12. 02.02.06

Definition 12.1. For families $\mathcal{A}, \mathcal{B}$, let $G_1(\mathcal{A}, \mathcal{B})$ denote the game of length ω, where at round $n \in \mathbb{N}$, player **I** picks $\mathcal{U}_n \in \mathcal{A}$ and player **II** responds with picking $U_n \in \mathcal{U}_n$.

$$\begin{array}{llllll}
\textbf{I}: & \mathcal{U}_1 \in \mathcal{A} & & \mathcal{U}_2 \in \mathcal{A} & & \dots \\
& \searrow & \nearrow & \searrow & & \\
\textbf{II}: & & U_1 \in \mathcal{U}_1 & & U_2 \in \mathcal{U}_2 & \dots
\end{array}$$

Player **II** *wins this game* if $\{U_n \mid n \in \mathbb{N}\} \in \mathcal{B}$, otherwise, player **I** wins the game.

Let $\mathrm{Seq}(A)$ denote the family of finite sequences (including the empty sequence) with elements from a given set A. If $s = \langle x_1, .., x_n \rangle \in \mathrm{Seq}(A)$ and $x \in A$, then $s^\frown x := \langle x_1, .., x_n, x \rangle$.

Definition 12.2. A function $\sigma : \mathrm{Seq}(\bigcup \mathcal{A}) \to \mathcal{A}$ is a *winning strategy* for player **I** in $G_1(\mathcal{A}, \mathcal{B})$ iff player **I** plays according to this strategy, then he wins the game:

$$\begin{array}{llllllll}
\textbf{I}: & \mathcal{U}_1 := \sigma(\emptyset) & & \mathcal{U}_2 := \sigma\langle U_1 \rangle & & \mathcal{U}_3 := \sigma\langle U_1, U_2 \rangle & \dots \\
& \searrow & \nearrow & \searrow & \nearrow & & \\
\textbf{II}: & & U_1 \in \mathcal{U}_1 & & U_2 \in \mathcal{U}_2 & & \dots
\end{array}$$

Definition 12.3. A function $\tau : \mathrm{Seq}(\mathcal{A}) \setminus \{\emptyset\} \to \bigcup \mathcal{A}$ is a *winning strategy* for player **II** in $G_1(\mathcal{A}, \mathcal{B})$ iff player **II** plays according to this strategy, then he wins the game:

$$\begin{array}{llllll}
\textbf{I}: & \mathcal{U}_1 \in \mathcal{A} & & \mathcal{U}_2 \in \mathcal{A} & & \dots \\
& \searrow & \nearrow & \searrow & & \\
\textbf{II}: & & U_1 := \tau\langle \mathcal{U}_1 \rangle & & U_2 := \tau\langle \mathcal{U}_1, \mathcal{U}_2 \rangle & \dots
\end{array}$$

Definition 12.4. For a given families $\mathcal{A}, \mathcal{B}$, we write $\mathbf{I} \uparrow G_1(\mathcal{A}, \mathcal{B})$ to denote that player **I** has winning strategy in $G_1(\mathcal{A}, \mathcal{B})$. We define $\mathbf{I} \not\uparrow G_1(\mathcal{A}, \mathcal{B}), \mathbf{II} \uparrow G_1(\mathcal{A}, \mathcal{B}), \mathbf{II} \not\uparrow G_1(\mathcal{A}, \mathcal{B})$ in the obvious fashion.

The game $G_1(\mathcal{A}, \mathcal{B})$ is said to be *determined* iff $\mathbf{I} \uparrow G_1(\mathcal{A}, \mathcal{B}) \vee \mathbf{II} \uparrow G_1(\mathcal{A}, \mathcal{B})$.

Note that both players cannot have a winning strategy in the same game.

Observation 12.5. *Suppose $\langle X, O \rangle$ is a topological space, and $\mathcal{A}, \mathcal{B}$ are given families.*
Then $X \models \mathbf{II} \uparrow G_1(\mathcal{A}, \mathcal{B})$ implies $X \models \mathbf{I} \not\uparrow G_1(\mathcal{A}, \mathcal{B})$ implies $X \models S_1(\mathcal{A}, \mathcal{B})$.

Notice that if $G_1(\mathcal{A}, \mathcal{B})$ is determined, then $X \models \mathbf{II} \uparrow G_1(\mathcal{A}, \mathcal{B})$ iff $X \models \mathbf{I} \not\uparrow G_1(\mathcal{A}, \mathcal{B})$.

Lemma 12.6. *Suppose $\langle X, O \rangle$ is a topological space.*
We define the following cardinal function invariant:

$$\delta(X) := \min\{\kappa + \aleph_0 \mid \exists \mathcal{F} \in [[O]^{\leq\kappa}]^{\leq\kappa}.\forall \mathcal{U} \in \mathcal{F}\Big(\bigcup \mathcal{U} = X\Big) \wedge \forall \phi \in \prod \mathcal{F}\Big(|\bigcap \mathrm{Im}(\phi)| \leq \kappa\Big)\}.$$

Lemma 12.7. *If $\langle X, d \rangle$ is a metric space, then $\delta(X) \leq d(X)$.*

Proof. Put $\kappa := \delta(X)$. Let $D := \{x_i \mid i < \kappa\}$ enumerate a dense subset of X.

Let $\mathcal{F} := \{\mathcal{U}_n \mid n \in \mathbb{N}\}$, where $\mathcal{U}_n := \{\mathsf{B}_r(x_i) \mid i < \kappa, r \in \mathbb{Q} \cap (0, \frac{1}{n+1})\}$ for all $n \in \mathbb{N}$.

Since $|\mathcal{U}_n| \leq \aleph_0 \cdot \kappa = \kappa$ for all $n \in \mathbb{N}$, we have that $\mathcal{F} \in [[O]^{\leq\kappa}]^{\aleph_0}$, where O denotes the family of all open sets in this metric space. In particular, $\mathcal{F} \in [[O]^{\leq\kappa}]^{\leq\kappa}$.

Since D is a dense subset, we also have that $\bigcup \mathcal{U}_n = X$ for all $n \in \mathbb{N}$.

Finally, if $\phi \in \prod \mathcal{F}$ is a choice function, then letting $U_n := \phi(\mathcal{U}_n)$ for all $n \in \mathbb{N}$, we get that $\lim_{n\to\infty} \mathrm{Diam}(U_n) = 0$, and hence $\bigcap \mathrm{Im}(\phi) \leq 1 \leq \kappa$. □

Theorem 12.8. *Suppose $\langle X, O\rangle$ is a topological space, $\mathcal{O} := \mathcal{O}_X$, and $X \models \mathbf{II} \uparrow G_1(\mathcal{O}, \mathcal{O})$.*

Then $|X| \leq \delta(X)$.

Proof. Let $\mathcal{F} \in [[O]^{\leq\kappa}]^{\leq\kappa}$ be a witness to the value of $\kappa := \delta(X)$. We shall examine the outcome of the game $G_1(\mathcal{O}, \mathcal{O})$ when player **II** plays with a winning strategy, τ, against members of $\mathcal{F}$. For any sequence $s \in \mathrm{Seq}(\mathcal{F})$, let $\mathcal{A}_s := \{\tau(s^\frown \mathcal{U}) \mid \mathcal{U} \in \mathcal{F}\}$. Since $\mathcal{U} \mapsto \tau(s^\frown \mathcal{U})$ defines a choice function on $\mathcal{F}$, we know that $|\mathcal{A}_s| \leq \kappa$.

Claim 12.9. *$A := \bigcup_{s \in \mathrm{Seq}(\mathcal{F})} \bigcap \mathcal{A}_s$ is of cardinality $\leq \kappa$.*

Proof. $|\mathcal{F}| \leq \kappa$, and the latter is an infinite cardinal number, thus $|\mathrm{Seq}(\mathcal{F})| \leq \kappa$.

It follows that A is the union of length at most κ of sets of at most cardinality κ. □

Claim 12.10. *$A = X$.*

Proof. Suppose not and pick $x \in X \setminus A$. It follows that for all $s \in \mathrm{Seq}(\mathcal{F})$, there exists some $\mathcal{U} \in \mathcal{F}$ such that $x \notin \tau(s^\frown \mathcal{U})$. This implies that we may define inductively, a sequence $\langle \mathcal{U}_n \in \mathcal{F} \mid n \in \mathbb{N}\rangle$ such that $x \notin \tau\langle \mathcal{U}_1, .., \mathcal{U}_n\rangle$ for all $n \in \mathbb{N}$. In particular $\bigcup_{n\in\mathbb{N}} \tau\langle \mathcal{U}_1, .., \mathcal{U}_n\rangle \neq X$, a contradiction to the assumption that τ is a winning strategy for **II** in $G_1(\mathcal{O}, \mathcal{O})$. □

It follows that $|X| = |A| \leq \kappa$. □

Corollary 12.11 (Telgarski)**.** *If $\langle X, d\rangle$ is a separable metric space, then $X \models \mathbf{II} \uparrow G_1(\mathcal{O}, \mathcal{O})$ iff X is countable.*

Proof. If X is countable, then it is easy to introduce a winning strategy for **II** in this game. For the other direction, we apply to Theorem 12.8 and Lemma 12.7 to conclude $|X| \leq \delta(X) \leq (X) = \aleph_0$. □

Define the game $G_{fin}(\mathcal{A}, \mathcal{B})$ in the obvious fashion, then:

Theorem 12.12 (Telgarski)**.** *For all $X \subseteq \mathbb{R}$, $X \models \mathbf{II} \uparrow G_{fin}(\mathcal{O}, \mathcal{O})$ iff X is σ-compact.*

Proof. Essentially the same as in the proof of 12.8. □

Theorem 12.13 (Pavlikowsky)**.** *For all $X \subseteq \mathbb{R}$, $X \models \mathbf{I} \not\uparrow G_1(\mathcal{O}, \mathcal{O})$ iff $X \models S_1(\mathcal{O}, \mathcal{O})$.*

Corollary 12.14. *It is consistent that the game $G_1(\mathcal{O}_X, \mathcal{O}_X)$ is determined for all $X \subseteq \mathbb{R}$.*

Proof. Assume the Borel conjecture 7.5 (Recall that BC is consistent). Fix $X \subseteq \mathbb{R}$.

If $X \models S_1(\mathcal{O}, \mathcal{O})$, then by Observation 8.1, $|X| \leq \aleph_0$, together with Corollary 12.11, we conclude that $\mathbf{II} \uparrow G_1(\mathcal{O}_x, \mathcal{O}_X)$.

Suppose $X \not\models S_1(\mathcal{O}, \mathcal{O})$, then by Theorem 12.13, we have $\mathbf{I} \uparrow G_1(\mathcal{O}_x, \mathcal{O}_X)$. □

Corollary 12.15 (Recław)**.** *It is consistent to have some $X \subseteq \mathbb{R}$ such that the game $G_1(\mathcal{O}_X, \mathcal{O}_X)$ is not determined.*

Proof. Let $L \subseteq \mathbb{R}$ be a Luzin set. $L \models S_1(\mathcal{O}, \mathcal{O})$, thus by Theorem 12.13, $L \models \mathbf{I} \not\uparrow G_1(\mathcal{O}, \mathcal{O})$.

L is uncountable, thus by Corollary 12.11, $L \models \mathbf{II} \not\uparrow G_1(\mathcal{O}, \mathcal{O})$. □