11. 26.01.06

Definition 11.1. An open cover $\mathcal{U}$ is an ω-cover of X iff:

- For every finite set $F \subseteq X$ there exist $U \in \mathcal{U}$ such that $F \subseteq U$.
- $X \notin \mathcal{U}$.

We denote the family all ω-covers of X by Ω.

Observation 11.2. *If $\mathcal{U}$ is an ω-cover of X, then for every finite subset $F \subseteq X$ there are infinitely many $U \in \mathcal{U}$ such that $F \subseteq U$. In particular, $\mathcal{U}$ is infinite.*

Proof. For all $U \in \mathcal{U}$, pick $x_U \in X \setminus U$ arbitrarily. Fix $F \in [X]^{<\omega}$.

We define an infinite family $\{U_n \mid n \in \mathbb{N}\}$ by induction. Let U_1 be such that $F \subseteq U_1$, and let U_{n+1} be such that $F \cup \{X_{U_1}, .., X_{U_n}\} \subseteq U_{n+1}$. □

We denote by $C(X)$ the set of all continuous functions from X to $\mathbb{R}$. We will consider this as a topological space, and the topology will be inherited from $\mathbb{R}^X \supseteq C(X)$. This topology is determined by pointwise convergence, that is, $f_n \to f$ iff $f_n(x) \to f(x)$ for all $x \in X$. This topological space is not metrizable, thus the closure operator is not easy to figure.

Definition 11.3. A topological space X satisfies the Frèchet-Urysohn (FU) property iff for every $A \subset X$ and every $a \in \overline{A}$ there exist a sequence $\langle a_n | n \in \mathbb{N}\rangle$ such that $a_n \to a$.[31]

Definition 11.4. A topological space satisfies the property $\binom{\mathcal{A}}{\mathcal{B}}$ iff for every $\mathcal{U} \in \mathcal{A}$ there exist $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathcal{V} \in \mathcal{B}$.

For example, denote by Φ all finite open covers. The property $\binom{\mathcal{O}}{\Phi}$ is compactness.

Theorem 11.5 (Gerlitz-Nagy)**.** *$C(X)$ satisfies the FU property iff $X \models \binom{\Omega}{\Gamma}$.*

The property $\binom{\Omega}{\Gamma}$ is also known as the γ-property and is equivalent to $S_1(\Omega, \Gamma)$:

Lemma 11.6. *$S_1(\Omega, \Gamma)$ implies $\binom{\Omega}{\Gamma}$.*

Proof. Suppose $\langle X, O\rangle$ is a topological space. $\Omega := \Omega_X, \Gamma := \Gamma_X$, and $X \models S_1(\Omega, \Gamma)$.

Fix $\mathcal{U} \in \Omega$. For all $n \in \mathbb{N}$, let $\mathcal{U}_n := \mathcal{U}$. It follows from $X \models S_1(\Omega, \Gamma)$ that there exists $\langle U_n \in \mathcal{U}_n \mid n \in \mathbb{N}\rangle$ such that $\{U_n \mid n \in \mathbb{N}\} \in \Gamma$. Since $\{U_n \mid n \in \mathbb{N}\} \subseteq \mathcal{U}$, we are done. □

Theorem 11.7. $S_1(\Omega, \Gamma) = \binom{\Omega}{\Gamma}$.

[31]In a general topological space, a sequence $\langle a_n \mid n \in \mathbb{N}\rangle$ converges to a iff every open set containing a, contains the tail of the sequence.

Definition 11.8. The *Rothberger space* is $[\mathbb{N}]^{\aleph_0} := \langle A \subseteq \mathbb{N} \mid |A| = \aleph_0 \rangle$.

- For $A, B \subseteq \mathbb{N}$: $A \subseteq^* B$ iff $|A \setminus B| < \omega$.
- $\mathcal{F} \subseteq [\mathbb{N}]^{\aleph_0}$ is *centered* iff every $A_1, ..., A_k \in \mathcal{F}$ satisfies $\bigcap_{i \leq k} A_k$ is infinite.
- $A \in [\mathbb{N}]^{\aleph_0}$ is *almost intersection* of $\mathcal{F}$ iff for every $B \in \mathcal{F}$, $A \subseteq^* B$.
- $\mathfrak{p} := \min \left\{ |\mathcal{F}| \mid \mathcal{F} \subseteq [\mathbb{N}]^{\aleph_0} \text{ is centered and } \mathcal{F} \text{ does not have an almost intersection} \right\}$.

Lemma 11.9. $\mathfrak{p} > \aleph_0$.

Proof. Suppose $\mathcal{F} := \{B_n \in [\mathbb{N}]^{\aleph_0} \mid n \in \mathbb{N}\}$ is centered. For all $n \in \mathbb{N}$, put $A_n := B_1 \cap ... \cap B_n$, by the hypothesis on $\mathcal{F}$, $A_n \neq \emptyset$, so pick $x_n \in A_n \setminus \{x_1, .., x_{n-1}\}$. It is now obvious that $A = \{x_n \mid n \in \mathbb{N}\}$ is an almost intersection of $\mathcal{F}$. □

Lemma 11.10. *Suppose $\langle X, O \rangle$ is a topological space, and the product space X^k is Lindelöf for all $k \in \mathbb{N}$, then any ω-cover contains a countable ω-cover.*

Theorem 11.11. $\mathsf{non}(\binom{\Omega}{\Gamma}) = \mathfrak{p}$.

Proof. Suppose $X \in [\mathbb{R}]^{<\mathfrak{p}}$ and $\mathcal{U} \in \Omega$. By the preceding lemma, we may assume an enumeration $\mathcal{U} := \{U_n \mid n \in \mathbb{N}\}$. For all $x \in X$, let $A_x := \{n \in \mathbb{N} \mid x \in U_n\}$. By Observation 11.2, $A_x \in [\mathbb{N}]^{\aleph_0}$ for all $x \in \mathbb{X}$ and $\mathcal{F} := \{A_x \mid x \in X\}$ is centered.

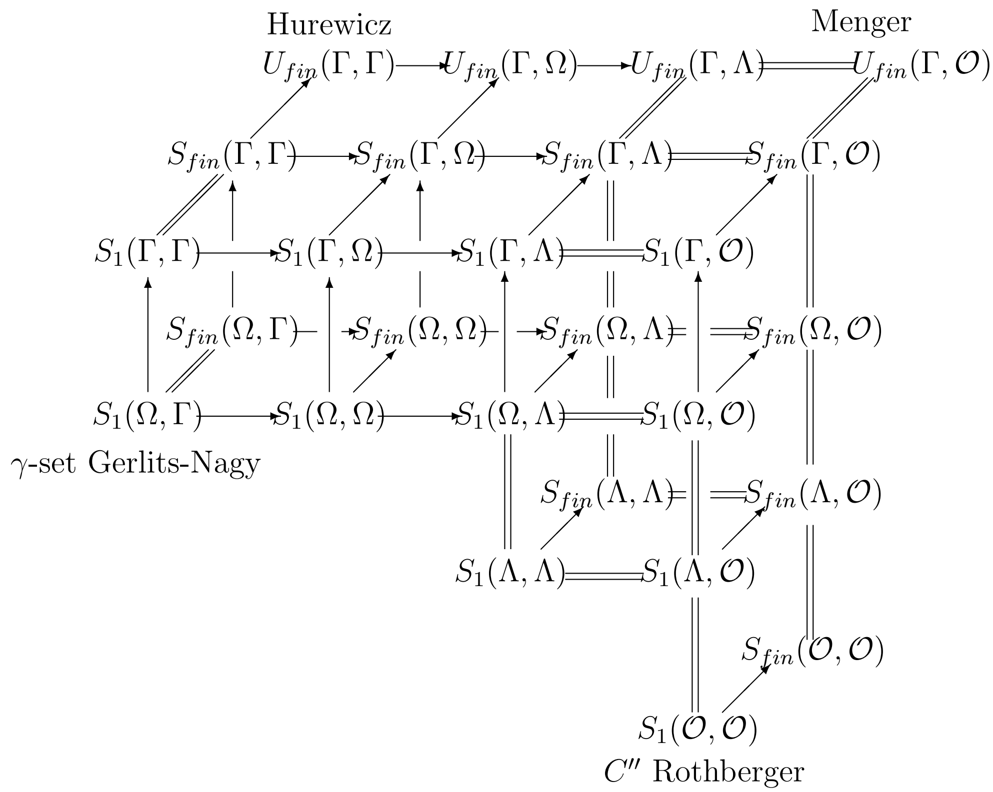
Since $|\mathcal{F}| \leq |X| < \mathfrak{p}$, we may pick an almost intersection $B \in [\mathbb{N}]^{\aleph_0}$.

We claim that $\{U_n \mid n \in B\} \in \Gamma$. Indeed, if $x \in X$ then $B \setminus A_x$ is finite, that is, $\{n \in B \mid x \notin U_n\}$ is finite.

We shall now introduce a set $X \subseteq \mathbb{N}^{\mathbb{N}}$ of cardinality $\mathfrak{p}$ with $X \not\models \binom{\Omega}{\Gamma}$.

By definition of $\mathfrak{p}$, there exists a centered family $X \subseteq [\mathbb{N}]^{\aleph_0}$ of cardinality $\mathfrak{p}$ with no almost intersection. For each $n \in \mathbb{N}$, let $U_n := \{A \in [\mathbb{N}]^{\aleph_0} \mid n \in A\}$, this is an open set and $\mathcal{U} := \{U_n \mid n \in \mathbb{N}\} \in \Omega_X$, because if $F \subseteq X$ is finite, then centeredness of X implies that $I = \bigcap F$ is infinite, and hence $I \subseteq \{n \in \mathbb{N} \mid F \subseteq U_n\}$.

Finally, suppose there exists a strictly increasing function $k : \mathbb{N} \to \mathbb{N}$ such that $\{U_{k(n)} \mid n \in \mathbb{N}\} \in \Gamma_X$. We claim that $B := \mathrm{Im}(k)$ is an almost-intersection of X which is a contradiction. Indeed, for $A \in X$, if $\{n \in \mathbb{N} \mid A \notin U_{k(n)}\}$ is finite, then $B \setminus A$ is finite. □


Hurewicz
Menger
$U_{fin}(\Gamma,\Gamma)$
$U_{fin}(\Gamma,\Omega)$
$U_{fin}(\Gamma,\Lambda)$
$U_{fin}(\Gamma,\mathcal{O})$
$S_{fin}(\Gamma,\Gamma)$
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$S_1(\Omega,\mathcal{O})$
γ-set Gerlits-Nagy
$S_{fin}(\Lambda,\Lambda)$
$S_{fin}(\Lambda,\mathcal{O})$
$S_1(\Lambda,\Lambda)$
$S_1(\Lambda,\mathcal{O})$
$S_{fin}(\mathcal{O},\mathcal{O})$
$S_1(\mathcal{O},\mathcal{O})$
C'' Rothberger

Figure 3.