IDENTITIES OF A FIVE-ELEMENT 0-SIMPLE SEMIGROUP

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In the theory of varieties of semigroups study of finite semigroups, especially with small number of elements, leads sometimes to unexpected results (see [6],[7], [8]). In our approach to the identities we use associated graphs. This clear and transparent approach proved often to be fruitful ([1], [2], [5], [9], [11]).

We consider a 5-element 0-simple semigroup with identities admitting descriptions in terms of the graph of adjacency and with a finite basis of identities in three variables. Our main object is the semigroup $A =< a, b|aba = a, bab = b, a^2 = a, b^2 = 0 >$. It has five elements a, b, ab, ba and 0, which is the zero of A. All elements of A except b are idempotents. A is a 0-simple non-orthodox semigroup. It arises repeatedly in investigations (see [3], [4], [9], [11]) and seems to be of some interest in itself.

Theorem 0.1 *The identities*

$$x^2 = x^3, xyx = xyxyx, xyxzx = xzxyx \tag{1}$$

form an irreducible basis of identities of $varA$.

This result was announced in [10] and generalized in [5]. Let us introduce necessary definitions and notations.

$varS$ is the variety of semigroups generated by a semigroup S;

identity $u = v$ is called normal if the words u and v consist of the same variables;

a word u is called simple if $u = x_1...x_n$, where x_i are variables, and $x_i \neq x_j$ for $i \neq j$;

F_X (or merely F) - the free semigroup over an alphabet X;

u^α is the class of a completely invariant congruence α on F that contains an element u.

Let ρ be a completely invariant congruence on F defined by A. With each word u over X we associate a graph $G(u)$, its vertex set is $X \cup \{0\}$, there is an arrow $x \to y$ if xy is a subword of u, $0 \to x$ for the first letter x, and $y \to 0$ for the last letter y of u.

A word u is called indecomposable if every two vertices of $G(u) \setminus \{0\}$ belong to an oriented cycle. Let Gp be a path without cycles in $G(u)$ and $a_1, a_2, ...a_n$ consecutive vertices of the path. A word u is called complete if for every such path Gp it contains the simple word $a_1, a_2, ...a_n$ as subword.

Now we use the following

Proposition [4]. Let u and v be words over the alphabet$X$. Then the identity $u = v$ holds in A if and only if $G(u) = G(v)$.

Identities 1 in A follow from that Proposition. Let β be the completely invariant congruence on F corresponding to identities 1. Let $s = f$ to be identity of $varA$ with $s = s_1...s_n$ and all s_i are maximal indecomposable subwords of s. Then $G(s)$ is the chain of subgraphs G_i corresponding to s_i ,$G_i = G(s_i)$. By Proposition, $f = f_1...f_n$, where $G(f_i) = G_i$. Therefore, all identities $s_i = f_i$ are valid in A and $s = f$ follows from the set of identities $s_i = f_i$. Now it is sufficient to prove that $u^\rho = u^\beta$ for any indecomposable word u.

By Proposition, if u is an indecomposable word, all words from u^ρ are indecomposable. Each congruence class of ρ contains a complete word. Let v be such a word from u^ρ. Now we show v is the beginning of some word from u^β, that is, vw belongs to u^β for some suitable word w. Consider the beginning subword of length n of the word v. We proceed by induction on n. For $n = 1$ it is obvious because the first letters of all the words in u^ρ coincide. Suppose now that for v' from u^β left subwords of lengths $n-1$ of v and v' coincide, $v = w_1txv_1$, $v' = w_1tyv_1'$ for some subwords w_1, v, v_1' and letters t, x, y. Here v_1', v, y may be empty. $G(v) = G(v')$ implies that v' contains the subword tx. If tx is contained in w_1, then we use the identity $xyx = xyxyx$ and transform v' into the word w_1txw_2 for some w_2 with x in the n-th place. Suppose now that tx belongs to v_1'. Since v' is indecomposable, the identity $xyx = xyxyx$ gives us an opportunity to repeat the subword tx. So we can find a letter t in three places: the first one is ty, then tx, and then tx. Now we can use the identity $xyxzx = xzxyx$ and obtain x in the n-th place.

Thus the word vw belongs to u^β for a complete word v from u^ρ. Because of $x^2 = x^3$, assume that w contains no degrees of subwords greater than two. All simple subwords of w belong to v. Therefore, using the identity $xyxzxyx = xyxzx$ that follows from 1 , we can reduce vw to the word v. So v from u^β and the identity $u = v$ follows from 1 for every complete word v from u^ρ.

Theorem is proved.

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