10. 19.01.06

Definition 10.1. A property P is a *topological invariant* iff every two homeomorphic spaces either both satisfy P, or they do not satisfy it.

It is easier to think about it in the sense that topological invariants are determined by the topology (which is determined up to an homeomorphism).

For example:

- Completeness of metrics is not a topological invariant. Despite the fact that the spaces $\mathbb{N}^{\mathbb{N}}$ and $\mathbb{R} \setminus \mathbb{Q}$ are homeomorphic, $\mathbb{N}^{\mathbb{N}}$ is a complete space but $\mathbb{R} \setminus \mathbb{Q}$ is not.
- We have already seen that SMZ is not a topological invariant.

Assume X, Y are homeomorphic, and let $\psi : X \to Y$ be an homeomorphism.

- First category is a topological invariant. Assume $M \subset X$ is meager, hence, $M \subseteq \bigcup_{n\in\mathbb{N}} F_n$, where F_n is closed and nowhere dense for every $n \in \mathbb{N}$. Now $\psi[M] \subseteq \psi\big[\bigcup_{n\in\mathbb{N}} F_n\big] = \bigcup_{n\in\mathbb{N}} \psi[F_n]$. Assume that for some $n \in \mathbb{N}$ $\psi[F_n]$ is not nowhere dense, that is, there is an open set $U \subset Y$ such that $U \subset \psi[F_n]$ meaning that $\psi^{-1}[U] \subset F_n$. But ψ^{-1} is continuous, thus $\psi^{-1}[U]$ is open, a contradiction to the fact that F_n is nowhere dense.
- Being a Luzin set is a topological invariant. Let $L \subset X$ be a Luzin set, and assume $M \subset Y$ is meager. knowing the last example $L \cap \psi^{-1}[M]$ is countable, but since ψ is an injection, so is $\psi\big[L \cap \psi^{-1}[M]\big] = \psi[L] \cap M$. The last equality holds since ψ is a bijection. Since M is an arbitrary meager set in Y, we get that $\psi[L]$ is a Luzin set.

Definition 10.2. Suppose P is a topological invariant property, let $\mathsf{non}(P)$ denote the minimal cardinality of a space that does not satisfy property P.

We sometime call $\mathsf{non}(P)$ as *the critical cardinality* of P.

The diagram from page 60 shows implications and has the property that any property $\pi(\mathcal{A}, \mathcal{B})$, where $\pi \in \{S_1, S_{fin}, U_{fin}\}$ and $\{\mathcal{A}, \mathcal{B}\} \subseteq \{\mathcal{O}, \Gamma\}$, is equivalent to one of the properties that appears in the diagram.

We now would like to show that this diagram is succinct, in the sense that there are no more equivalent properties in this diagram. We obtain our goal by analyzing their critical cardinalities.

Observation 10.3. $\mathsf{non}(S_{fin}(\mathcal{O}, \mathcal{O})) = \mathfrak{d}$.

Proof. By Theorem 4.10. □

Observation 10.4. $\mathsf{non}(U_{fin}(\mathcal{O}, \Gamma)) = \mathfrak{b}$.

Proof. By Theorem 5.18. □

Observation 10.5. $\mathsf{non}(S_1(\mathcal{O},\mathcal{O})) = \mathsf{cov}(\mathcal{M})$.

Proof. By Corollary 7.22, Theorem 7.19 and Fact 7.34. □

Observation 10.6. *Suppose P, Q are topological properties, and $P \to Q$, that is, for any space X, $X \models P$ only if $X \models Q$. then $\mathsf{non}(P) \leq \mathsf{non}(Q)$.*

Lemma 10.7. $\mathsf{non}\left(S_1(\Gamma,\mathcal{O})\right) = \mathfrak{d}$.

Proof. By the preceding observation, by $S_1(\Gamma,\mathcal{O}) \to S_{fin}(\mathcal{O},\mathcal{O})$, and by $\mathsf{non}(S_{fin}(\mathcal{O},\mathcal{O})) = \mathfrak{d}$, it suffices to show that if $\langle X, O\rangle$ is a topological space and $|X| < \mathfrak{d}$, then $X \models S_1(\Gamma, O)$.

Suppose $\langle \mathcal{U}_n \in \Gamma \mid n \in \mathbb{N}\rangle$ are given. By Observation 5.9, we may assume an enumeration $\mathcal{U}_n = \{U_n^k \mid k \in \mathbb{N}\}$ for all $n \in \mathbb{N}$. For all $x \in X$, define $f_x \in \mathbb{N}^{\mathbb{N}}$, by letting for all $n \in \mathbb{N}$:

$$f_x(n) := \min\{m \in \mathbb{N} \mid \forall k \geq m(x \in U_n^k)\}.$$

By $|X| < \mathfrak{d}$, we may pick some $g \in \mathbb{N}^{\mathbb{N}}$ such that $g \not\leq^* f_x$ for all $x \in X$.

For all $n \in \mathbb{N}$, let $U_n := U_n^{g(n)}$. We claim that $\{U_n \mid n \in \mathbb{N}\} \in \mathcal{O}$. To see this, fix $x \in X$.

Let $n \in \mathbb{N}$ be such that $f_x(n) < g(n)$, then, by definition of f_x, $x \in U_n^{g(n)} = U_n$. □

Thus, we obtain the analogue of Corollary 7.24.

Corollary 10.8. *If $X \subseteq \mathbb{R}$ is $\mathfrak{d}$-concentrated at one of its countable subsets, then $X \models S_1(\Gamma,\mathcal{O})$.*

Proof. Divide to odds and evens like in the proof of Observation 3.17. □

Lemma 10.9. $\mathsf{non}(S_1(\Gamma,\Gamma)) = \mathfrak{b}$.

Proof. By $S_1(\Gamma,\Gamma) \to U_{fin}(\mathcal{O},\Gamma)$, and $\mathsf{non}(U_{fin}(\mathcal{O},\Gamma)) = \mathfrak{b}$, it suffices to show that if $\langle X, O\rangle$ is a topological space and $|X| < \mathfrak{b}$, then $X \models S_1(\Gamma,\Gamma)$.

Suppose $\langle \mathcal{U}_n \in \Gamma \mid n \in \mathbb{N}\rangle$ are given. By Observation 5.9, we may assume an enumeration $\mathcal{U}_n = \{U_n^k \mid k \in \mathbb{N}\}$ for all $n \in \mathbb{N}$. For all $x \in X$, define $f_x \in \mathbb{N}^{\mathbb{N}}$, by letting for all $n \in \mathbb{N}$:

$$f_x(n) := \min\{m \in \mathbb{N} \mid \forall k \geq m(x \in U_n^k)\}.$$

By $|X| < \mathfrak{b}$, we may pick some $g \in \mathbb{N}^{\mathbb{N}}$ such that $\{f_x \mid x \in X\} \subseteq \underline{\{g\}}$. For all $n \in \mathbb{N}$, let $U_n := U_n^{g(n)}$. We claim that $\{U_n \mid n \in \mathbb{N}\} \in \mathcal{O}$. To see this, fix $x \in X$.

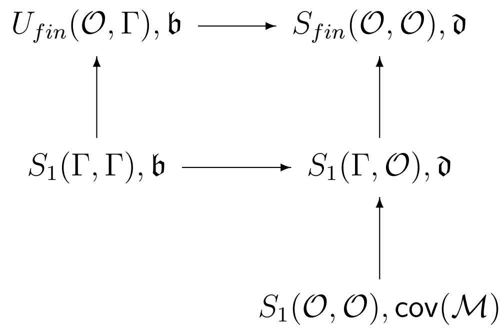
Let $m \in \mathbb{N}$ be such that $f_x(n) \leq g(n)$ for all $m \geq n$, then, by definition of f_x, we have that $x \in U_n^{g(n)} = U_n$ for all $n \geq m$ and we are done. □

By Lemma 1.9, $\mathfrak{b} \leq \mathfrak{d}$, and by Observation 5.9, $\mathsf{cov}(\mathcal{M}) \leq \mathfrak{d}$. Assuming CH they are all equal, but it is also consistent to have $\mathfrak{b} < \mathfrak{d}$ or $\mathsf{cov}(\mathcal{M}) < \mathfrak{d}$. Thus:

Corollary 10.10. *$S_1(\Gamma,\Gamma) \nrightarrow S_1(\mathcal{O},\mathcal{O})$, $S_1(\Gamma,\mathcal{O}) \nrightarrow S_{fin}(\mathcal{O},\Gamma)$ and $S_1(\mathcal{O},\mathcal{O}) \nrightarrow U_{fin}(\mathcal{O},\Gamma)$.*

Also recall Theorem 6.13 that shows that $S_{fin}(\mathcal{O},\mathcal{O}) \nrightarrow U_{fin}(\mathcal{O},\Gamma)$.

Thus, to claim that the diagram is succinct, we still need to seperate $S_1(\Gamma,\Gamma)$ from $U_{fin}(\mathcal{O},\Gamma)$ and $S_1(\Gamma,\mathcal{O})$ from $S_{fin}(\mathcal{O},\mathcal{O})$.



Theorem 10.11 (Scheepers-Just-Miller-Szeptycki)**.** *The cantor space satisfies $S_{fin}(\mathcal{O},\mathcal{O})$ and $U_{fin}(\mathcal{O},\Gamma)$ but does not satisfy $S_1(\Gamma,\mathcal{O})$ and $S_1(\Gamma,\Gamma)$.*

Proof. Let $X := \{0,1\}^{\mathbb{N}}$ be the cantor space. X is compact, so by Lemma 5.16, $X \models U_{fin}(\mathcal{O},\Gamma)$, and hence also $X \models S_{fin}(\mathcal{O},\mathcal{O})$.

To show that $X \models \neg S_1(\gamma,\mathcal{O}) \wedge \neg S_1(\Gamma,\Gamma)$, it suffices to show that $X \not\models S_1(\Gamma,\mathcal{O})$. We first need the following lemma:

Lemma 10.12. *There exist a matrix $A = \langle A^n_m \mid m,n \in \mathbb{N}\rangle$ satisfying :*

1. *Each element of the matrix is closed subset of the cantor space.*
2. *Fixing $m \in \mathbb{N}$, $\langle A^n_m \mid n \in \mathbb{N}\rangle$ are disjoint.*
3. *For different $m_1,...,m_k \in \mathbb{N}$, $\cap A^{n_1}_{m_1} \cdots \cap A^{n_k}_{m_k} \neq \emptyset$, for all $n_1,...,n_k \in \mathbb{N}$.*

Proof. Omitted. ☐

Now, for each $m \in \mathbb{N}$, let $\mathcal{U}_m := \{X \setminus A^m_n \mid n \in \mathbb{N}\}$. By property (1), members of $\mathcal{U}_m$ are open sets. Together with property (2), we get that $\mathcal{U}_m \in \Gamma$.

Finally, assume a sequence $\langle U_m \in \mathcal{U}_m \mid m \in \mathbb{N}\rangle$. For all $m \in \mathbb{N}$, there exists some $n_m \in \mathbb{N}$ such that $U_m = X \setminus A^{n_m}_m$. By property (3), $\mathcal{F} := \{A^{n_m}_m \mid m \in \mathbb{N}\}$ satisfies the finite intersection property. Together with with property (1), we obtain that $\bigcap \mathcal{F} \neq \emptyset$, and hence $\{U_m \mid m \in \mathbb{N}\} \notin \mathcal{O}$. ☐

Corollary 10.13. *For all $X \subseteq \mathbb{R}$, if X contains a perfect subset, then $X \not\models S_1(\Gamma,\mathcal{O})$.*

Proof. If X contains a perfect set, then it contains a closed subset which is homeomorphic to the cantor space. Now, it is easy to see that $S_1(\Gamma,\mathcal{O})$ is a closed-hereditary property. ☐

Corollary 10.14. *If $X \subseteq \mathbb{R}$ is an uncountable F_σ set, then $X \not\models S_1(\Gamma,\mathcal{O})$.*

Proof. Since any uncountable F_σ set contains a closed perfect subset. □

Theorem 10.15. *It is consistent that* $\mathfrak{b} = \mathsf{cov}(\mathcal{M})$*, while* $U_{fin}(\mathcal{O},\Gamma) \neq S_1(\mathcal{O},\mathcal{O})$*.*

First proof. By the arguments of Observation 5.23, if $\mathsf{cov}(\mathcal{M}) = \mathsf{cof}(\mathcal{M})$, then there exists a set $X \subseteq \mathbb{N}^\mathbb{N}$ which is $\leq^*$-unbounded and $\mathsf{cov}(\mathcal{M})$-concentrated on its dense countable subset, by Corollary 7.24, $X \models S_1(\mathcal{O},\mathcal{O})$, and by Theorem 5.19, $X \not\models U_{fin}(\mathcal{O},\Gamma)$.

Finally, assuming CH, we indeed have $\mathfrak{b} = \mathsf{cov}(\mathcal{M}) = \mathsf{cof}(\mathcal{M})$. □

The essence of the preceding proof is Corollary 4.5 that implies that any Luzin subset of $\mathbb{N}^\mathbb{N}$ is $\leq^*$-unbounded. Also notice that since any Luzin set $L \subseteq \mathbb{R}$ satisfies $S_1(\mathcal{O},\mathcal{O})$, and the latter implies SMZ, then there must exist some dense subset of $\mathbb{R}$ which is disjoint from L.

Observation 10.16. *A Sierpinski does not satisfy* $S_1(\mathcal{O},\mathcal{O})$*.*

Proof. By Observation 7.14 and Proposition 7.3, if $S \models S_1(\mathcal{O},\mathcal{O})$, then S is a null set. A Sierpinski set is an uncountable set have countable intersection with any null set, so it cannot be itself a null set. □

Lemma 10.17 (Scheepers-Just-Miller-Szeptycki)**.** *Any Sierpinski, S, satisfies* $S_1(\Gamma,\Gamma)$*.*

Proof. Suppose $\langle \mathcal{U}_n \in \Gamma \mid n \in \mathbb{N}\rangle$ are given. By Observation 5.9, we may assume an enumeration $\mathcal{U}_n = \{U_n^k \mid k \in \mathbb{N}\}$ for all $n \in \mathbb{N}$. For all $x \in X$, define $f_x \in \mathbb{N}^\mathbb{N}$, by letting for all $n \in \mathbb{N}$:

$$f_x(n) := \min\{m \in \mathbb{N} \mid \forall k \geq m(x \in U_n^k)\}.$$

We claim that $x \overset{\psi}{\mapsto} f_x$ is a Borel map. Fix a finite function $\sigma : \{1,..,m\} \to \mathbb{N}$, we need to show that $A := \psi^{-1}[\sigma^\uparrow]$ is a Borel subset of S. Indeed, $A = \bigcap\{A_1^n, A_2^n \mid 1 \leq n \leq m\}$, where:

$$A_1^n = \{x \in S \mid \forall k \geq \sigma(n)(x \in U_n^k)\} = \bigcap_{k=\sigma(n)}^{\infty} U_n^k,$$

$$A_2^n = \{x \in S \mid \exists k < \sigma(n)(x \notin U_n^k)\} = \bigcup_{k<\sigma(n)} S \setminus U_n^k.$$

If follows Claim 5.29 that we may pick some $g \in \mathbb{N}^\mathbb{N}$ such that $\{f_x \mid x \in X\} \subseteq \underline{\{g\}}$. For all $n \in \mathbb{N}$, let $U_n := U_n^{g(n)}$. We claim that $\{U_n \mid n \in \mathbb{N}\} \in \mathcal{O}$. To see this, fix $x \in X$.

Let $m \in \mathbb{N}$ be such that $f_x(n) \leq g(n)$ for all $m \geq n$, then, by definition of f_x, we have that $x \in U_n^{g(n)} = U_n$ for all $n \geq m$ and we are done. □

Corollary 10.18. *It is consistent that* $\mathfrak{b} = \mathsf{cov}(\mathcal{M})$*, while* $S_1(\Gamma,\Gamma) \neq S_1(\mathcal{O},\mathcal{O})$*.*

Proof. By Corollary 3.8, assuming CH, there exists a Sierpinski set, S, and also $\mathfrak{b} = \mathsf{cov}(\mathcal{M})$. □