FORCING THEORY: EXERCISE 3

BOAZ TSABAN

Definitions. We make the definition of $\mathrm{Def}(A)$ given in class rigorous. First, define for each n:

(a) $\mathrm{Diag}_=(A^n, i, j) = \{s \in A^n : s(i) = s(j)\}$.
(b) $\mathrm{Diag}_\in(A^n, i, j) = \{s \in A^n : s(i) \in s(j)\}$.
(c) $\mathrm{Proj}(A^n, R) = \{s \in A^n : (\exists t \in R)\ s \subseteq t\}$.

By induction on k, define for all n simultaneously:

$$\begin{aligned} D_0(A^n) &= \{\mathrm{Diag}_=(A^n, i, j), \mathrm{Diag}_\in(A^n, i, j) : i, j < n\}; \\ D_{k+1}(A^n) &= \{R, A^n \setminus R, R \cap S : R, S \in D_k(A^n)\} \cup \{\mathrm{Proj}(A^n, R) : R \in D_k(A^{n+1})\}; \\ D(A^n) &= \bigcup_{k<\omega} D_k(A^n). \end{aligned}$$

1.

(a) Prove that $D(A^n)$ is closed under taking complements in A^n, intersections of finitely many elements, and for each $R \in D(A^{n+1})$, $\mathrm{Proj}(A^n, R) \in D(A^n)$.
(b) Let $\varphi(x_1, \dots, x_n)$ be a formula. Prove: For each A, $\{s \in A^n : A \models \varphi(s)\} \in D(A^n)$.
Hint. Induction on the structure of φ, simultaneously for all n.

Definition. For sequences a, b, $a\hat{\ }b$ denotes their concatenation. Define

$$\mathrm{Def}'(A) = \{X \subseteq A : (\exists n)(\exists s \in A^n)(\exists R \in D(A^{n+1}))\ X = \{x \in A : s\hat{\ }x \in R\}\}.$$

2. Let $\varphi(v_1, \dots, v_n, x)$ be a formula. Prove: For each A and all $a_1, \dots, a_n \in A$,

$$\{b \in A : A \models \varphi(a_1, \dots, a_n, b)\} \in \mathrm{Def}'(A).$$

Note that the other direction, that every element of $\mathrm{Def}'(A)$ is defineable in A, is immediate (as a metamathematical statement).

Reading assignment. Read Kunen, Chapter VI Section 1 (pages 165–169). In the following exercises, if needed, you may quote any fact you need from this book.

3. Kunen, Page 180, Exercise 2.

4. Kunen, Page 180, Exercise 3.

Good luck!

Department of Mathematics, The Weizmann Institute of Science, Rehovot 76100, Israel
E-mail address: boaz.tsaban@weizmann.ac.il
URL: http://www.cs.biu.ac.il/~tsaban