# FORCING THEORY: EXERCISE 2

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It's more fun solving without hints. When needed, hints are available on the next page.

**1.** Let $\kappa$ be a measurable cardinal, and $\mathcal{U}$ be a nonprincipal $\kappa$-complete ultrafilter on $\kappa$. Prove:

(a) For each $f \in \mathbf{V}^\kappa$, $\{g \in \mathbf{V}^\kappa : g =^* f\}$ is a proper class.
(b) $\mathbf{E}$ is well-defined, set-like, and well-founded on $\mathbf{V}^\kappa/\mathcal{U}$.

**2.** Prove **Łos' Theorem**: Let $\kappa$ be a measurable cardinal, and $\mathcal{U}$ be a measure on $\kappa$. For each formula $\varphi(x_1, \dots, x_n)$, and all $f_1, \dots, f_n \in V^\kappa$,

$$V^\kappa/\mathcal{U} \models \varphi([f_1], \dots, [f_n]) \quad \Leftrightarrow \quad \{\alpha < \kappa : \varphi(f_1(\alpha), \dots, f_n(\alpha))\} \in \mathcal{U}.$$

*Reading assignment.* Read Chapter IV, Section 3 (Pages 117–124, but better start on Page 110), of Kunen's book. It is not necessary, but you can quote results from there when solving the following exercises.

*Reminders.* $i : M_1 \to M_2$ is an *embedding* if $i : M_1 \to i[M_1]$ is an isomorphism. Thus, $i$ is injective, and for each formula $\varphi(x_1, \dots, x_n)$ and all $a_1, \dots, a_m \in M_1$, $M_1 \models \varphi(a_1, \dots, a_n) \Leftrightarrow i[M_1] \models \varphi(i(a_1), \dots, i(a_n))$.

$i : M_1 \to M_2$ is an *elementary embedding* if it is an embedding, and $i[M_1] \prec M_2$. Thus, $i$ is injective, and for each formula $\varphi(x_1, \dots, x_n)$ and all $a_1, \dots, a_m \in M_1$,

$$\begin{aligned} M_1 \models \varphi(a_1, \dots, a_n) &\Leftrightarrow i[M_1] \models \varphi(i(a_1), \dots, i(a_n)) \\ &\Leftrightarrow M_2 \models \varphi(i(a_1), \dots, i(a_n)). \end{aligned}$$

We have proved that for each measurable $\kappa$, there is a transitive $M$ and an embedding $i : V \to M$ such that $\kappa$ is the first ordinal $\alpha$ such that $\alpha < i(\alpha)$. In the following two exercises you prove that the converse also holds.

**3.** Assume that $i : V \to M$ is an elementary embedding which is not the identity mapping, and $M$ is transitive.

(a) There is an ordinal $\alpha$ such that $i(\alpha) > \alpha$.
(b) For each $n$, $i(n) = n$, and $i(\omega) = \omega$.

**4.** Assume that $i : V \to M$ is an elementary embedding which is not the identity mapping, and $M$ is transitive. Then $\kappa := \min\{\alpha : \alpha < i(\alpha)\}$ is a measurable cardinal.

**Hint for 1(b).** Set-like: Fix $[g]$. If $[f]\mathbf{E}[g]$, let $h(\alpha)=f(\alpha)$ when $f(\alpha)\in g(\alpha)$, and $h(\alpha)=0$ otherwise. $[h]=[f]$. $\{[f]:[f]\mathbf{E}[g]\}=\{[h]:h\in^* g,\ (\forall\alpha<\kappa)\ h(\alpha)\in g(\alpha)\cup\{0\}\}$.
Well-founded: If $\dots[f_2]\mathbf{E}[f_1]\mathbf{E}[f_0]$, take $\alpha\in\bigcap_{n\in\mathbb{N}}\{\alpha<\kappa:f_{n+1}(\alpha)\in f_n(\alpha)\}$.

**Hint for 2.** Induction on the structure of $\varphi$.

**Hint for 3(a).** For each $x$, $i(\mathrm{rank}(x))=\mathrm{rank}(i(x))$. Assume that for each $\alpha$, $i(\alpha)=\alpha$. Then $\mathrm{rank}(i(x))=\mathrm{rank}(x)$ for all $x$. By induction on the rank, $i(x)=x$ for all $x$. ($y\in i(x)$ implies $y=i(y)$.)

**Hint for 3(b).** Induction on $n$ for the first assertion; definition of $\omega$ for the second. ($V\models\varphi(\omega)$, where $\varphi(x)=0\in x\wedge\forall y\in x(y\neq 0\rightarrow\exists z\in x(y=z\cup\{z\}))$. $i[V]\models\varphi(i(\omega))$. $\varphi(i(\omega))$ holds.)

**Hint for 4.** $\kappa>\omega$. Let

$$\mathcal{U}=\{X\subseteq\kappa:\kappa\in i(X)\}.$$

$\kappa\in\mathcal{U}$. $\emptyset\notin\mathcal{U}$.
$X\subseteq Y\rightarrow i(X)\subseteq i(Y)$ (express $X\subseteq Y$ as a formula).
$i(\kappa\setminus X)=i(\kappa)\setminus i(X)$.
For all $\alpha<\kappa$, $i(\{\alpha\})=\{i(\alpha)\}=\{\alpha\}$.
$\kappa$-completeness: Fix $\gamma<\kappa$ and $f:\gamma\rightarrow\mathcal{U}$. You must show that $\bigcap_{\alpha<\gamma}f(\alpha)\in\mathcal{U}$:
For each $\alpha<\gamma$, if $X=f(\alpha)$, then $i(X)=i(f)(i(\alpha))$. (Express $X=f(\alpha)$ as a formula $\varphi(X,f,\alpha)$.)
As $\alpha<\gamma$, $i(f(\alpha))=i(X)=i(f)(\alpha)$.
Express "$A=\bigcap_{\alpha<\gamma}f(\alpha)$" as a formula $\varphi(A,\gamma,f)$.

$$i(A)=\bigcap_{\alpha<i(\gamma)}i(f)(\alpha)=\bigcap_{\alpha<\gamma}i(f(\alpha))\ni\kappa.$$

*Good luck!*

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