# FORCING THEORY: EXERCISE 4

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**1.** Prove: If $V = L$, then for each $\alpha \geq \omega$, $P(L_\alpha) \subseteq L_{\alpha^+}$.
(Thus, $L \models \forall \alpha \geq \omega,\ P(L_\alpha) \subseteq L_{\alpha^+}$.)
*Hint.* Use the sketch shown in class, or read the proof in Kunen.

**2.** Prove: $V = L \to GCH$. (Thus, $L \models ZFC + GCH$.)
*Hint.* By Question 1, $V = L \to P(\kappa) \subseteq L_{\kappa^+}$.
Note: Now you can prove Question 4 of Exercise 3 easily.

**3.** Prove *Scott's Theorem*: $L \models$ "There are no measurable cardinals".
*Hint.* Work in $L$. Assume there is a measurable cardinal, and let $\kappa$ be the minimal one.
Then there is an elementary embedding $j : V \to M$ such that $M$ is transitive and $j(\kappa) > \kappa$.
As $j$ is an elementary embedding, $M \models V = L$, thus $M = L$. As $V = L$, $j : L \to L$.
$V = L \models$ "$\kappa$ is the first measurable cardinal", thus $M = L \models$ "$j(\kappa)$ is the first measurable cardinal".
A contradiction.

*Reminders.*
$\kappa > \aleph_0$ is *weakly inaccessible* if it is a regular ($\kappa = \mathrm{cf}(\kappa)$) limit cardinal ($\lambda < \kappa \to \lambda^+ < \kappa$).
$\kappa > \aleph_0$ is *inaccessible* if it is a regular and *strong* limit ($\lambda < \kappa \to 2^\lambda < \kappa$).

**4.** Prove:
(1) Assume that $N \subseteq M$ are transitive models of $ZFC$, then:
  (a) If $M \models$ "$\kappa$ is a cardinal", then so does $N$.
  (b) If $N \models$ "$\kappa$ is a successor cardinal", and $M \models$ "$\kappa$ is a cardinal", then $M \models$ "$\kappa$ is a successor cardinal".
(2) $Con(ZFC + \exists\text{inaccessible } \kappa) \leftrightarrow Con(ZFC + \exists\text{weakly inaccessible } \kappa)$.

*Hint.* (1b) If $N \models \kappa = \lambda^+$, then $N \models (\forall \alpha)\ \lambda \leq \alpha < \kappa \to |\alpha| = \lambda$, and therefore so does $M$.
(2) If $M \models$ "$\kappa$ is weakly inaccessible", then by (1) $L^M \models$ "$\kappa$ is weakly inaccessible". $L^M \models GCH$, thus $L^M \models$ "$\kappa$ is inaccessible".

*Good luck!*

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