FORCING THEORY: EXERCISE 2

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Definitions. We make the definition of $\mathrm{Def}(A)$ given in class rigorous. First, define for each n:

(a) $\mathrm{Diag}_=(A^n, i, j) = \{s \in A^n : s(i) = s(j)\}$.
(b) $\mathrm{Diag}_\in(A^n, i, j) = \{s \in A^n : s(i) \in s(j)\}$.
(c) $\mathrm{Proj}(A^n, R) = \{s \in A^n : (\exists t \in R)\ s \subseteq t\}$.

By induction on k, define for all n simultaneously:

$$\begin{aligned} D_0(A^n) &= \{\mathrm{Diag}_=(A^n,i,j), \mathrm{Diag}_\in(A^n,i,j) : i,j < n\}; \\ D_{k+1}(A^n) &= \{R, A^n \setminus R, R \cap S : R, S \in D_k(A^n)\} \cup \{\mathrm{Proj}(A^n, R) : R \in D_k(A^{n+1})\}; \\ D(A^n) &= \bigcup_{k<\omega} D_k(A^n). \end{aligned}$$

1. Prove:

(a) $D(A^n)$ is closed under taking complements in A^n.
(b) $D(A^n)$ is closed under intersections of finitely many elements.
(c) For each $R \in D(A^{n+1})$, $\mathrm{Proj}(A^n, R) \in D(A^n)$.

2. Let $\varphi(x_1, \dots, x_n)$ be a formula. Prove: For each A, $\{s \in A^n : A \models \varphi(s)\} \in D(A^n)$.
Hint. Induction on the structure of φ, simultaneously for all n.

Definition. For sequences a, b, $a\hat{\ }b$ denotes their concatenation. Define

$$\mathrm{Def}'(A) = \{X \subseteq A : (\exists n)(\exists s \in A^n)(\exists R \in D(A^{n+1}))\ X = \{x \in A : s\hat{\ }x \in R\}\}.$$

3. Let $\varphi(v_1, \dots, v_n, x)$ be a formula. Prove: For each A and all $a_1, \dots, a_n \in A$,

$$\{b \in A : A \models \varphi(b, a_1, \dots, a_n)\} \in \mathrm{Def}'(A).$$

Note that the other direction, that every element of $\mathrm{Def}'(A)$ is defineable in A, is immediate (as a metamathematical statement).

4. Kunen, Page 180, Exercise 2.

Good luck!

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