FORCING THEORY: EXERCISE 1

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1. Compute the ranks of:

(a) $\bigcup x$, $P(x)$, and $\{x\}$, in terms of $\mathrm{rank}(x)$.
(b) $x \times y$, $x \cup y$, $\{x, y\}$, (x, y), and ${}^y x$, in terms of $\mathrm{rank}(x)$ and $\mathrm{rank}(y)$.
(c) $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$.

2. Let $\alpha < \aleph_2$. Prove that $\mathbf{V}_{\aleph_2}$ "knows" that $|\alpha| \leq \aleph_1$ in the sense that there is $f \in \mathbf{V}_{\aleph_2}$ such that $f : \aleph_1 \to \alpha$ is surjective.

3. Prove the *Mostowski Collapse Theorem*:
Assume that $\mathbf{E}$ is well-founded, set-like, and extensional on the class $\mathbf{P}$. Then:

(a) There is a transitive class $\mathbf{M}$ such that $(\mathbf{M}, \in) \cong (\mathbf{P}, \mathbf{E})$.
(b) $\mathbf{M}$ and the isomorphism are unique.

Hint. (a) By $\mathbf{E}$-recursion, define $\pi(x) = \{\pi(y) : y\mathbf{E}x\}$, and $\mathbf{M} = \pi[\mathbf{P}]$.
(b) Use $\mathbf{E}$-induction.

Terminology. The transitive class $\mathbf{M}$ in the Mostowski Collapse Theorem will be called the *collapse* of $(\mathbf{P}, \mathbf{E})$.

4. Assume that $\mathbf{E}$ well-orders $\mathbf{P}$, and is set-like and extensional on $\mathbf{P}$. Prove:

(a) If $\mathbf{P}$ is a set, then the collapse of $(\mathbf{P}, \mathbf{E})$ is an ordinal, namely, the order type of $(\mathbf{P}, \mathbf{E})$.
(b) If $\mathbf{P}$ is a proper class, then the collapse of $(\mathbf{P}, \mathbf{E})$ is $\mathbf{ON}$, the class of all ordinal numbers.

Good luck!

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