INFINITE COMBINATORIAL TOPOLOGY

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Abstract. We summarize our view on the course given by Dr. Boaz Tsaban at the Weizmann Institute of Science, Fall 2006.

1. 03.11.05

Definition 1.1. Define the *Baire space* to be the family of all functions from $\mathbb{N}$ to $\mathbb{N}$, and denote it by $\mathbb{N}^{\mathbb{N}}$.

Definition 1.2. Assume X is a set. A family $\mathcal{I} \subseteq \mathcal{P}(X)$ is an *ideal over* X iff it satisfies:

- $\emptyset \in \mathcal{I}$.
- $A \in \mathcal{I} \Longrightarrow \mathcal{P}(A) \subseteq \mathcal{I}$.
- $A, B \in \mathcal{I} \Longrightarrow A \cup B \in \mathcal{I}$.

The ideal is said to be *non-trivial* if, additionally :

- $\big\{\{x\} \mid x \in X\big\} \subseteq \mathcal{I}$.

If $\mathcal{I} \neq \mathcal{P}(X)$ (equivalently, if $X \notin \mathcal{I}$) we say that $\mathcal{I}$ is a *proper ideal*.

Definition 1.3. Assume $\mathcal{I}$ is an ideal over $\mathbb{N}$, for $f, g \in \mathbb{N}^{\mathbb{N}}$, put:

$$f \leq_{\mathcal{I}} g \text{ iff } \{n \in \mathbb{N} \mid f(n) > g(n)\} \in \mathcal{I}.$$

Let $\mathcal{I}_{fin} := \{X \subseteq \mathbb{N} \mid |X| < \aleph_0\}$ be the ideal of finite subsets of $\mathbb{N}$ and $\mathcal{J} := \{\emptyset\}$.

Define two binary relations on $\mathbb{N}^{\mathbb{N}}$: $\leq^* := \leq_{\mathcal{I}_{fin}}$ and $\leq := \leq_{\mathcal{J}}$, i.e., $f \leq^* g$ iff there exists some $m \in \mathbb{N}$ such that $f(n) \leq g(n)$ for all $n > m$, and $f \leq g$ iff $f(n) \leq g(n)$ holds for all n.

Lemma 1.4. *$\langle \mathbb{N}^{\mathbb{N}}, \leq^* \rangle$ is a quasi-ordered set, that is, $\leq^*$ is a reflexive and a transitive binary relation on $\mathbb{N}^{\mathbb{N}}$.*

Definition 1.5. For a set $A \subseteq \mathbb{N}^{\mathbb{N}}$, define the *downward closure* of A:

$$\underline{A} := \{f \in \mathbb{N}^{\mathbb{N}} \mid \exists g \in A (f \leq^* g)\}.$$

Let the *external cofinality* of A be $\operatorname{ecf}(A) := \min\{|D| \mid D \subseteq \mathbb{N}^{\mathbb{N}} \text{ and } A \subseteq \underline{D}\}$.

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By "our view" we mean that sometimes we omit material given in class, sometimes we give alternative definitions or proofs, and sometimes we include our own additional propositions. However, we are always *consistent* with the material given in class.

It is obvious that $\operatorname{ecf}(\underline{A}) = \operatorname{ecf}(A) \leq |A|$ for all $A \subseteq \mathbb{N}^{\mathbb{N}}$.

Definition 1.6. A subset $B \subseteq \mathbb{N}^{\mathbb{N}}$ is said to be *bounded* iff $\operatorname{ecf}(B) \leq 1$.

As expected, we say that B is *unbounded* iff $\operatorname{ecf}(B) > 1$.

Definition 1.7. A subset $D \subseteq \mathbb{N}^{\mathbb{N}}$ is said to be *dominating* (or *cofinal*) iff $\underline{D} = \mathbb{N}^{\mathbb{N}}$.

Definition 1.8. We define three important cardinals:

(i) $\mathfrak{b} := \min\{|B| \mid B \subseteq \mathbb{N}^{\mathbb{N}} \text{ and } \operatorname{ecf}(B) > 1\}$.
(ii) $\mathfrak{d} := \operatorname{ecf}(\mathbb{N}^{\mathbb{N}})$.
(iii) $\mathfrak{c} := |\mathbb{N}^{\mathbb{N}}|$.

Lemma 1.9. $\aleph_0 < \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c} = 2^{\aleph_0}$.

Proof. To see that $\mathfrak{b}$ is uncountable, we pick an arbitrary family $A = \{f_n \in \mathbb{N}^{\mathbb{N}} \mid n < \omega\}$ and then find some $g \in \mathbb{N}^{\mathbb{N}}$ witnessing $\operatorname{ecf}(A) = 1$.

Define $g = g_A$ as follows, for all $n \in \mathbb{N}$: $g(n) = \max\{f_i(n) \mid 0 \leq i \leq n\}$. It is now easy to see that $A \subseteq \underline{\{g\}}$ and that $\operatorname{ecf}(A) = 1$.

To see that $\mathfrak{b} \leq \mathfrak{d}$, it suffices to prove that if $D \subseteq \mathbb{N}^{\mathbb{N}}$ is cofinal, then D is unbounded. Towards a contradiction, assume there exists some dominant $D \subseteq \mathbb{N}^{\mathbb{N}}$ such that $\operatorname{ecf}(D) = 1$. Pick $g \in \mathbb{N}^{\mathbb{N}}$ such that $D \subseteq \underline{\{g\}}$. It follows that $\mathbb{N}^{\mathbb{N}} \subseteq \underline{D} \subseteq \underline{\{g\}}$, i.e., that $\operatorname{ecf}(\mathbb{N}^{\mathbb{N}}) = 1$, which is an absurd.[1] □

Corollary 1.10. *If CH holds (that is, if* $\mathfrak{c} = \aleph_1$*), then* $\mathfrak{b} = \mathfrak{d} = \mathfrak{c} = \aleph_1$.

It is worth mentioning that an unbounded family is not necessarily cofinal, e.g., take $\{f \in \mathbb{N}^{\mathbb{N}} \mid \forall n \in \mathbb{N}(f(2n) = 0)\}$.

Lemma 1.11. *There exists a* $\mathfrak{b}$*-scale, that is, a sequence* $\langle f_\alpha \in \mathbb{N}^{\mathbb{N}} \mid \alpha < \mathfrak{b}\rangle$*, such that:*

(a) $\operatorname{ecf}\{f_\alpha \mid \alpha < \mathfrak{b}\} > 1$;
(b) $\alpha < \beta < \mathfrak{b}$ *implies* $f_\alpha \leq^* f_\beta$.

Proof. By definition of $\mathfrak{b}$, we may pick an unbounded family $B = \{g_\alpha \in \mathbb{N}^{\mathbb{N}} \mid \alpha < \mathfrak{b}\}$.

We now define the $\mathfrak{b}$-scale by induction on $\alpha < \mathfrak{b}$. Put $f_0 := g_0$.

Assume now $\{f_\beta \mid \beta < \alpha\}$ had already been defined. Since $\alpha < \mathfrak{b}$, $\operatorname{ecf}(\{f_\beta \mid \beta < \alpha\}) = 1$, we may pick an exemplifying $h \in \mathbb{N}^{\mathbb{N}}$. Put $f_\alpha := \max\{g_\alpha, h\}$.[2] End of the construction.

Put $B' := \{f_\alpha \mid \alpha < \mathfrak{b}\}$. Since $g_\alpha \leq^* f_\alpha$ for all relevant α, we get that $B \subseteq \underline{B'}$, thus, $1 < \operatorname{ecf}(B) \leq \operatorname{ecf}(B')$ and property (a) is satisfied. Property (b) follows immediately from the construction. □

[1] For each $f \in \mathbb{N}^{\mathbb{N}}$: $f \leq^* (f+1)$ and $(f+1) \not\leq^* f$, where $(f+1)(n) = f(n) + 1$ for all $n \in \mathbb{N}$.

[2] Here, max denotes the pointwise-maximum function between two functions of the same domain.

Lemma 1.12. *There exists a* $\mathfrak{d}$*-scale, that is, a sequence* $\langle f_\alpha \in \mathbb{N}^\mathbb{N} \mid \alpha < \mathfrak{d}\rangle$*, such that:*

(a) $\{f_\alpha \mid \alpha < \mathfrak{d}\}$ *is cofinal;*
(b) $\alpha < \beta < \mathfrak{d}$ *implies* $f_\beta \not\leq^* f_\alpha$*.*

In particular, for all $g \in \mathbb{N}^\mathbb{N}$*, there exists some* $\alpha < \mathfrak{d}$ *such that* $f_\beta \not\leq^* g$ *whenever* $\alpha < \beta < \mathfrak{d}$*.*

Proof. By definition of $\mathfrak{d}$, we may pick a family $D = \{g_\alpha \mid \alpha < \mathfrak{d}\}$ such that $\underline{D} = \mathbb{N}^\mathbb{N}$.

We now define the $\mathfrak{d}$-scale by induction on $\alpha < \mathfrak{d}$. Put $f_0 := g_0$.

Assume now $\{f_\beta \mid \beta < \alpha\}$ had already been defined. Since $\alpha < \mathfrak{d}$, we may pick $h_\alpha \in \mathbb{N}^\mathbb{N}$ such that $h_\alpha \notin \underline{\{f_\beta \mid \beta < \alpha\}}$. Put $f_\alpha := \max\{g_\alpha, h_\alpha\}$. End of the construction.

Just like in the preceding proof, we put $D' := \{f_\alpha \mid \alpha < \mathfrak{b}\}$ and notice that the two properties holds for D'. Being user-friendly, we now give a direct proof for the last property. Fix $g \in \mathbb{N}^\mathbb{N}$.

By $\mathbb{N}^\mathbb{N} = \underline{D} \subseteq \underline{D'} \subseteq \mathbb{N}^\mathbb{N}$, we may pick $\alpha < \mathfrak{d}$ such that $g \leq^* f_\alpha$.

Suppose there exists $\beta > \alpha$ such that $f_\beta \leq^* g$, then, in particular $f_\beta \leq^* f_\alpha$. It follows from $\alpha < \beta$ that $\alpha \in \{\gamma \mid \gamma < \beta\}$ and $f_\beta \in \underline{\{f_\gamma \mid \gamma < \beta\}}$. A moment's reflection make it clear that this implies $h_\beta \in \underline{\{f_\beta\}} \subseteq \underline{\{f_\gamma \mid \gamma < \beta\}}$ which is obviously a contradiction to the choice of h_β. □

Claim 1.13. $\mathfrak{b}$ *is a* regular cardinal*, that is,* $\mathrm{cf}(\mathfrak{b}) = \mathfrak{b}$*.*

Proof. It is obvious that $\mathrm{cf}(\mathfrak{b}) \leq \mathfrak{b}$, as this is true for any infinite cardinal number.

Fix an increasing sequence of ordinals $\langle \alpha_i < \mathfrak{b} \mid i < \mathrm{cf}(\mathfrak{b})\rangle$ converging to $\mathfrak{b}$. Let $\langle f_\alpha \mid \alpha < \mathfrak{b}\rangle$ be a $\mathfrak{b}$-scale. Put $B := \{f_{\alpha_i} \mid i < \mathrm{cf}(\mathfrak{b})\}$. We shall show that $\mathrm{ecf}(B) > 1$, and then - by definition/minimality of $\mathfrak{b}$ - we would have to conclude that $\mathfrak{b} \leq |B| \leq \mathrm{cf}(\mathfrak{b})$.

Assume there exists some $g \in \mathbb{N}^\mathbb{N}$ such that $B \subseteq \underline{\{g\}}$, we reach a contradiction by showing that $f_\alpha \leq^* g$ for all $\alpha < \mathfrak{b}$.

Indeed, pick $\alpha < \mathfrak{b}$ and pick $i < \mathrm{cf}(\mathfrak{b})$ such that $\alpha < \alpha_i$. We get that $f_\alpha \leq^* f_{\alpha_i} \leq^* g$. □

Claim 1.14. $\mathfrak{b} \leq \mathrm{cf}(\mathfrak{d})$.

Proof. Fix a $\mathfrak{d}$-scale $\langle f_\alpha \in \mathbb{N}^\mathbb{N} \mid \alpha < \mathfrak{d}\rangle$, and an increasing sequence $\langle \alpha_i \mid i < \mathrm{cf}(\mathfrak{d})\rangle$ converging to $\mathfrak{d}$. Put $B := \{f_{\alpha_i} \mid i < \mathrm{cf}(\mathfrak{b})\}$. We claim that $\mathrm{ecf}(B) > 1$.

Suppose not, and let $g \in \mathbb{N}^\mathbb{N}$ be such that $B \subseteq \underline{\{g\}}$. Pick $\alpha < \mathfrak{d}$ such that $g \leq^* f_\alpha$ and $i < \mathrm{cf}(\mathfrak{d})$ such that $\alpha < \alpha_i$. We get from one hand that $B \ni f_{\alpha_i} \leq^* g \leq^* f_\alpha$, while on the other hand $f_{\alpha_i} \not\leq^* f_\alpha$. A contradiction. □

Corollary 1.15. $\aleph_1 \leq \mathrm{cf}(\mathfrak{b}) = \mathfrak{b} \leq \mathrm{cf}(\mathfrak{d}) \leq \mathfrak{d} \leq \mathfrak{c}$.

It is worth mentioning that the latter is all one can *prove*. That's because for all cardinal numbers $\kappa, \lambda, \mu, \theta$ with $\aleph_1 \leq \mathrm{cf}(\kappa) = \kappa \leq \lambda = \mathrm{cf}(\mu) \leq \theta$ and $\mathrm{cf}(\theta) > \aleph_0$, there exists a model of set theory satisfying $\mathfrak{b} = \kappa, \mathfrak{d} = \mu, \mathrm{cf}(\mathfrak{d}) = \lambda$ and $\mathfrak{c} = \theta$.

Definition 1.16 (Menger's Basis property)**.** A metric space $\langle X, d\rangle$ is said to satisfy *Menger's Basis property* iff for each basis $\mathcal{B}$, there exists a sequence $\langle B_n \in \mathcal{B} \mid n \in \mathbb{N}\rangle$ such that $X = \bigcup_{n\in\mathbb{N}} B_n$ and $\lim_{n\to\infty} \mathrm{Diam}(B_n) = 0$.

Observation 1.17. *Menger's Basis property is closed hereditary.*[3]

Notation 1.18. *For a metric space $\langle X, d\rangle$, $x \in X$ and $\delta \in \mathbb{R}^+$, let $\mathsf{B}_\delta(x) := \{y \in X \mid d(x, y) < \delta\}$ denote the open ball of radius δ, centered at x.*

Definition 1.19. The *canonical base* for a metric space $\langle X, d\rangle$ is $\{\mathsf{B}_\delta(x) \mid \delta \in \mathbb{R}^+, x \in X\}$.

Fact 1.20. *Suppose $\mathcal{B}$ is a family of open sets in a metric space $\langle X, d\rangle$, satisfying:*

($\star$) *For all relevant x, y, δ with $y \in \mathsf{B}_\delta(x)$, there exists $U \in \mathcal{B}$ satisfying $y \in U \subseteq \mathsf{B}_\delta(x)$.*

Then $\mathcal{B}$ is a basis for $\langle X, d\rangle$.

Lemma 1.21. *A space that satisfies Menger's Basis property is Lindelöf.*

Proof. Suppose $\langle X, d\rangle$ satisfies Menger's Basis property and $\mathcal{U}$ is a given open cover. Put $\mathcal{B} := \{U \cap \mathsf{B}_{\frac{1}{n}}(x) \mid U \in \mathcal{U}, n \in \mathbb{N}^+, x \in X\}$. Since $\mathcal{B}$ is a basis, we can find some $\mathcal{F} \in [\mathcal{B}]^{\aleph_0}$ such that $\bigcup \mathcal{F} = X$. Finally, for each $G \in \mathcal{F}$, pick a single $G' \in \mathcal{U}$ such that $G \subseteq G'$, then $\mathcal{V} := \{G' \mid G \in \mathcal{F}\}$ is a countable subcover of $\mathcal{U}$. $\square$

Corollary 1.22. *The discrete space $\langle X, d\rangle$ satisfies Menger's Basis property iff $|X| \leq \aleph_0$.*

Lemma 1.23. *If $\langle X, d\rangle$ is a compact metric space, then it satisfies Menger's Basis property.*

Proof. Suppose $\mathcal{B}$ is a basis for the space. X is a metric space, thus, it easy to find a family $\{A_n \in \mathcal{B} \mid n \in \mathbb{N}\}$ such that $\lim_{n\to\infty} \mathrm{Diam}(A_n) = 0$.

By compactness, we may pick $\mathcal{U} \in [\mathcal{B}]^{<\omega}$ such that $X = \bigcup \mathcal{U}$. Now, let $\{B_n \mid k \leq n\}$ enumerate $\mathcal{U}$, and for al $n > k$, put $B_n := A_n$. $\square$

Definition 1.24. A space $\langle X, O\rangle$ is said to be σ-compact iff there exists a family of compact subsets $\langle K_n \subseteq X \mid n \in \mathbb{N}\rangle$ such that $X = \bigcup_{n\in\mathbb{N}} K_n$.

It is obvious that a finite union of compact subspaces is compact, hence, we may always assume that the family $\langle K_n \mid n \in \mathbb{N}\rangle$ is increasing with respect to inclusion. For instance $\langle \mathbb{R}, d\rangle$ is σ-compact, as it is the countable union of the compact intervals:

$$\mathbb{R} = \bigcup_{n\in\mathbb{N}} [-n, n].$$

[3]A property p is said to be *closed hereditary*, if for any topological space $\langle X, O\rangle$ and any closed subset $Y \subseteq X$: $X \models p$ implies $Y \models p$.

Claim 1.25. *If $\langle X, d\rangle$ is a σ-compact metric space, then it satisfies Menger's Basis property.*

Proof. Suppose $\mathcal{B}$ is a basis for the space. It follows that for all $n \in \mathbb{N}$ and $x \in K_n$, we may find $B_{x,n} \in \mathcal{B}$ with $x \in B_{x,n}$ and $\mathrm{Diam}(B_{x,n}) < \frac{1}{n+1}$. Fix $n \in \mathbb{N}$.

Evidently, $K_n \subseteq \bigcup_{x\in K_n} B_{x,n}$, so by compactness, there exists $f(n) \in \mathbb{N}$ and a family $\{B_{m,n} \in \mathcal{B} \mid m \leq f(n)\} \subseteq \{B_{x,n} \mid x \in K_n\}$ s.t. $K_n \subseteq \bigcup_{m\leq f(n)} B_{m,n}$ and $\mathrm{Diam}(B_{m,n}) < \frac{1}{n+1}$.

Finally, let $\psi : \mathbb{N} \leftrightarrow \{(m, n) \mid n \in \mathbb{N}, m \leq f(n)\}$ be the order-preserving bijection.[4]

We have that $X = \bigcup_{n\in\mathbb{N}} K_n = \bigcup_{n\in\mathbb{N}} \bigcup_{m\leq f(n)} B_{m,n} = \bigcup_{n\in\mathbb{N}} B_{\psi(n)}$ and $\lim_{n\to\infty} \mathrm{Diam}(B_{\psi(n)}) = \lim_{n\to\infty} \frac{1}{n+1} = 0$, that is, $\{B_{\psi(n)} \mid n \in \mathbb{N}\}$ witnesses Menger's Basis property. □

Definition 1.26 (Menger's covering)**.** For a topological space $\langle X, O\rangle$, we denote by $S_{fin}(\mathcal{O}, \mathcal{O})$ the property that for any countable sequence of open covers of X, $\langle \mathcal{U}_n \subseteq O \mid n \in \mathbb{N}\rangle$, there exists some $\langle \mathcal{F}_n \in [\mathcal{U}_n]^{<\omega} \mid n \in \mathbb{N}\rangle$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ is an open cover of X.

Observation 1.27. *Menger's covering is closed hereditary.*

Observation 1.28. *If $\langle X, O\rangle$ satisfies $S_{fin}(\mathcal{O}, \mathcal{O})$, then X is Lindelöf.*

Proof. Suppose $\mathcal{U}$ is an open cover. Put $\mathcal{U}_n := \mathcal{U}$ for all $n \in \mathbb{N}$. For $\mathcal{F}_n \in [\mathcal{U}_n]^{<\omega}$ witnessing $S_{fin}(\mathcal{O}, \mathcal{O})$, then $\mathcal{V} := \bigcup \mathcal{F}_n$ is a countable subcover of $\mathcal{U}$. □

Lemma 1.29. *If $\langle X, O\rangle$ is a σ-compact topological space, then $X \models S_{fin}(\mathcal{O}, \mathcal{O})$.*

Proof. Suppose $X = \bigcup_{n\in\mathbb{N}} K_n$ where each K_n is compact. Assume $\langle \mathcal{U}_n \subseteq O \mid n \in \mathbb{N}\rangle$ is a given family of covers. In particular $K_n \subseteq \bigcup \mathcal{U}_n$ for all $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$.

By compactness, we may pick $\mathcal{F}_n \in [\mathcal{U}_n]^{<\omega}$ such that $K_n \subseteq \bigcup \mathcal{F}_n$.

Evidently, $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ is an open cover of X. □

Conjecture 1.30 (Menger)**.** *$S_{fin}(\mathcal{O}, \mathcal{O})$ is equivalent to σ-compactness.*

Observation 1.31. *For a space $\langle X, O\rangle$, and a sequence $\langle \mathcal{B}_n \mid n \in \mathbb{N}\rangle$ of bases to X, TFAE:*

(a) $X \models S_{fin}(\mathcal{O}, \mathcal{O})$.

(b) *For any countable sequence of open covers of X, $\langle \mathcal{V}_n \subseteq \mathcal{B}_n \mid n \in \mathbb{N}\rangle$, there exists some $\langle \mathcal{F}_n \in [\mathcal{V}_n]^{<\omega} \mid n \in \mathbb{N}\rangle$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ is an open cover of X.*

Proof. We assume (b) and prove (a). Suppose $\langle \mathcal{U}_n \subseteq O \mid n \in \mathbb{N}\rangle$ is a given family of covers.

Fix $n \in \mathbb{N}$. Let $\psi_n : O \to \mathcal{P}(\mathcal{B}_n)$ be a function such that $U = \bigcup \psi_n(U)$ for all $U \in O$.[5]

Put $\mathcal{V}_n := \bigcup\{\psi_n(U) \mid U \in \mathcal{U}_n\}$. Clearly, $\mathcal{V}_n \subseteq \mathcal{B}_n$ and $\bigcup \mathcal{V}_n = \bigcup \mathcal{U}_n = X$.

Now, by the hypothesis (b), we yield $\mathcal{F}_n \in [\mathcal{V}_n]^{<\omega}$ for all $n \in \mathbb{N}$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ covers X. Finally, for each $n \in \mathbb{N}$ and $G \in \mathcal{F}_n$, pick a single $G' \in \mathcal{U}_n$ such that $G \subseteq G'$ and put $\mathcal{F}'_n := \{G' \mid G \in \mathcal{F}_n\}$. It follows that $|\mathcal{F}'_n| \leq |\mathcal{F}_n| < \aleph_0$ and $\bigcup_{n\in\mathbb{N}} \mathcal{F}'_n$ covers X. □

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[4] Recall the lexicographic order on $\mathbb{N} \times \mathbb{N}$: $(m_1, n_1) < (m_2, n_2)$ iff $(n_1 < n_2)$ or $((n_1 = n_2) \wedge (m_1 < m_2))$.

[5] By definition, an open set is a union of basis-elements.