup \mathcal{C}' \subseteq C_\alpha$, contradicting $(\star)$.

Put $\mu := \mathsf{non}(\mathcal{I}), \sigma := \mathrm{cf}(\mu)$ and pick some $D \in [X]^\mu$ such that $D \notin \mathcal{I}$. By $|D| = \mu$, there exists a family of sets $\{D_i \in [D]^{<\mu} \mid i < \sigma\}$ such that $D = \bigcup_{i<\sigma} D_i$. Now, by $|D_i| < \mathsf{non}(\mathcal{I})$ for all i, we know that $\{D_i \mid i < \sigma\} \subseteq \mathcal{I}$, thus, if $\sigma < \mathsf{add}(\mathcal{I})$, then $D = \bigcup_{i<\sigma} D_i \in I$. A contradiction.

□

Corollary 3.6. *Suppose $\mathcal{I}$ is a non-trivial proper ideal over some infinite set X, then:* $\aleph_0 \le \mathrm{cf}(\mathsf{add}(\mathcal{I})) = \mathsf{add}(\mathcal{I}) \le \min\big\{\mathsf{cov}(\mathcal{I}), \mathrm{cf}(\mathsf{non}(\mathcal{I})), \mathrm{cf}(\mathsf{cof}(\mathcal{I}))\big\} \le \mathsf{cov}(\mathcal{I}) \le \mathsf{cof}(\mathcal{I}) \le 2^{|X|}$.

Theorem 3.7. *Assume $\mathcal{I}$ is a non-trivial proper ideal over an infinite set X.*

Suppose $\mathsf{cov}(I) = \mathsf{cof}(I) = \kappa$*, then there exists some set $A \subseteq X$ such that $|A| = \kappa$ and for all $B \in \mathcal{I}$, $|B \cap A| < \kappa$.*

Proof. Fix $\langle B_\alpha \mid \alpha < \kappa\rangle$ witnessing $\mathsf{cof}(\mathcal{I}) = \kappa$. We define $A = \{a_\alpha \mid \alpha < \kappa\}$ by induction on $\alpha < \kappa$. Assume $\{a_\beta \mid \beta < \alpha\}$ had already been defined. Since $\mathcal{I}$ is non-trivial, $\{a_\beta\} \in \mathcal{I}$ for all $\beta < \alpha$. It follows from $\alpha < \mathsf{cov}(\mathcal{I})$ and properness of $\mathcal{I}$ that $(\bigcup_{\beta<\alpha}\{a_\beta\} \cup \bigcup_{\beta<\alpha} B_\beta) \neq X$, so let us pick $a_\alpha \in X \setminus (\{a_\beta \mid \beta < \alpha\} \cup \bigcup_{\beta<\alpha} B_\beta)$. End of the construction.

Clearly, the construction ensures that $|A| = \kappa$. To see the other property, fix $B \in \mathcal{I}$.

By defining properties of $\langle B_\alpha \mid \alpha < \kappa\rangle$, there exists some $\beta < \kappa$ such that $B \subseteq B_\beta$. By the construction, for all $\alpha < \kappa$ with $\alpha > \beta$, $a_\alpha \in X \setminus B_\beta$ and hence $B \cap A \subseteq \{a_\delta \mid \delta \le \beta\}$, that is, $|B \cap A| \le |\beta| < \kappa$. □

Corollary 3.8. *If $\mathfrak{c} = \aleph_1$, then there exists a* Sierpinski set*, that is, an uncountable set $S \subseteq \mathbb{R}$ such that for any null set $N \subseteq \mathbb{R}$: $|L \cap N| \le \aleph_0$.*

Proof. Trivially, $\mathcal{N}$ is a proper ideal. Applying $\mathsf{add}(\mathcal{N}) \ge \aleph_1$ and Corollary 3.6, we get that:

$$\aleph_1 \le \mathsf{add}(\mathcal{N}) \le \mathsf{cov}(\mathcal{N}) \le \mathsf{cof}(\mathcal{N}) \le \mathfrak{c} = \aleph_1.$$

□

Corollary 3.9 (Luzin)**.** *If $\mathfrak{c} = \aleph_1$, then there exists a Luzin set.*

Proof. By now, the only missing ingredient is the following. □

Theorem 3.10 (Baire)**.** *$\mathcal{M}$ is a proper ideal.*

Proof. We give a proof in a wider context. See Theorem 3.16. □

Thus, we yield the consistency of existence of a Luzin set. It is worth mentioning that the non-existence of a Luzin set is also consistent.

Definition 3.11. A set A is *comeager* iff A^c is meager.

Remark: Assume that A is meager, then there exist a sequence of nowhere dense sets $\{F_i\}_{i\in\mathbb{N}}$ such that $A=\bigcup_{i\geq 1}F_i$, therefore $A\subseteq\bigcup_{i\geq 1}\overline{F_i}$. We conclude that $\bigcap_{i\geq 1}\overline{F_i}^c\subseteq A^c$, where $\{\overline{F_i}^c\}_{i\in\mathbb{N}}$ are dense and open.

Since the converse is also true, we get that a set is comeager iff it contains a G_δ subset, such that each open set in the intersection is dense. We will see that in complete metric spaces, such sets are dense.

Definition 3.12. A metric space is *complete* iff every Cauchy sequence converges.

Lemma 3.13. *Every compact subspace of a metric space is complete.*

Proof. If C is compact, then any sequence from C has a converging subsequence, in particular if the sequence is Cauchy, its (unique) limit is in C. □

Lemma 3.14. *Every closed set in a complete space is complete.*

Proof. Assume X is complete, $F\subseteq X$ is closed, and $\{f_n\}_{n\in\mathbb{N}}\subseteq F$ is Cauchy.

$\{f_n\}_{n\in\mathbb{N}}\subseteq X$ is also Cauchy (since the metric on F is induced by the metric on X), thus converges to some $x\in X$. On the other hand, F is closed, so x must be in F. □

Definition 3.15. X is a *Baire space* iff the intersection of any countable family of dense open sets in X is dense.[10]

A generalization of Theorem 3.10 is the following.

Theorem 3.16. *Every complete metric space is a Baire space.*

Proof. Assume $\langle F_i\mid i\in\mathbb{N}\rangle$ is a family of closed and nowhere dense subsets in a complete metric space $\langle X,d\rangle$. We will show that $G:=\left(\bigcup F_i\right)^c$ is dense in X.

Pick an arbitrary open ball B. Now, $B\setminus F_1\neq\emptyset$ (since F_1 is nowhere dense and has no interior), so we pick $x_1\in B\setminus F_1$. X is metric hence regular, therefore there exist an open ball B_1, such that $x_1\in B_1\subseteq\overline{B_1}\subseteq B\setminus F_1$, and $\mathrm{Diam}(B_1)<\frac{1}{2}$. Once again, $B_1\setminus F_2\neq\emptyset$, $x_1\in B_1\setminus F_2$ is picked and we can find some open ball B_2 that satisfies $x_2\in B_2\subseteq\overline{B_2}\subseteq B_1\setminus F_2$ and $\mathrm{Diam}(B_2)<\frac{1}{3}$.

We continue likewise and construct a downward chain $\{B_n\}_{n\in\mathbb{N}}$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$, such that $\mathrm{Diam}(B_n)<\frac{1}{n+1}$, and $x_n\in B_n$ for all $n\in\mathbb{N}$. $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy in $\overline{B_1}$ which is a complete space, thus converges to some $x\in\overline{B_1}$. Now, $x\in B\cap G$, and since B is an arbitrary ball, we get that G is dense. □

[10]Notice that a Baire space can not be a countable union of nowhere dense sets.

Observation 3.17. *Suppse $\langle X, O\rangle$ is a topological space and $Y \subseteq X$ is such that :*

- $Y \models S_{fin}(\mathcal{O}, \mathcal{O})$*;*
- *If U is an open set containing Y, then $X \setminus U \models S_{fin}(\mathcal{O}, \mathcal{O})$*

then $X \models S_{fin}(\mathcal{O}, \mathcal{O})$.

Proof. Assume X, Y are like in the statement. Let $\langle \mathcal{U}_n \subseteq O \mid n \in \mathbb{N}\rangle$ be a countable family of open covers of X. By $Y \models S_{fin}(\mathcal{O}, \mathcal{O})$ and $\langle \mathcal{U}_{2n} \subseteq O \mid n \in \mathbb{N}\rangle$ being a countable family of open covers of Y, there exists some $\langle \mathcal{F}_{2n} \in [\mathcal{U}_{2n}]^{<\omega} \mid n \in \mathbb{N}\rangle$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_{2n}$ is an open cover of Y. Put $U := \bigcup\bigcup_{n\in\mathbb{N}} \mathcal{F}_{2n}$. Finally, since $Y \subseteq U$ and $\langle \mathcal{U}_{2n+1} \subseteq O \mid n \in \mathbb{N}\rangle$ is an open cover of $X \setminus U$, there exists $\langle \mathcal{F}_{2n+1} \in [\mathcal{U}_{2n+1}]^{<\omega} \mid n \in \mathbb{N}\rangle$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_{2n+1}$ is an open cover of $X \setminus U$ and it follows that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ is an open cover of X exemplifying $S_{fin}(\mathcal{O}, \mathcal{O})$. □

Definition 3.18. Suppose $\langle X, O\rangle$ is a topological space and κ is an infinite cardinal number. For $Y \subseteq X$, we say that X is *κ-concentrated at Y* iff for any open $U \supseteq Y$: $|X \setminus U| < \kappa$.

Corollary 3.19. *Suppose $\langle X, O\rangle$ is a topological space and $Y \subseteq X$ is such that:*

- $Y \models S_{fin}(\mathcal{O}, \mathcal{O})$*;*
- *X is* concentrated *(i.e. $\aleph_1$-concentrated) at Y.*

then $X \models S_{fin}(\mathcal{O}, \mathcal{O})$.

Proof. By Observation 3.17 and the fact that any countable set satisfies Menger's property. □

In special cases, we can prove a stronger result. We first need another definition.

Definition 3.20. For a topological space $\langle X, O\rangle$, we denote by $S_1(\mathcal{O}, \mathcal{O})$ the property that for any countable sequence of open covers of X, $\langle \mathcal{U}_n \subseteq O \mid n \in \mathbb{N}\rangle$, there exists some $\langle U_n \in \mathcal{U}_n \mid n \in \mathbb{N}\rangle$ such that $X = \bigcup_{n\in\mathbb{N}} U_n$.

Observation 3.21. *Suppose $\langle X, O\rangle$ is a topological space and $Y \subseteq X$ is such that:*

- $Y \models S_1(\mathcal{O}, \mathcal{O})$*;*
- *X is concentrated at Y.*

then $X \models S_1(\mathcal{O}, \mathcal{O})$.

Proof. Same as 3.17. □

Corollary 3.22. *Suppose $\langle X, O\rangle$ is a topological space and is concentrated at some countable $Y \subseteq X$, then $X \models S_1(\mathcal{O}, \mathcal{O})$.*

It is worth mentioning that $S_1(\mathcal{O},\mathcal{O})$ is indeed stronger than $S_{fin}(\mathcal{O},\mathcal{O})$. $[0,1] \subseteq \mathbb{R}$ is compact, hence, satisfies Menger's property. However, for any family of open covers $\langle \mathcal{U}_n \mid n \in \mathbb{N}\rangle$ with $\mathrm{Diam}(U) < \frac{1}{2^{n+17}}$ for all $n \in \mathbb{N}$ and $U \in \mathcal{U}_n$, we get that $\sum_{n\in\mathbb{N}} \mathrm{Diam}(U_n) < 1 = \mathrm{Diam}([0,1])$ for all $\langle U_n \in \mathcal{U}_n \mid n \in \mathbb{N}\rangle$. In particular $[0,1]$ cannot satisfy $S_1(\mathcal{O},\mathcal{O})$.

Lemma 3.23. *If $X \subseteq \mathbb{R}$ is uncountable and σ-compact, then X contains a perfect set.*

Proof. Assuming $X = \bigcup_{n\in\mathbb{N}} K_n$, where $\langle K_n \mid n \in \mathbb{N}\rangle$ are compact, we know that there must exist some $m \in \mathbb{N}$, with $|K_m| > \aleph_0$, thus, K_m is an uncountable closed set. Applying Theorem 2.24, we conclude that K_m (and hence, also X) contains a perfect subset. □

Theorem 3.24. *Megner's conjecture 1.30 is consistently false.*

Proof. Since the existence of a Luzin set is consistent, it suffices to prove that a Luzin set $L \subseteq \mathbb{R}$ satisfies Menger's property but is not σ-compact.

Claim 3.25. *L is concentrated at some $A \in [L]^{\leq\aleph_0}$.*
In particular, $L \models S_1(\mathcal{O},\mathcal{O})$.

Proof. Since $L \subseteq \mathbb{R}$, we have that $w(L) \leq w(\mathbb{R}) \leq \aleph_0$. It follows from Lemma 2.6 that L is separable, so let $A \subseteq L$ be a countable dense subset of L. To see that L is concentrated at A, pick some open set $U \subseteq \mathbb{R}$ with $U \supseteq A$. To see $|L \setminus U| \leq \aleph_0$, notice that $L \setminus U = L \cap (\overline{A} \setminus U)$. Now, $\mathbb{R} \setminus \overline{(\overline{A} \setminus U)} = \mathbb{R} \setminus (\overline{A} \setminus U) = (\mathbb{R} \setminus \overline{A}) \cup (\overline{A} \cap U) \supseteq (\mathbb{R} \setminus \overline{A}) \cup A$, and the latter is surely dense in $\mathbb{R}$.[11] It follows from Lemma 2.19 that $\overline{A} \setminus U$ is nowhere dense. Recalling that L is a Luzin set, we conclude that $L \cap (\overline{A} \setminus U)$ is countable. □

It follows that $L \models S_{fin}(\mathcal{O},\mathcal{O})$. We are left with showing that L is not σ-compact. Using Lemma 3.23, this reduces to showing that L does not contain a perfect subset. In the following, we prove that any perfect set contains a meager subset of cardinality $\mathfrak{c}$, and hence, L cannot contain a perfect subset. □

Lemma 3.26. *If $P \subseteq \mathbb{R}$ is perfect, then there exists some $X \subseteq P$ such that:*

- *X is perfect;*
- *X is a null set.*
- *X is nowhere dense and homeomorphic to the product space $\{0,1\}^{\mathbb{N}}$;*

In particular, any perfect subset of $\mathbb{R}$ is of cardinality $\mathfrak{c}$.

Proof. We first need the following Observation:

[11]Simply because $\overline{(\mathbb{R} \setminus \overline{A}) \cup A} = \overline{(\mathbb{R} \setminus \overline{A})} \cup \overline{A} = \mathbb{R}$.

Observation 3.27. *Suppose $\langle L, \leq \rangle$ is a linearly-ordered set.*

Put $\mathcal{B}_{\leq} := \{(\alpha, \beta) \mid \alpha, \beta \in L, \alpha < \beta\}$,[12] *and let $\langle L, \mathcal{O}_{\leq} \rangle$ be the topological space generated by the base $\mathcal{B}_{\leq}$ (This is called the* interval topology*).*

For any perfect $P \subseteq L$ and a closed interval $I \subseteq L$ with $I \cap P \neq \emptyset$, there exists some closed interval $J \subseteq I$ such that $J \cap P$ is perfect.

Proof. Assume P is perfect and $I = [a, b]$ is an interval with $P \cap I \neq \emptyset$. If $P \cap I$ is perfect, we are done, so assume this is not the case, that is, at least one of the elements a, b are isolated at $P \cap I$. Note that no elements of (a, b) can be isolated in $[a, b] \cap P$. If a is isolated (and b is not), then we can find some $a < c < b$ such that $[c, b] \cap P = I \cap P \setminus \{a\}$, so take $J := [c, b]$. If b is isolated (and a is not), then we can find some $a < d < b$ such that $[a, d] \cap P = I \cap P \setminus \{b\}$, so take $J := [a, d]$. If both a and b are isolated we can find $a < c < d < b$ such that $[c, d] \cap P = I \cap P \setminus \{a, b\}$, so take $J := [c, d]$. □

Assume $P \subset \mathbb{R}$ is a perfect set.

Let $\mathcal{S} := \{s : \{1, .., k\} \to \{0, 1, 2\} \mid k \in \mathbb{N}\}$ denote the family of finite ternary sequences. Define a function $\varphi : \mathcal{S} \to \{I \subseteq \mathbb{R} \mid I \text{ is a closed interval}\}$. By induction on n - the length of $s \in \mathcal{S}$. For $s \in \mathcal{S}$, we sometime write I_s for $\varphi(s)$ whenever defined.

Induction base ($n = 1$): Let $s_0 = \{(1, 0)\}, s_1 = \{(1, 1)\}, s_2 = \{(1, 2)\}$, and find a family of mutually disjoint intervals $\{I_{s_1}, I_{s_2}, I_{s_3}\}$ such that $\text{Diam}(I_{s_i}) < \frac{1}{3}$ and $I_{s_i} \cap P$ is perfect for all $i \in \{0, 1, 2\}$. (E.g. take some interval $I \subseteq P$. Since P is prefect, I is infinite, so split it into three mutually disjoint intervals, and apply the preceding observation on each one of them).

Induction step ($n + 1$): For $s \in \mathcal{S}$ of length n, find a family of mutually disjoint intervals $\mathcal{F} = \{I_{s^\frown 1}, I_{s^\frown 2}, I_{s^\frown 3}\}$ such that $\mathcal{F} \subseteq \mathcal{P}(I_s)$ and $\text{Diam}(I_{s^\frown i}) < (\frac{1}{3})^i$ for all $i \in \{0, 1, 2\}$.

Put $\varphi(s^\frown i) := I_{s^\frown i}$ for all $i \in \{0, 1, 2\}$.

Finally, we define a fucntion $\psi : \{0, 2\}^{\mathbb{N}} \to P$. For $f \in \{0, 2\}^{\mathbb{N}}$, $\cap_{n=1}^{\infty} I_{f \restriction \{1,..,n\}}$ is a single element of P, so let $\psi(f)$ be this single element. Clearly, ψ is one-to-one.

Viewing $\{0, 2\}^{\mathbb{N}}$ as the product of length ω of the discrete space $\{0, 2\}$, we already met the type of arguments justifying why ψ is an homeomorphism on $M := \text{Im}(\psi)$ (see, e.g., Lemma 2.30). Furthere more, it is not hard to see that $int(M) = \emptyset$. Since M is closed, it is also nowhere dense. The choice of diameters in the definition of φ also ensures that M is a null set.

Finally, to see that M is perfect, assume towards a contradiction there exists some $f \in \{0, 2\}^{\mathbb{N}}$ and interval $(a, b) \subseteq \mathbb{R}$ such that $M \cap (a, b) = \{x\}$ where $x = \psi(f)$. However, by the choice of x, there exists some length $n \in \mathbb{N}$ such that $x \in I_{f \restriction \{1,..,n\}} \subseteq (a, b)$ and $I_{f \restriction \{1,..,n\}} \cap P$ is perfect. A contradiction. □

[12] $(\alpha, \beta) := \{\gamma \in L \mid \alpha < \gamma < \beta\}$ is the open interval. $[\alpha, \beta] := \{\gamma \in L \mid \alpha \leq \gamma \leq \beta\}$ is a closed interval, and so on..

Proposition 3.28. *The Cantor set is homeomorphic to* $\{0,1\}^{\mathbb{N}}$.

Remark: Once the proposition is proved, we get that the cantor set is a subspace of the Baire space.

Proof. Fix $x \in C$. $x = \sum_{n\geq 1} \frac{x_n}{3^n}$, where for all $n \in \mathbb{N}$, $x_n \in \{0,2\}$.

Define $\psi : C \to \{0,1\}^{\mathbb{N}}$ by $\psi(x) := \{\frac{x_n}{2}\}_{n\geq 1}$. ψ is obviously a bijection. Using similar methods from the proof of Lemma 2.30, we get that ψ is open and continuous as well. □

A more probabilistic point of view of the set $\{0,1\}^{\mathbb{N}}$ is the following: a coin with equiprobable outcome is tossed endlessly. We define Ω to be all infinite sequences of coin tosses, i.e., $\Omega = [0,1]$ (where *heads* is 1 and *tails* is 0, and we consider the binary representation of elements of $[0,1]$). The event "the first outcome is 0" is of probability 1/2. The event "the first two outcomes are 0" is of probability 1/4, etc.

It follows that $P([a,b)) = b - a$ whenever $0 \leq a \leq b \leq 1$ and a, b are of the form $k/2^n$. Such numbers are dense, and using monotonicity of probability measure we get that $P([a,b)) = b - a$ whenever $0 \leq a \leq b \leq 1$. This is of course the Lebesgue measure.

Example 3.29. Is $\mathbb{Q}$ a G_δ set?

Assume $\mathbb{Q} = \bigcap_{n\geq 1} G_n$ where G_n is open for all $n \in \mathbb{N}$. Obviously, G_n is dense for all $n \in \mathbb{N}$, since $\mathbb{Q} \subseteq G_n$. We get that $\mathbb{R} \setminus \mathbb{Q} = \bigcup_{n\geq 1} G_n^c$ where G_n^c is nowhere dense for all $n \in \mathbb{N}$, thus $\mathbb{R} \setminus \mathbb{Q}$ is meager. But, $\mathbb{Q}$ is also meager, hence $\mathbb{R}$ is meager, a contradiction to Baire's Theorem 3.10.

Definition 3.30. Assume X is a set. A family $F \subseteq \mathcal{P}(X)$ is a *filter over* X iff it satisfies:

- $X \in F$, and $\emptyset \notin F$.
- $A \in F$ and $A \subseteq B \subseteq X \Longrightarrow B \in F$.
- $A, B \in F \Longrightarrow A \cap B \in F$.

Intuitively, a filter is a collection of "fat" sets. It is not hard to see that if I is a proper ideal over X, then $I^* := \{X \setminus A \mid A \in I\}$ forms a filter.

It is very often that we call sets that comes from an ideal as "sets of measure zero", sets the comes from a filter as "sets of measure one", and sets that comes from outside a given ideal as "sets of positive measure".

However, this terminology might sometimes be misleading. In the following we show that it is possible for a set to be "of measure zero" from one ideal's point of view, and "of measure one" in the view of another filter.

Proposition 3.31. *$\mathcal{N} \cap \mathcal{M}^* \neq \emptyset$, that is, $\mathbb{R}$ can be decomposed as $\mathbb{R} = D \uplus M$, where M is meager and D is a null set.*

Proof. Write $\mathbb{Q}$ as $\{q_n\}_{n\geq 1}$. Let $\{\varepsilon_k\}_{k\geq 1}$ be a sequence converging to 0. For all $k \in \mathbb{N}$ pick a sequence $\{r_{k,n}\}_{n\geq 1}$ such that $r_{k,n} < r_{k-1,n}$ for all $n \in \mathbb{N}$ and $\sum_n r_{k,n} < \varepsilon_k$.

For every $k \in \mathbb{N}$, define $D_k := \bigcup_n \mathsf{B}_{r_{k,n}}(q_n)$. $D := \bigcap_k D_k$ is a null set and is comeager (By the above example). Now, define $M := \mathbb{R} \setminus D$. □

The example we give next is typical of an existence theorem based on the Baire's theorem. We show that some element of a space must have a given property by showing that the space is second category while the elements which do not have a given property form a set of first category.

Definition 3.32. For an interval $I \subseteq \mathbb{R}$, let $C(I)$ denote the family of all continuous real-valued function on I.

It is a well-known fact that a uniform limit of continuous function is continuous, thus, if we regard $C(I)$ as a metric space with $\rho(f,g) := \sup_{x\in I} |f(x) - g(x)|$ (for all $f, g \in C(I)$, then $\langle C(I), \rho\rangle$ is a complete metric space.

It is nice to see that if $\langle f_1, f_2, ...\rangle$ is a Cauchy sequence in $C(I)$, then, for each $x \in I$, $\{f_n(x)\}_{n\geq 1}$ is a Cauchy sequence of real numbers, hence converges.

Theorem 3.33. *There is a continuous real-valued functions on I (some closed interval) having a derivative at no point.*

Proof. Denote by $\mathcal{D}$ the set of all functions in $C(I)$ having a derivative somewhere.

Define for all $n \in \mathbb{N}$:

$$\mathcal{D}_n := \Big\{ f \in C(I) \Big| \text{ for some } x \in [0, \frac{n-1}{n}], \text{ whenever } h \in (0, 1/n], \Big| \frac{f(x+h) - f(x)}{h} \Big| \leq n \Big\}.$$

If $f \in C(I)$ has a derivative at some point, then for some large enough $n \in \mathbb{N}$, $f \in \mathcal{D}_n$. Hence $\mathcal{D} = \bigcup \mathcal{D}_n$. By showing that $\mathcal{D}_n$ is closed and has no interior (for all n) we will conclude that $C(I) \setminus \mathcal{D}$ is of the second category.

1. $\mathcal{D}_n$ has no interior: Given $f \in \mathcal{D}_n$ we will find a continuous function $g \notin \mathcal{D}_n$ such that $d(f,g) < \varepsilon$, that is, for all $x \in [0, \frac{n-1}{n}]$ there is some $h \in (0, 1/n]$ with $\left|\frac{g(x+h)-g(x)}{h}\right| > n$. Find a polynomial function $P(x)$ on $[0,1]$ such that $d(f, P) < 1/2$ (that is possible since polynomials functions are dense in $C(I)$ with the uniform metric). Let M be the maximum slope of P in $[0,1]$, and let $Q(x)$ be a continuous function consisting of straight line segments of slope $\pm(M + n + 1)$ constrained so that $\mid Q(x) \mid < \varepsilon/2$. Now, define $g(x) := P(x) + Q(x)$. Then $d(f,g) < d(f,P) + d(P,Q) < \varepsilon$ and:

$$\Big|\frac{g(x+h) - g(x)}{h}\Big| = \Big|\frac{P(x+h) + Q(x+h) - P(x) - Q(x)}{h}\Big| \geq \Big|\frac{Q(x+h) - Q(x)}{h}\Big| - \Big|\frac{P(x+h) - P(x)}{h}\Big|$$

But for $x \in [0, \frac{n-1}{n}]$, an $h \in (0, 1/n]$ can be found for which the latter is greater than $(M + n + 1) - M = n + 1$. Thus, $g \notin \mathcal{D}_n$.

2. $\mathcal{D}_n$ is closed: The map $e : C(I) \times I \to \mathbb{R}$ defined be $e(f, x) := f(x)$ is continuous. It follows that if h_0 is a fixed element of $(0, 1/n]$, the map $E_{h_0} : C(I) \times [0, \frac{n-1}{n}] \to \mathbb{R}$ defined by $E_{h_0}(f, x) := \left|\frac{f(x+h_0)-f(x)}{h_0}\right|$ is continuous. Thus $E_{h_0}^{-1}[0, n]$ is closed in $C(I) \times [0, \frac{n-1}{n}]$. Define $D_{h_0} := \left\{f \in C(I) \middle| (f, x) \in E_{h_0}^{-1}[0, n), \text{for some } x \in [0, \frac{n-1}{n}]\right\}$. Then D_{h_0} is closed in $C(I)$. For if $\{f_m\}_m \subseteq D_{h_0}$ where $f_m \to f$, then $\{x_m\}_m \subseteq [0, 1 - 1/n]$ such that $\{f_m, x_m\}_m \subseteq E_{h_0}^{-1}[0, n]$ has a cluster point x. Now, $(f, x) \in E_{h_0}^{-1}[0, n]$, so that $f \in D_{h_0}$.

Now, $\mathcal{D}_n = \bigcap_{h_0 \in (0,1/n]} D_{h_0}$, establishing that $\mathcal{D}_n$ is closed.

$\square$