2. 10.11.05

Lemma 2.1. *$S_{fin}(\mathcal{O},\mathcal{O})$ is a topological property, that is, whenever $\langle X_1,O_1\rangle,\langle X_2,O_2\rangle$ are topological spaces, and $f: X_1\rightarrow X_2$ is a continuous surjection, then $X_1\models S_{fin}(\mathcal{O},\mathcal{O})$ implies $X_2\models S_{fin}(\mathcal{O},\mathcal{O})$.*

Proof. Suppose $\langle \mathcal{U}_n\subseteq O_2\mid n\in\mathbb{N}\rangle$ is a family of open covers of X_2. For any relevant n, put $\mathcal{V}_n:=\{f^{-1}[U]\mid U\in\mathcal{U}_n\}$. By continuity of f, $\langle \mathcal{V}_n\subseteq O_1\mid n\in\mathbb{N}\rangle$ is a family of open covers of X_1. If $X_1\models S_{fin}(\mathcal{O},\mathcal{O})$, then there exists a witness in the form of $\langle \mathcal{G}_n\in[\mathcal{V}_n]^{<\omega}\mid n\in\mathbb{N}\rangle$. Finally, put $\mathcal{F}_n:=\{U\mid f^{-1}[U]\in\mathcal{G}_n\}$ and notice that $\langle\mathcal{F}_n\in[\mathcal{U}_n]^{<\omega}\mid n\in\mathbb{N}\rangle$ exemplifies $S_{fin}(\mathcal{O},\mathcal{O})$ for X_2. □

Definition 2.2. For a topological space $\langle X,O\rangle$, put:
- $d(X):=\min\{|D|\mid D\subseteq X \text{is dense in } X\}+\aleph_0$,
- $w(X):=\min\{|\mathcal{B}|\mid \mathcal{B} \text{ is a basis to } \langle X,O\rangle\}+\aleph_0$,
- $L(X):=\min\{\mu\in\text{ICN}\mid \text{ every open cover of } X \text{ contains a subcover of cardinality } \leq\mu\}$.[6]

In the above terminology, a space $\langle X,O\rangle$ is *separable* iff $d(X)=\aleph_0$, is *seocond-countable* iff $w(X)=\aleph_0$, and is *Lindelöf* iff $L(X)=\aleph_0$.

Lemma 2.3. *For any topological space $\langle X,O\rangle$: $d(X)\leq w(X)$ and $L(X)\leq w(X)$.*

Proof. Fix a basis $\mathcal{B}\in[O]^{w(X)}$. For any choice function $f\in\prod_{U\in O}U$, $\mathrm{Im}(f)$ is a dense subset (since its intersection with any non-trivial open sets is never empty). Also $|\mathrm{Im}(f)|\leq w(X)$.

To see that $L(X)\leq w(X)$, fix an open cover $\mathcal{U}$. Pick $\psi: O\rightarrow\mathcal{B}$ such that $U=\bigcup\psi(U)$ for all $U\in O$. Now $\mathcal{V}:=\bigcup\{\psi(U)\mid U\in\mathcal{U}\}\subseteq\mathcal{B}$ is a cover of X and $|\mathcal{V}|\leq|\mathcal{B}|$. For each $G\in\mathcal{V}$, pick $G'\in\mathcal{U}$ such that $G'\subseteq G$.

Finally, $\{G'\mid G\in\mathcal{V}\}\subseteq\mathcal{U}$ is a subcover of cardinality $\leq|\mathcal{B}|=w(X)$. □

To complete the picture, we include the following two observations:

Observation 2.4. *There exists a topological space $\langle X,\tau\rangle$ with $\aleph_0=d(X)<w(X)=\aleph_1$.*

Proof. Take $X:=\omega_1$ and $\tau:=\big\{\{0,\alpha\}\mid\alpha<\omega_1\big\}$. Evidently $\{0\}$ is a dense subset. Notice that if $\mathcal{B}$ is a basis to X, then $\mathcal{B}=\tau$. It follows that $w(X)=\aleph_1$. □

Observation 2.5. *There exists a topological space $\langle X,\tau\rangle$ with $\aleph_0=L(X)<w(X)=\aleph_1$.*

Proof. Put $X=\omega_1$ and $\tau:=\{\alpha^{\uparrow}\mid\alpha<\omega_1\}$, where $\alpha^{\uparrow}:=\{\beta<\omega_1\mid\beta>\alpha\}$. Since a basis to this space induces an unbounded set in ω_1 and a countable union of countable sets is countable, $w(X)$ must equal $\aleph_1$. To see that $L(X)=\aleph_0$, fix a cover $\mathcal{U}$ of X.

[6] ICN stands for the class of infinite cardinal numbers.

Put $\gamma := \min\{\alpha < \omega_1 \mid \exists U \in \mathcal{U}(\alpha^\uparrow \subseteq U)\}$ and let U_γ be an exemplifying set, i.e., $\gamma^\uparrow \subseteq U_\gamma$. Now, for all $\beta < \gamma$ (there are only countable many!), find $U_\beta \in \mathcal{U}$ such that $\beta \in U_\beta$.

It follows that $\{U_\beta \mid \beta \leq \gamma\} \subseteq \mathcal{U}$ is a countable subcover for X. □

It is not by chance that the two spaces mentioned above are not metric:

Lemma 2.6. *If $\langle X, d\rangle$ is a metric space, then $w(X) = d(X) = L(X)$.*

Proof. Fix a dense subset $D \in [X]^{d(X)}$. Put $\mathcal{B} := \{\mathsf{B}_{\frac{1}{n}}(x) \mid x \in D, n \in \mathbb{N}^+\}$. We shall show that $\mathcal{B}$ is a basis, and conclude that $w(X) \leq |\mathcal{B}| = |D| = d(x)$. Fix $y \in X$ and $\delta \in \mathbb{R}^+$. Since D is dense, we may find $x \in D \cap \mathsf{B}_\delta(y)$. Since $x \in \mathsf{B}_\delta(y)$ and the latter is open, then x is an interior point, and hence for a large enough $n \in \mathbb{N}$, we have that $\mathsf{B}_\delta(y) \supseteq \mathsf{B}_{\frac{1}{n}}(x) \in \mathcal{B}$ and we are done.

We now show $d(X) \leq L(X)$. For $n \in \mathbb{N}^+$, it is clear that $\{\mathsf{B}_\delta(x) \mid x \in X, \delta \in (0, \frac{1}{n})\}$ is an open cover of X. Now, by definition of $L(X)$, for all $n \in \mathbb{N}^+$, there exists two families $\{x_{i,n} \in X \mid i < L(X)\}$ and $\{\delta_{i,n} \in (0, \frac{1}{n}) \mid i < L(X)\}$ s.t. $\{\mathsf{B}_{\delta_{i,n}}(x_{i,n}) \mid i < L(X)\}$ covers X.

Put $D := \{x_i^n \mid n \in \mathbb{N}^+, i < L(X)\}$. Evidently, $|D| \leq L(X)$. We are left with showing that D is dense, that is, to show that every member of X is a limit point of D. Fix $y \in X$.

Since the above families covers X, for all $n \in \mathbb{N}^+$, there exists i_n such that $y \in \mathsf{B}_{\delta_{i_n,n}}(x_{i_n,n})$, in particular, $d(y, x_{i_n,n}) < \frac{1}{n}$, hence, $\lim_{n\to\infty} d(y, x_{i_n,i}) = 0$. Since $\{x_{i_n,n} \mid n \in \mathbb{N}^+\} \subseteq D$, then we conclude that y is a limit point of D. □

Definition 2.7. For a topological space $\langle X, O\rangle$, let $I(X) := \{x \in X \mid \{x\} \in O\}$ denote the family of all isolated points of X.

It is obvious that for all $Y \subseteq X$, if $\exists z \in I(X) \setminus Y$, then $z \notin \overline{Y}$ as well. Hence:

Lemma 2.8. *If $\langle X, O\rangle$ is a topological space and $D \subseteq X$ is a dense subset, then $I(X) \subseteq D$.*

In particular, $|I(X)| \leq d(X)$.

Theorem 2.9 (Hurewicz, Lelek)**.** *Suppose $\langle X, d\rangle$ is a metric space.*

Then $X \models S_{fin}(\mathcal{O}, \mathcal{O})$ iff X satisfies Menger's Basis property.

Proof. ($\Rightarrow$) Suppose $\mathcal{B}$ is a basis for the space. It follows that for all $x \in X$ and $n \in \mathbb{N}$, we may find $B_{x,n} \in \mathcal{B}$ with $x \in B_{x,n}$ and $\mathrm{Diam}(B_{x,n}) < \frac{1}{n+1}$. Now apply $S_{fin}(\mathcal{O}, \mathcal{O})$ to $\langle \{B_{x,n} \mid x \in X\} \mid n \in \mathbb{N}\rangle$ and find $\mathcal{F}_n \in [\{B_{x,n} \mid x \in X\}]^{<\omega}$ such that X is covered by $\mathcal{F}_n$ for all $n \in \mathbb{N}$. The proof now continues in the same fashion of Claim 1.25, we find an enumeration $\{B_n \mid n \in \mathbb{N}\}$ of $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ such that $\lim_{n\to\infty} \mathrm{Diam}(B_n) = 0$.

($\Leftarrow$) Fix a family of open covers $\langle \mathcal{U}_n \mid n \in \mathbb{N}\rangle$.

For $x, y \in X$ and $\delta \in \mathbb{R}^+$, put $\mathsf{DB}_\delta(x, y) := \mathsf{B}_\delta(x) \cup \mathsf{B}_\delta(y)$. Let $J(X) := \big\{\{x\} \mid x \in I(X)\big\}$.

For all $n \in \mathbb{N}$, define $\mathcal{V}_n$ to be:

$$\left\{\mathsf{DB}_\delta(x,y) \;\middle|\; x,y \in X, d(x,y) > \frac{1}{n+1}, \delta \in \mathbb{R}^+, \exists\{U',U''\} \in [\mathcal{U}_n]^{\leq 2}\big(\mathsf{DB}_\delta(x,y) \subseteq U' \cup U''\big)\right\}.$$

Claim 2.10. *$\mathcal{B} := \bigcup_{n\in\mathbb{N}} \mathcal{V}_n \cup J(X)$ is a basis to $\langle X, d\rangle$.*

Proof. Fix $x \in X, \varepsilon \in \mathbb{R}^+$ and $y \in \mathsf{B}_\varepsilon(x)$. We shall find $U \in \mathcal{B}$ with $y \in U \subseteq \mathsf{B}_\varepsilon(x)$.

Since $J(X) \subseteq \mathcal{B}$, we may assume $y \neq x$. Pick $n \in \mathbb{N}$ large enough such that $d(x,y) > \frac{1}{n+1}$. Now, since $X = \bigcup \mathcal{U}_n$, there exists $\{U',U''\} \in [\mathcal{U}_n]^{\leq 2}$ such that $x \in U', y \in U''$. Since U' is open and $U'' \cap \mathsf{B}_\varepsilon(x)$ is open, we may find some positive $\delta < \varepsilon$ small enough such that $\mathsf{B}_\delta(x) \subseteq U'$ and $\mathsf{B}_\delta(y) \subseteq U'' \cap \mathsf{B}_\varepsilon(x)$. By the choice of δ, we have $\mathsf{B}_\delta(x) \cup \mathsf{B}_\delta(y) \subseteq \mathsf{B}_\varepsilon(x)$.

It now follows that $U := \mathsf{B}_\delta(x) \cup \mathsf{B}_\delta(y) = \mathsf{DB}_\delta(x,y) \in \mathcal{B}$ and $y \in U \subseteq \mathsf{B}_\varepsilon(x)$. $\square$

Assume that X satisfies Menger's basis property. By Lemmas 1.21,2.6,2.8, we may enumerate $I(X) = \{x_i \mid i \in \mathbb{N}\}$. Also, the hypothesis implies the existence of a family $\mathcal{F} = \{B_n \in \mathcal{B} \mid n \in \mathbb{N}\}$ such that $X = \bigcup_{n\in\mathbb{N}} B_n$ and $\lim_{n\to\infty} \mathrm{Diam}(B_n) = 0$.

Fix $n \in \mathbb{N}$ and let $\mathcal{F}_n := \mathcal{F} \cap \mathcal{V}_n$. Since $\lim_{n\to\infty} \mathrm{Diam}(B_n) = 0$ and $\mathrm{Diam}(U) > \frac{1}{n+1}$ for all $U \in \mathcal{V}_n$, we must conclude that $\mathcal{F}_n$ is finite. Also, by the definition of $\mathcal{B}$ and $\mathcal{F}$:

$$X = \bigcup \mathcal{F} = \bigcup\Big(\bigcup_{n\in\mathbb{N}} \mathcal{F}_n \cup J(X)\Big) = \bigcup\bigcup_{n\in\mathbb{N}} \big(\mathcal{F}_n \cup \{x_n\}\big).$$

Now, for all $U \in \mathcal{F}_n$, find $U', U'' \in \mathcal{U}_n$ such that $U \subseteq U' \cup U''$, and also find $G_n \in \mathcal{U}_n$ such that $x_n \in G_n$. Put $\mathcal{F}'_n := \{U', U'' \mid U \in \mathcal{F}_n\} \cup \{G_n\} \subseteq \mathcal{U}_n$.

It easy to see that $|\mathcal{F}'_n| \leq 2 \cdot |\mathcal{F}_n| + 1 < \aleph_0$ and that $\bigcup_{n\in\mathbb{N}} \mathcal{F}'_n$ covers X. $\square$

Corollary 2.11. *Menger's basis property does not depend on the choice of metric for any given metric space.*

Definition 2.12. Suppose I is some index set and $\langle X_i \mid i \in I\rangle$ is a sequence of sets.

The *Cartesian product* of $\langle X_i \mid i \in I\rangle$ is:

$$\prod_{i\in I} X_i = \Big\{f : I \to \bigcup_{i\in I} X_i \mid f(i) \in X_i \text{ for all } i \in I\Big\}$$

In practice, for $x \in \prod_{i\in I} X_i$, we usually write x_i instead of $x(i)$, and x_i is referred as the *i-th coordinate* of x.

The map $\pi_j : \prod_{i\in I} X_i \to X_j$, defined by $\pi_j(x) = x_j$, is called the *projection map* of $\prod_{i\in I} X_i$ on X_j.

Remark: we need the axiom of choice to ensure that the cartesian product of a non-empty collection of non-empty sets is indeed non-empty.

Definition 2.13. Suppose A is some index set. Assume that $\langle\langle X_\alpha, O_\alpha\rangle \mid \alpha \in A\rangle$ is a family of topological spaces. The *product topology* (or *Tychonoff topology*) on $\prod_{\alpha\in A} X_\alpha$ is obtained by taking as a (canonical) base for the space $\langle \prod_{\alpha\in A} X_\alpha, O\rangle$, the family :

$$\mathcal{B} := \left\{ \prod_{\alpha\in A} U_\alpha \middle| \begin{array}{c} U_\alpha \in O_\alpha \text{ for each } \alpha \in A \\ \{\alpha \in A \mid U_\alpha \neq X_\alpha\} \text{ is finite} \end{array} \right\}.$$

Notice that the set $\prod_{\alpha\in A} U_\alpha$, where $U_\alpha = X_\alpha$ except for $\alpha = \alpha_1, ..., \alpha_n$, can be written as:

$$\prod U_\alpha = \pi^{-1}_{\alpha_1}(U_{\alpha_1}) \cap \cdots \cap \pi^{-1}_{\alpha_n}(U_{\alpha_n}),$$

Thus, the product topology is precisely that topology which has for a subbase the collection $\{\pi^{-1}_\alpha(U_\alpha) \mid \alpha \in A, U_\alpha \text{ is open in } X_\alpha\}$. Moreover, the sets U_α can be restricted to be taken from some fixed subbases for each of the spaces $\langle X_\alpha, O_\alpha\rangle$ (think why?).

Example 2.14. Consider now the Baire space $\mathbb{N}^\mathbb{N} := \prod_{n\in\mathbb{N}} \mathbb{N}$ where $\mathbb{N}$ is equipped with the discrete topology. A subbase for this product topology is of the form $\{\pi^{-1}_n(\{k\}) \mid n, k \in \mathbb{N}\}$. The canonical base for $\mathbb{N}^\mathbb{N}$ is $\mathcal{B} := \{\sigma^\uparrow \mid \exists I \in [\mathbb{N}]^{<\omega}(\sigma \text{ is a function from } I \text{ to } \mathbb{N})\}$, where $\sigma^\uparrow := \{g \in \mathbb{N}^\mathbb{N} \mid g \restriction \mathrm{dom}(\sigma) = \sigma\}$. It is a nice observation that the following is also a base:

$$\Big\{\{(n_1, \ldots, n_k)\} \times \mathbb{N}^\mathbb{N} \Big| n_1, \ldots, n_k, k \in \mathbb{N}\Big\} = \Big\{\sigma^\uparrow \Big| k \in \mathbb{N}(\sigma \text{ is a function from } \{1, .., k\} \text{ to } \mathbb{N})\Big\}.$$

An easy proposition to formulate is the following,

Proposition 2.15. *The βth projection is continuous and open, and the Tychonoff topology is the weakest topology on $\prod X_\alpha$ for which each projection π_β is continuous.*

Proof. The first part is trivial by definitions. Let O be a topology on the product in which each projection is continuous, then for each β, if U_β is open in X_β, we get that $\pi^{-1}_\beta(U_\beta) \in O$. Thus, the members of a subbase for the Tychonoff topology all belong to O, hence the Tychonoff topology is contained in O. □

Definition 2.16. Suppose $\langle X, O\rangle$ is a topological space and some $A \subseteq X$.
- A is G_δ iff it is the countable intersection of open sets.
- A is F_σ iff it is the countable union of closed sets.

Evidently, an open set is G_δ and a closed set is F_σ. In metric spaces, closed set is also G_δ.

Definition 2.17. Let $\langle X, O\rangle$ be a topological space. A set $A \subseteq X$ is *nowhere dense* in X iff $int(\overline{A}) = \emptyset$. A set $A \subseteq X$ is of the *first category* (or *meager*) iff $A = \bigcup_{n\in\mathbb{N}} A_n$ where A_n is nowhere dense for all $n \in \mathbb{N}$. All other subsets of X are said to be of the *second category.*[7]

[7] $int(A)$ stands for the interior of A, that is, the family of all interior points of A.

Remark: It is by definition that A is nowhere dense iff $\overline{A}$ is nowhere dense. Consider a meager set A. Now, $A = \bigcup_{n\geq 1} A_n \subseteq \bigcup_{n\geq 1} \overline{A_n}$, and we conclude that every meager set is a subset of some meager F_σ set.

Fact 2.18. *Suppose $\langle X, O\rangle$ is a topological space and some $A \subseteq X$. Then:*
- $bnd(X \setminus A) = bnd(A)$.[8]
- $\overline{A} = A \cup bnd(A)$.
- $X = int(A) \uplus bnd(A) \uplus int(X \setminus A)$.

Lemma 2.19. *Suppose $\langle X, O\rangle$ is a topological space and some $A \subseteq X$.*
Then A is nowhere dense iff $(X \setminus \overline{A})$ is dense in X.

Proof. Suppose A is nowhere dense. By $X = int(\overline{A}) \cup bnd(\overline{A}) \cup int(X \setminus \overline{A})$, we get that:

$$X = bnd(\overline{A}) \cup int(X \setminus \overline{A}) = bnd(X \setminus \overline{A}) \cup int(X \setminus \overline{A}) = \overline{X \setminus \overline{A}},$$

i.e., that $X \setminus \overline{A}$ is dense in X. The other direction is similar. □

Example 2.20 (The Cantor set)**.** Beginning with the unit interval $I = [0, 1]$, we will define closed subsets $I_1 \supset I_2 \supset \cdots$ in I as follows. We obtain I_1 by removing the interval $(\frac{1}{3}, \frac{2}{3})$ from I. I_2 is obtained by removing from I_1 the intervals $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$. In general, having I_{n-1}, I_n is obtained by removing the open middle third of the 2^{n-1} closed intervals that make I_{n-1}.

The Cantor set is obtained by intersecting all these closed sets, $C := \bigcap_{n\in\mathbb{N}} I_n$.

We develop an interesting alternative description of the cantor set. Each $x \in I$ has an expansion $(x_1, x_2, ...)$ in trinary form, that is $x_i \in \{0, 1, 2\}$ for all $i \in \mathbb{N}$, and $x = \sum_{n\in\mathbb{N}} \frac{x_n}{3^n}$. These expressions are unique, except that any number (but 1) expressible in an expansion ending in a sequence of 2's can be re-expressed in an expansion ending in a sequence of 0's. For example, $\frac{1}{3}$ can be written as $(0, 2, 2, 2, ...)$ and also as $(1, 0, 0, 0, ...)$. We agree to use only expressions of the first type. Then the Cantor set is precisely the set of points in I having a trinary expansion without 1's.

The Cantor set is closed, so in order to show that it is nowhere dense we are left with showing that it has no interior. Every base set$(a, b) \subset [0, 1]$ contains some element with 1 in it's trinary decomposition. Hence $(a, b) \nsubseteq C$, thus C is nowhere dense.

Another way of showing that is the following: assume $(a, b) \subset C$ for some $0 \leq a < b \leq 1$. From monotonicity of the Lesbegue measure m, we get that $b - a = m((a, b)) \leq m(C) = 0$, a contradiction. To see that indeed $m(C) = 0$ notice that $m(C) = \lim_{n\to\infty}(\frac{2}{3})^n$.

[8] $bnd(A)$ stands for the boundary of A.

Definition 2.21. Let A be a set in a topological space $\langle X, O\rangle$.

A point $x \in X$ is an *accumulation point* of A iff any $U \in O$ with $x \in U$, satisfies $U \cap A \neq \{x\}$.

A point $x \in A$ is an *isolated point* of A iff $x \in A \backslash A^d$, where A^d is the set of all accumulation points of A

Definition 2.22. A set F is *perfect* iff F is closed, non-empty, and dense in itself; i.e., each point of F is an accumulation point of F (F does not contain isolated points).

Definition 2.23. $x \in X$ is a *Condensation point* of A if $A \cap U_x$ is not countable for all $U_x \in O$ with $x \in U_x$. We denote by $\mathrm{cond}(A)$ the set of all condensation points of A.

Remark: Notice that $I(A) \subseteq A \setminus \mathrm{cond}(A)$.

Theorem 2.24 (Cantor-Bendixson)**.** *Suppose $\langle X, O\rangle$ is a second-countable topological space (i.e., $w(X) = \aleph_0$). Then every closed set F can be written as the decomposition $F = P \uplus N$, where P is perfect, and N is countable.*

Proof. The proof is technical and non-trivial. We will formulate results (and prove some of them) towards the theorem's proof.

Lemma 2.25. *Any topological space $\langle X, O\rangle$ can be decomposed as $X = P \uplus N$, where P is perfect and N is* scattered *(that is, N doesn't contain any set which is dense in itself).*

Proof. Put $\mathcal{A} := \{A \subseteq X \mid A \text{ is dense in itself}\}$ and $P := \bigcup \mathcal{A}$. We claim that P is perfect.

Suppose first that $\bigcup \mathcal{A}$ is not dense in itself, thus, there exists a point $x \in \bigcup \mathcal{A}$ which is isolated in the relative topology of $\bigcup \mathcal{A}$. In particular, $x \in A_0$ for some $A_0 \in \mathcal{A}$.

Now, there is an open set U_x such that $U_x \cap (\bigcup \mathcal{A}) = \{x\}$, therefore $U_x \cap A_0 = \{x\}$, a contradiction to the fact that A_0 is dense in itself.

We know that P is dense in itself and left with showing that P is closed. We will do that by proving that the closure of a set dense in itself, is a set dense in itself.

Assume A is dense in itself and $x \in \overline{A} \setminus A$, that is, $U \cap A \neq \{x\}$ for every open set U containing x. It follows that $\overline{A}$ is dense in itself.

Put $N := X \setminus P$. By the definition of P, N must be scattered. □

Lemma 2.26. $\mathrm{cond}(A)$ *is a closed set and* $\mathrm{cond}(A \cup B) = \mathrm{cond}(A) \cup \mathrm{cond}(B)$.

Lemma 2.27. *In a second-countable space,* $A \setminus \mathrm{cond}(A)$ *is countable and* $\mathrm{cond}(\mathrm{cond}(A)) = \mathrm{cond}(A)$.

Proof. Fix a countable base $\mathcal{B}$ and a point $x \in A \setminus \mathrm{cond}(A)$. There exists $U_x \in O$ such that $U_x \cap A$ is countable, hence $B_x \cap A$ is countable for all $B_x \in \mathcal{B}$ such that $B_x \subseteq U_x$. Now $\{B_x \mid x \in A \setminus \mathrm{cond}(A)\}$ is countable, thus $A \setminus \mathrm{cond}(A)$ is countable.

Now, $A = (A \setminus \mathrm{cond}(A)) \cup \mathrm{cond}(A)$. Using the previuos lemma we obtain:

$\mathrm{cond}(A) = \mathrm{cond}(A \setminus \mathrm{cond}(A)) \cup \mathrm{cond}(\mathrm{cond}(A)) = \emptyset \cup \mathrm{cond}(\mathrm{cond}(A)) = \mathrm{cond}(\mathrm{cond}(A))$.

□

Define $P := \mathrm{cond}(X)$ and $N := X \setminus P$. By definition, P is dense in itself, and from previous results it is closed, and N is countable. Hence the theorem is proved. □

Definition 2.28. Assume that X and Y are topological spaces. A function $f : X \to Y$ is an *homeomorphism* iff f is a continuous open bijection.

If there exists an homeomorphism from X to Y, we say that X and Y are *homeomorphic*.

Remark: two spaces are homeomorphic if they are equipped with the "same" topology.

Theorem 2.29. *The Baire space* $\mathbb{N}^{\mathbb{N}}$ *is homeomorphic to* $(0,1) \setminus \mathbb{Q}$.

Proof. We break the proof into several lemmas.

Lemma 2.30. *The Baire space is homeomorphic to* $(0,1) \setminus \left\{\frac{k}{2^n} \mid n \in \mathbb{N}, k < 2^n\right\}$.

Proof. Put $\omega := \mathbb{N} \cup \{0\}$, $D := \left\{\frac{k}{2^n} \mid n \in \mathbb{N}, k < 2^n\right\}$ and let $A := (0,1) \setminus D$.

Suppose $B \subseteq A$ is a subset of the form $B = (\frac{n}{2^k}, \frac{n+1}{2^k})$ where $n \in \omega, k \in \mathbb{N}$ and $n < 2^k$. For $m \in \omega$, let $B_m := (\frac{n}{2^k} + \frac{m}{2^{k+m}}, \frac{n}{2^k} + \frac{m+1}{2^{k+m+1}})$. Since $D \cap A = \emptyset$, it is easily seen that $B = \biguplus_{m=0}^{\infty} B_m$. For m_1, m_2, we write B_{m_1,m_2} for $(B_{m_1})_{m_2}$, and so forth..

We shall now define an homeomorphism $\psi : A \to \omega^{\omega}$.[9] Fix $x \in A$.

For notational simplicity, denote $f_x := \psi(x)$. We define $f_x(n)$ by recursion on $n \in \omega$.

For $n = 0$, let $f_x(1)$ be the unique $m \in \omega$ such that $x \in A_m$. For the recursive step, let $f_x(n+1)$ be the unique $m \in \omega$ such that $x \in A_{f(0),..,f(n),m}$.

Evidently, the above defines a bijection. We prove that ψ is open and leave the proof of continuity for the reader, since the idea of the proof is essentially the same.

Pick an open set $U \subseteq A$ and $f \in \psi[U]$. We shall show that f is an interior point of $\psi[U]$. Let $x := \psi^{-1}(f)$. Since x is an interior point of U, we may pick $n \in \omega, k \in \mathbb{N}$ such that $x \in (\frac{n}{2^k}, \frac{n+1}{2^k}) \subseteq U$. Since $\{x\}$ equals the intersection of the decreasing chain $\{A_{f(0),..,f(m)} \mid m \in \omega\}$, there must exist some $m \in \omega$ such that $A_{f(0),..,f(m)} = (\frac{n}{2^k}, \frac{n+1}{2^k})$. Now, put $\sigma := f \restriction \{0,..,m\}$. Clearly, $f \in \sigma^{\uparrow} \subseteq \psi[U]$, where $\sigma^{\uparrow}$ is like in Example 2.14. □

Lemma 2.31 (Cantor)**.** *Any two dense countable sets in* $(0,1)$ *are homeomorphic.*

Proof. Suppose $D = \{d_n\}_{n \geq 1}$ and $E = \{e_n\}_{n \geq 1}$ are dense in $(0,1)$.

We define by induction on $n \in \mathbb{N}$ an increasing chain of partial functions $\{\psi_n : D_n \to E \mid n \in \mathbb{N}\}$ where $D_n \in [D]^n$ for any relevant n.

[9]Clearly ψ would induce an homeomorphism from A to $\mathbb{N}^{\mathbb{N}}$.

Induction base: for $n = 1$, let $D_1 := \{d_1\}$ and $\psi_1(d_1) := e_1$.

Induction hypothesis : ψ_n is order-preserving.

Inductive step: We divide into two case.

For $n+1$ where n is even, Put $j := \min\{j \in \mathbb{N} \mid d_j \notin D_n\}$, and let j_1, j_2 be such that:

$$d_{j_2} := \min\{d \in D_n \mid d > d_j\} \text{ and } d_{j_1} := \max\{d \in D_n \mid d < d_j\}.$$

Now, since E is a dense subset, $\big(\psi_n(d_{j_1}), \psi_n(d_{j_2})\big) \cap E$ is non-empty. So let $i := \min\{i \in \mathbb{N} \mid e_i \in \big(\psi_n(d_{j_1}), \psi_n(d_{j_2})\big)\}$. Let $D_{n+1} := D_n \cup \{d_j\}$ and extend ψ_n to ψ_{n+1} such that $\psi_{n+1}(d_j) = e_i$. By the hypothesis, ψ_n is order-preserving bijection, thus ψ_{n+1} is order-preserving, $e_i \notin \mathrm{Im}(\psi_n)$, and ψ_{n+1} is bijective.

For $n+1$ where n is odd, Put $j := \min\{j \in \mathbb{N} \mid e_j \notin \mathrm{Im}(\psi(D_n)\}$, and let j_1, j_2 be such that:

$$d_{j_2} := \min\{d \in D_n \mid \psi(d) > e_j\} \text{ and } d_{j_1} := \max\{d \in D_n \mid \psi(d) < e_j\}.$$

Now, since D is a dense subset, we may define $i := \min\{i \in \mathbb{N} \mid d_i \in \big(d_{j_1}, d_{j_2}\big)\}$. Let $D_{n+1} := D_n \cup \{d_j\}$ and extend ψ_n to ψ_{n+1} such that $\psi_{n+1}(d_j) = e_i$. End of the construction.

Clearly, the construction ensures that for all $d \in D$ and $e \in E$, there exists some large enough $n \in \mathbb{N}$ such that $d \in \mathrm{dom}(\psi_n)$ and $e \in \mathrm{Im}(\psi_n)$ and we are done by letting $\psi := \bigcup_{n \in \mathbb{N}} \psi_n$.

Finally, since ψ is an order-preserving bijection, then ψ is also an homeomorphism.

□

Lemma 2.32. *The complements of two dense countable sets in* $(0,1)$ *are homeomorphic.*

Proof. Let D^c and E^c be the complements of some two dense countable sets in $(0,1)$, and let $\psi : D \to E$ be an homeomorphism.

We shall now define an homeomorphism $\varphi : D^c \to E^c$. Fix $x \in D^c$. Fix a convergent sequence $\{d_n\}_{n \geq 1} \subseteq D$ such that $\lim d_n = x$ and let $\varphi(x) := \lim \psi(d_n)$. Now, $\{\psi(d_n)\}_{n \geq 1}$ is Cauchy in E, but assume that $\lim \psi(d_n) \in E$. ψ is an homeomorphism hence ψ^{-1} is continuous, so $\psi^{-1}(\lim \psi(d_n)) = \lim(\psi^{-1}\psi(d_n)) = \lim d_n = x \notin D$, a contradiction to the fact that the range of ψ^{-1} is D.

φ is well defined (think why?), and since it is an order-preserving bijection, then ψ is an homeomorphism. □

This completes the proof of 2.29. □