Verification Tools for Checking some kinds of Testability

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Abstract
A locally testable language $L$ is a language with the property that for some nonnegative integer $k$, called the order of local testability, whether or not a word $u$ is in the language $L$ depends on (1) the prefix and suffix of the word $u$ of length $k-1$ and (2) the set of intermediate substrings of length $k$ of the word $u$. For given $k$ the language is called $k$-testable. The local testability has a wide spectrum of generalizations.

A set of procedures for deciding whether or not a language given by its minimal automaton or by its syntactic semigroup is locally testable, right or left locally testable, threshold locally testable, strictly locally testable, or piecewise testable was implemented in the package TESTAS written in C/C++. The bounds on order of local testability of transition graph and order of local testability of transition semigroup are also found. For given $k$, the $k$-testability of transition graph is verified.

We consider some approaches to verify these procedures and use for this aim some auxiliary programs. The approaches are based on distinct forms of presentation of a given finite automaton and on algebraic properties of the presentation.

New proof and fresh wording of necessary and sufficient conditions for local testability of deterministic finite automata is presented.

Keywords: deterministic finite automaton, locally testable, algorithm, graph, semigroup

Introduction
The concept of local testability is connected with languages, finite automata, graphs and semigroups. It was introduced by McNaughton and Papert (1971) and by Brzozowski and Simon (1973). Our investigation is based on both graph and semigroup representation of automaton accepting formal language. Membership of a long text in a locally testable language just depends on a scan of short subpatterns of the text. It is best understood in terms of a kind of computational procedure used to classify a two-dimensional image: a window of relatively small size is moved around on the image and a record is made of the various attributes of the image that are detected by what is observed through the window. No record is kept of the order in which the attributes are observed, where each attribute occurs, or how many times it occurs. We say that a class of images is locally testable if a decision about whether a given image belongs to the class can be made simply on the basis of the set of attributes that occur.

How many times the attributes are observed is essential in the definition of locally threshold testable language, but it is considered only for number of occurrences less than given threshold.

The order in which the attributes are observed from left [from right] is essential in the definition of the left [right] locally testable languages.

The definition of strictly locally testable language differs from definition of locally testable language only by the length of prefixes and suffixes: in this case they have the same length $k$ as substrings.

Piecewise testable languages are the finite Boolean combinations of languages of the form $A^*a_1A^*a_2A^*...A^*a_kA^*$ where $k \geq 0$, $a_i$ is a letter from the alphabet $A$ and $A^*$ is the free monoid over $A$. 
A language is piecewise testable iff its syntactic semigroup is $J$-trivial (distinct elements generate distinct ideals) (Simon 1975).

The considered languages form subclasses of star-free languages and have a lot of applications. Regular languages and picture regular languages can be described by help of a strictly locally testable languages (Birget 1991, Hinz 1990). Local automata (a kind of locally testable automaton) are heavily used to construct transducers and coding schemes adapted to constrained channels (Beal Senellart 1998). Locally testable languages are used in the study of DNA and informational macromolecules in biology (Head 1987).

The locally threshold testable languages (Beauquier Pin 1989) generalize the concept of locally testable language and have been studied extensively in recent years. An important reason to study locally testable languages is the possibility of being used in pattern recognition (Ruiz Espana Garcia 1998). Stochastic locally threshold testable languages, also known as $n$-grams are used in pattern recognition, particularly in speech recognition, both in acoustic-phonetics decoding and in language modelling (Vidal Casacuberta Garcia 1995).

The implementation of algorithms concerning distinct kinds of testability of finite automata was begun by Caron (1998). Our package TESTAS (testability of automata and semigroups), written in C/C++, presents most wide list of different kinds of testability. The package contains a set of procedures for deciding whether or not a language given by its minimal automaton or by its syntactic semigroup is locally testable, right or left locally testable, threshold locally testable, strictly locally testable, bilateral locally testable, or piecewise testable. The bounds on order of local testability of transition graph and order of local testability of transition semigroup are also found. For given $k$, the $k$-testability of transition graph is verified. The transition semigroups of automata are studied in the package for the first time.

We present in this paper the theoretical background of the algorithms, giving sometimes fresh wording of results. In particular, we give new short proof for necessary and sufficient conditions for the local testability of DFA.

The complexity of the algorithms and programs is estimated here in detail.

We consider here some approaches to verify the procedures of the package and use for this aim some auxiliary programs. The approaches are based on distinct forms of presentation of a given finite automaton and on the algebraic properties of the presentation.

**PRELIMINARIES**

Let $\Sigma$ be an alphabet and let $\Sigma^+$ denote the free semigroup on $\Sigma$. If $w \in \Sigma^+$, let $|w|$ denote the length of $w$. Let $k$ be a positive integer. Let $i_k(w)$ denote the prefix of $w$ of length $k$ or $w$ if $|w| < k$. Let $F_k(w)$ denote the set of factors of $w$ of length $k$. A language $L$ is called $k$-testable if there is an alphabet $\Sigma$ such that for all $u, v \in \Sigma^+$, if $i_{k-1}(u) = i_{k-1}(v)$, then either both $u$ and $v$ are in $L$ or neither is in $L$. An automaton is $k$-testable if the automaton accepts a $k$-testable language. A language $L$ [an automaton $A$] is locally testable if it is $k$-testable for some $k$.

The definition of strictly locally testable language is analogous, only the length of prefix and suffix is equal to $k$, in the definition of strongly locally testable language prefix and suffix are omitted at all.

The number of nodes of the graph $\Gamma$ is denoted $|\Gamma|$.

The direct product of $k$ copies of the graph $\Gamma$ denoted by $\Gamma^k$ consists of states $(p_1, \ldots, p_k)$ where $p_i$ from $\Gamma$ and edges $(p_1, \ldots, p_k) \rightarrow (p_1 \sigma, \ldots, p_k \sigma)$ labeled by $\sigma$ for every $\sigma$ from $\Sigma$.

A strongly connected component of the graph will be denoted for brevity $SCC$, a deterministic finite automaton will be denoted DFA.

A node from a cycle will be called for brevity $C-node$. $C-node$ can be defined also as a node that has right unit in the transition semigroup of the automaton.

If an edge $p \rightarrow q$ is labeled by $\sigma$ then let us denote the node $q$ as $p \sigma$.

We shall write $p \succeq q$ if $p = q$ or the node $q$ is reachable from the node $p$ (there exists a directed path from $p$ to $q$).

In the case $p \succeq q$ and $q \succeq p$ we write $p \sim q$ (that is $p$ and $q$ belong to one $SCC$).
The graph with only trivial SCC (loops) will be called acyclic.

The stabilizer $\Sigma(q)$ of the node $q$ from $\Gamma$ is the subset of letters $\sigma \in \Sigma$ such that any edge from $q$ labeled by $\sigma$ is a loop $q \rightarrow q$.

Let $\Gamma(\Sigma_i)$ be the directed graph with all nodes from the graph $\Gamma$ and edges from $\Gamma$ with labels only from the subset $\Sigma_i$ of the alphabet $\Sigma$.

So, $\Gamma(\Sigma(q))$ is a directed graph with nodes from the graph $\Gamma$ and edges from $\Gamma$ that are labeled by letters from stabilizer of $q$.

A semigroup without non-trivial subgroups is called aperiodic.

A semigroup $S$ has a property $\rho$ locally if for any idempotent $e \in S$ the subsemigroup $eSe$ has the property $\rho$.

So, a semigroup $S$ is called locally idempotent if $eSe$ is an idempotent subsemigroup for any idempotent $e \in S$.

### Complexity Measures

The state complexity of the transition graph $\Gamma$ of a deterministic finite automaton is equal to the number of his nodes $|\Gamma|$. The measures of the complexity of the transition graph $\Gamma$ are connected also with the sum of the numbers of the nodes and the edges of the graph $a$ and the size of the alphabet $g$ of the labels on the edges (the number of generators of the transition semigroup). The value of $a$ can be considered sometimes as a product $(g + 1)|\Gamma|$. Let us notice that $(g + 1)|\Gamma| \geq a$.

The input of the graph programs of the package is a rectangular table: nodes $X$ labels. So the space complexity of the algorithms considering the transition graph of an automaton is not less than $|\Gamma|g$. The graph programs use usually a table of reachability defined on the nodes of the graph. The table of reachability is a square table and so we have $|\Gamma|^2$ space complexity.

The number of the nodes of $\Gamma^k$ is $|\Gamma|^k$, the alphabet is the same as in $\Gamma$. So the sum of the numbers of the nodes and the edges of the graph $\Gamma^k$ is not greater than $(g + 1)|\Gamma|^k$. Some algorithms of the package use the powers $\Gamma^2$, $\Gamma^3$ and even $\Gamma^4$. So the space complexity of the algorithms reaches in these cases $|\Gamma|^2g$, $|\Gamma|^3g$ or $|\Gamma|^4g$.

The main measure of complexity for semigroup $S$ is the size of the semigroup $|S|$ denoted by $n$. Important characteristics are also the number of generators (size of alphabet) $g$ and the number of idempotents $i$.

The input of the semigroup programs of the package is the Cayley graph of the semigroup presented by a rectangular table: elements $X$ generators. So the space complexity of the algorithms considering the transition semigroup of an automaton is not less than $O(ng)$. Algorithms of the package dealing with the transition semigroup of an automaton use the multiplication table of the semigroup of $O(n^2)$ space. Another arrays used by the package present subsemigroups or subsets of the transition semigroup. So we have usually $O(n^2)$ space complexity.

### 1 Verification Tools of the Package

A deterministic finite automaton can be presented by its syntactic semigroup or by the transition graph of the automaton. The package TESTAS includes programs that analyze:

1) an automaton of the language presented by oriented labeled graph;
2) an automaton of the language presented by its syntactic semigroup.

Some auxiliary programs ensure verification of the algorithms used in the package.

An important verification tool is the possibility to study both transition graph and transition semigroup of a given automaton and compare the results. The algorithms for graphs and for semigroups are completely different.

An auxiliary program, written in C, finds the syntactic semigroup from the transition graph of the automaton. The program finds distinct mappings of the graph of the automaton induced by the letters of the alphabet of the labels. Any two mappings must to be compared, so we have $O(n(n - 1)/2)$ steps. These mappings form the set of semigroup elements. The set of generators coincides usually with the alphabet of the labels, but in some singular cases a proper subset of the alphabet is obtained. On this way, the syntactic semigroup of the automaton and the minimal
set of semigroup generators is constructed. The time complexity of the considered procedure is $O(|\Gamma|gn^2)$ with $O(|\Gamma|n)$ space complexity.

Let us notice that the size of the syntactic semigroup is in general not polynomial in the size of the transition graph. For example, let us consider a graph with 28 nodes and 33 edges (Kim McNaughton McCloskey 1991) and the following modification of this graph (Trahtman 2000a) obtained by adding one edge.

The syntactic semigroup of given automaton has over 22 thousand elements. The verifying of local testability and finding the order of local testability for this semigroup needs an algorithm of $O(n^2)$ time complexity (Trahtman 1998). So in semigroup case, we have $O(22126^2)$ time complexity for both checking the local testability and finding the order.

In the graph case, checking the local testability needs an algorithm of $O(28^2)$ time complexity (Kim McNaughton 1994, Trahtman 2001), but the finding the order of local testability is in general non-polynomial (Kim McNaughton 1994). However, in our case, the subprogram that finds the lower and upper bounds for the order of local testability finds equal lower and upper bounds and therefore gives the final answer for the graph more fast. The time complexity of the subprogram is $O(|\Gamma|^2g)$ (Trahtman 2000b), whence the algorithm in this case is polynomial and has only $O(28^2)$ time complexity.

In many cases the difference between the size of the semigroup and the graph is not so great in spite of the fact that the size of the syntactic semigroup is in general not polynomial in the size of the transition graph. Therefore the passage to the syntactic semigroup is useful because the semigroup algorithms are in many cases more simple and more rapid.

The checking of the algorithms is based also on the fact that some of the considered objects form a variety (quasivariety, pseudovariety) and therefore they are closed under direct product. For instance, $k$-testable semigroups form variety (Zalcstein 1973), locally threshold testable semigroups (Beauquier Pin 1989) and piecewise testable semigroups (Simon 1975) form pseudovariety. Left [right] locally testable semigroups form quasivariety because they are locally idempotent and satisfy locally some identities (Garcia Ruiz 2000). Let us mention also Eilenberg classical variety theorem (Eilenberg 1976).

Two auxiliary programs, written in C, that find the direct product of two semigroups and of two graphs belong to the package. The input of the semigroup program consists of two semigroups presented by their Cayley graph with generators at the beginning of the element list. The result is presented in the same form and the set of generators of the result is placed in the beginning of the list of elements. The number of generators of the result is $n_1g_2 + n_2g_1 - g_1g_2$ where $n_i$ is the size of the $i$-th semigroup and $g_i$ is the number of its generators.

The components of the direct product of graphs are considered as graphs with common alphabet of edge labels. The labels of both graphs are identified according to their order. The number of labels is not necessarily the same for both graphs, but the result alphabet uses only common labels from the beginning of both alphabets.

Big size semigroups and graphs with predesigned properties can be obtained by help of these programs. Any direct power of a semigroup or graph keeps important properties of the origin.

For example, let us consider the following semigroup

$$A_2 = \langle a, b \mid aba = a, bab = b, a^2 = a, b^2 = 0 \rangle$$

It is a 5-element 0-simple semigroup, $A_2 = \{a, b, ab, ba, 0\}$, only $b$ is not an idempotent. The key role of the semigroup $A_2$ in the theory of locally testable semigroups explains the following theorem:
Theorem 1.1  (Trahtman 1999) The semigroup $A_2$ generates the variety of 2-testable semigroups. Every $k$-testable semigroup is a $(k - 1)$-nilpotent extension of a 2-testable semigroup.

Any variety of semigroups is closed in particular under direct product. Therefore any direct power of a semigroup $A_2$ is 2-testable, locally testable, threshold locally testable, left and right locally testable, locally idempotent. These properties can be checked by the package.

The possibility to use distinct independent algorithms of different nature with various measures of complexity to the same object gives us a powerful verification tool.

2 Background of the Algorithms

2.1 The Necessary and Sufficient Conditions of Local Testability for DFA

Local testability plays an important role in the study of distinct kinds of star-free languages. Necessary and sufficient conditions of local testability for reduced deterministic finite automaton were found by Kim, McNaughton and McCloskey (1991).

Let us present a new short proof and fresh wording of these conditions:

Theorem 2.1 Reduced DFA $A$ with state transition graph $\Gamma$ and transition semigroup $S$ is locally testable iff for any $C$-node $(p, q)$ of $\Gamma^2$ such that $p \succeq q$ we have

1. If $q \succeq p$ then $p = q$.
2. For any $s \in S$ holds $ps \succeq q$ iff $qs \succeq q$.

Proof. Suppose $A$ is locally testable. Then the transition semigroup $S$ of the automaton is finite, aperiodic and for any idempotent $e \in S$ the subsemigroup $eSe$ is commutative and idempotent (Zalcstein 1973).

Let us consider the $C$-node $(p, q)$ from $\Gamma^2$ such that $p \succeq q$. Then for some element $e \in S$ we have $qe = q$ and $pe = p$. We have $qe^i = q$, $pe^i = p$ for any integer $i$. Therefore in view of aperiodicity and finiteness of $S$ we can consider $e$ as an idempotent. Let us notice that for some $a$ from $S$ we have $pa = q$.

Suppose first that $q \succeq p$. Then for some $b$ from $S$ we have $q^b = p$. Hence, $peae = q$, $qbe = p$. So $p = peae = p(eaeb)^i$ for any integer $i$. There exists a natural number $n$ such that in the finite aperiodic semigroup $S$ we have $(eae)^n = (eae)^{n+1}$. Commutativity of $eSe$ implies $eaeb = beae$. We have $p = peae = p(eaeb)^n = p(eae)^n(ebe)^n = p(eae)^{n+1}(ebe)^n = p(eae)^n(ebe)^n eae = p = q$. So $p = q$. Thus the condition 1 holds.

Let us go to the second condition. For any $s \in S$ we have $p = p = q$. Hence, $peae = q$, $qbe = p$. In idempotent subsemigroup $eSe$ we have $q = (qbe)^2$. Therefore $q = q = q = q = q$. And $q = q = q = q = q$.

If we assume that $p \succeq q$, then for some $b$ from $S$ holds $ps \succeq q$, whence $pse = q$. In idempotent subsemigroup $eSe$ we have $esbe = (esbe)^2$. Therefore $q = q = q = q = q$. And $q = q = q = q = q$.

Suppose now that the conditions 1 and 2 are valid for any $C$-node from $\Gamma^2$ such that his second component is successor of the first. We must to prove that the subsemigroup $eSe$ is idempotent and commutative for any idempotent $e$ from transition semigroup $S$.

Let us consider an arbitrary node $p$ from $\Gamma$ and an arbitrary element $s$ from $S$ such that the node $pse$ exists. The node $(pe, pse)$ is a $C$-node from $\Gamma^2$ and $pe \succeq pse$. We have $(pe, pse)e = (pse, pse)^2$. Therefore, by condition 2, $(pse)^2 \succeq pse$ and the node $pse$ exists too. The node $(pse, pse)^2$ is a $C$-node from $\Gamma^2$, whence by condition 1, $pse = pse$. The node $p$ is an arbitrary node, therefore $e = (e)^2$. Thus the semigroup $eSe$ is an idempotent subsemigroup.

Let us consider now arbitrary elements $a, b$ from idempotent subsemigroup $eSe$ and an arbitrary node $p$ such that the node $pab$ exists. We have $ab = (ab)^2$ and $pab = pab$. So, $pab \succeq pab$, whence $pba \sim pab$. The node $(pba, pab)$ is a $C$-node from $\Gamma^2$, whence, by condition 1, $pba = pab$. Therefore $pba = pab$ in view of $b^2 = b$. 

Notice that \((p, pa b) \succeq (p b a, pa b a)\) in \(\Gamma^2\). In view of \(pa b a = pa b\) and the condition 2, we have \(p b a \succeq p a b\). Therefore the node \(p b a\) exists.

We can prove now analogously that \(p a b \succeq p b a\), whence \(p b a \sim p a b\). Because the node \((p b a, p a b)\) is a \(C\)-node, we have, by the condition 1, \(p b a = p a b\). So \(eSe\) is commutative.

Let us go to the algorithms for local testability and to measures of complexity of the algorithms. Polynomial-time algorithm for the local testability problem for the transition graph (Trahtman 2001) of order \(O(n^2)\) (or \(O(|\Gamma|^2 g)\)) is implemented in the package TESTAS. The space complexity of the algorithm is also \(O(|\Gamma|^2 g)\). A polynomial-time algorithm of \(O(|\Gamma|^2 g)\) time and of \(O(|\Gamma|^2 g)\) space is used for finding the bounds on order of local testability for a given transition graph of the automaton (Trahtman 2000b, Trahtman 2000a). An algorithm of worst case \(O(|\Gamma|^4 g)\) time complexity and of \(O(|\Gamma|^2 g)\) space complexity checked the 2-testability (Trahtman 2000b). The 1-testability is verified by help of algorithm (Kim McNaughton 1994) of order \(O(|\Gamma|^2 g^k)\). Checking the \(k\)-testability for fixed \(k\) is polynomial but growing with \(k\). For checking the \(k\)-testability (Trahtman 2000b), we use an algorithm of worst case asymptotic cost \(O(|\Gamma|^2 g^{k-1})\) of time complexity with \(O(|\Gamma|^2 g)\) space complexity. The time complexity of the last algorithm is growing with \(k\) and on this way we obtain non-polynomial algorithm for finding the order of local testability. However, \(k\) is not greater than \(\log_2 M\) where \(M\) is the maximal size of the integer in the computer memory.

### 2.2 The Necessary and Sufficient Conditions of Local Testability for Finite Semigroup

The best known description of necessary and sufficient conditions of local testability was found independently by Brzozowski and Simon (1973), McNaughton (1974) and Zalcstein (1973):

Finite semigroup \(S\) is locally testable iff its subsemigroup \(eSe\) is commutative and idempotent for any idempotent \(e \in S\).

The class of \(k\)-testable semigroups forms a variety (Zalcstein 1973). This variety has a finite base of identities (Trahtman 1999). The variety of 2-testable semigroups is generated by 5-element semigroup and any \(k\)-testable semigroup is a nilpotent extension of 2-testable semigroup (Trahtman 1999).

We present here necessary and sufficient conditions of local testability of semigroup in new form and from another point of view.

**Theorem 2.2** For finite semigroup \(S\), the following four conditions are equivalent:

1. \(S\) is locally testable.
2. \(eSe\) is 1-testable for every idempotent \(e \in S\) (\(eSe\) is commutative and idempotent).
3. \(Se\) is 2-testable for every idempotent \(e \in S\).
4. \(Se\) is 2-testable for every idempotent \(e \in S\).

Proof. Equivalency of 1) and 2) is well known (Brzozowski Simon 1973, McNaughton 1974, Zalcstein 1973).

3) \(\rightarrow\) 2). \(Se\) satisfies identities of 2-testability: \(xyx = xyxyx, x^2 = x^3, xyx = xxyx\) (Trahtman 1999), whence \(ese = ese\) an \(ese = e\) for any idempotent \(e \in S\) and for any \(s, t \in S\). Therefore \(eSe\) is commutative and idempotent.

2) \(\rightarrow\) 4). Identities of 1-testability in \(eSe\) may be presented in the following form

\[
exe = exe, exey = exye
\]

for arbitrary \(x, y \in S\). Therefore for any \(u, v, w\) divided by \(e\) we have

\[
uu = uu, uvu = uvuvu, uww = uwuuw
\]

So identities of 2-testability are valid in \(Se\).

4) \(\rightarrow\) 3). \(Se \subseteq Se\) whence identities \(SeSe\) are valid in \(Se\).

Let us go now to the semigroup algorithms. The situation here is more favorable than in graphs. We implement in the package TESTAS a polynomial-time algorithms of \(O(n^2)\) time and space complexity for local testability problem and for finding the order of local testability for a given semigroup (Trahtman 1998). In spite of the fact that the last algorithm is more complicated and essentially more prolonged, the time complexity of both algorithms is the same.
The verification of associative low needs algorithm of $O(n^3)$ time complexity. Some modification of this algorithm known as Light test (Lidl Pilz 1984) works in $O(n^2 g)$ time. The equality $(ab)j = a(jb)$ where $a, b$ are elements and $j$ is a generator is tested in this case. This algorithm is used also in the package TESTAS. Relatively small gruppoids are checked by the package automatically, verification of big size objects can be omitted by user.

2.3 Threshold Local Testability

Let $\Sigma$ be an alphabet and let $\Sigma^+$ denote the free semigroup on $\Sigma$. If $w \in \Sigma^+$, let $|w|$ denote the length of $w$. Let $k$ be a positive integer. Let $i_k(w)$ [$t_k(w)$] denote the prefix [suffix] of $w$ of length $k$ or $w$ if $|w| < k$. Let $F_{k,j}(w)$ denote the set of factors of $w$ of length $k$ with at least $j$ occurrences. A language $L$ is called $l$-threshold $k$-testable if for all $u, v \in \Sigma^+$, if $i_{k-1}(u) = i_{k-1}(v)$, $t_{k-1}(u) = t_{k-1}(v)$ and $F_{k,j}(u) = F_{k,j}(v)$ for all $j \leq l$, then either both $u$ and $v$ are in $L$ or neither is in $L$.

An automaton is $l$-threshold $k$-testable if the automaton accepts a $l$-threshold $k$-testable language.

A language $L$ [an automaton $A$] is locally threshold testable if it is $l$-threshold $k$-testable for some $k$ and $l$.

The syntactic characterization of locally threshold testable languages was given by Beauquier and Pin (1989). Necessary and sufficient conditions of local threshold testability for the transition graph of DFA (Trahmtan 2001, Trahtman 2003) follow from their result. First polynomial-time algorithm for the local threshold testability problem for the transition graph of the language used previously in the package was based on the necessary and sufficient conditions from (Trahmtan 2001). The time complexity of this graph algorithm was $O(|\Gamma|^3 g)$ with $O(|\Gamma|^4 g)$ space. This algorithm is replaced now in the package TESTAS by a new algorithm of worst case asymptotic cost $O(|\Gamma|^4 g)$ of the time complexity (Trahmtan 2003). The algorithm works as a rule more quickly. The space complexity of the new algorithm is $O(|\Gamma|^3 g)$. The algorithm is based on the following concepts and result.

Définition 2.3 Let $p, q, r_1$ be nodes of graph $\Gamma$ such that $(p, r_1)$ is a $C$-node, $p \succeq q$ and for some node $r$ ($q, r$) is a $C$-node and $p \succeq r \succeq r_1$.

For such nodes $p, q, r_1$ let $T_3SCC(p, q, r_1)$ be the SCC of $\Gamma$ containing the set $T(p, q, r_1) := \{t \mid (p, r_1) \succeq (q, t), q \succeq t \text{ and } (q, t) \text{ is a } C\text{-node}\}$

$T_3SCC$ is not well defined in general case, but an another situation holds for local threshold testability.

Theorem 2.4 (Trahmtan 2003) DFA $A$ with state transition complete graph $\Gamma$ (or completed by sink state) is locally threshold testable iff

1) for every $C$-node $(p, q)$ of $\Gamma_2$ $\sim q$ implies $p = q$.

2) for every four nodes $p, q, t, r_1$ of $\Gamma$ such that

- the node $(p, r_1)$ is a $C$-node,

- $(p, r_1) \succeq (q, t)$,

- there exists a node $r$ such that $p \succeq r \succeq r_1$ and $(r, t)$ is a $C$ - node

holds $q \succeq t$. 
The algorithm for semigroups is based on the following modification of Beauquier and Pin ((1989) result):

**Theorem 2.5** A language $L$ is locally threshold testable if and only if the syntactic semigroup $S$ of $L$ is aperiodic and for any two idempotents $e$, $f$ and elements $a$, $b$ of $S$ we have $eafuebf = ebfa$. The direct use of this theorem gives us only $O(n^3)$ time complexity algorithm, but there exists a way to reduce the time. So the time complexity of the semigroup algorithm is $O(n^3)$ with $O(n^2)$ space complexity (Trahtman 2001).

### 2.4 Right [Left] Local Testability

Let $\Sigma$ be an alphabet and let $\Sigma^+$ denote the free semigroup on $\Sigma$. If $w \in \Sigma^+$, let $|w|$ denote the length of $w$. Let $k$ be a positive integer. Let $i_k(w)$ [$t_k(w)$] denote the prefix [suffix] of $w$ of length $k$ or $w$ if $|w| < k$. Let $F_k(w)$ denote the set of factors of $w$ of length $k$. A language $L$ is called right [left] $k$-testable if for all $u, v \in \Sigma^+$, if $i_{k-1}(u) = i_{k-1}(v), t_{k-1}(u) = t_{k-1}(v)$, $F_k(u) = F_k(v)$ and the order of appearance of these factors in prefixes [suffixes] in the word coincide, then either both $u$ and $v$ are in $L$ or neither is in $L$.

An automaton is right [left] $k$-testable if the automaton accepts a right [left] $k$-testable language.

A language $L$ [an automaton $A$] is right [left] locally testable if it is right [left] $k$-testable for some $k$.

Right [left] local testability was introduced and studied by König (1985) and by Garcia and Ruiz (2000). Algorithms for right local testability, for left local testability for the transition graph and corresponding algorithm for the transition semigroup of an automaton (Trahtman 2002) used in the package TESTAS are based on the results of the paper (Garcia Ruiz 2000). The time complexity of the semigroup algorithm for both left and right local testability is $O(ni)$. The left and right locally testable semigroups are locally idempotent. The package TESTAS checks also local idempotency and the time of corresponding simple algorithm is $O(ni)$ (Trahtman 2002).

The situation in the case of the transition graph is more complicated. The algorithms for right and left local testability for the transition graph are essentially distinct, moreover, the time complexity of algorithms differs. The left local testability algorithm for the transition graph
needs the algorithm for local idempotency. Thus the graphs of automata with locally idempotent transition semigroup (Trahtman 2002) are checked by the package too and the time complexity of the algorithm is $O(|\Gamma|^3 g)$ (Trahtman 2002). The graph algorithm for the left local testability problem needs in the worst case $O(|\Gamma|^5 g)$ time and $O(|\Gamma|^3 g)$ space (Trahtman 2002).

The following two theorems illustrate the difference between necessary and sufficient conditions for right and left local testability in the case of the transition graph.

**Theorem 2.6**  (Trahtman 2002) Let $S$ be transition semigroup of a deterministic finite automaton with state transition graph $\Gamma$.

Then $S$ is left locally testable iff
1. $S$ is locally idempotent,
2. for any $C$-node $(p, q)$ of $\Gamma^2$ such that $p \preceq q$ and for any $s \in S$ we have $ps \succeq q$ iff $qs \succeq q$ and
3. If for arbitrary nodes $p, q, r \in \Gamma$ the node $(p, q, r)$ is $C$-node of $\Gamma^3$, $(p, r) \succeq (q, r)$ and $(p, q) \succeq (r, q)$ in $\Gamma^2$, then $r = q$.

The graph $\Gamma^3$ is used in the theorem. However, the following theorem for right local testability does not use it.

**Theorem 2.7**  (Trahtman 2002) Let $S$ be transition semigroup of deterministic finite automaton with state transition graph $\Gamma$. Then $S$ is right locally testable iff
1. for any $C$-node $(p, q)$ from $\Gamma^2$ such that $p \sim q$ holds $p = q$.
2. for any $C$-node $(p, q) \in \Gamma^2$ and $s \in S$ from $ps \succeq q$ follows $qs \succeq q$.

The time complexity of the graph algorithm for the right local testability problem is $O(|\Gamma|^2 g)$ (Trahtman 2002). This algorithm has $O(|\Gamma|^2 g)$ space complexity. The last algorithm does not call the test of local idempotency used only for the left local testability problem.

### 2.5 Piecewise Testability

Piecewise testable languages introduced by Simon are finite boolean combinations of the languages of the form $A^+a_1A^+a_2A^+...A^+a_kA^+$ where $k \geq 0$, $a_i$ is a letter from the alphabet $A$ and $A^+$ is a free monoid over $A$ (Simon 1975).

An efficient algorithm for piecewise testability implemented in the package is based on the following theorem:

**Theorem 2.8**  (Trahtman 2001) Let $L$ be a regular language over the alphabet $\Sigma$ and let $\Gamma$ be a minimal automaton accepting $L$. The language $L$ is piecewise testable if and only if the following conditions hold

(i) $\Gamma$ is a directed acyclic graph;
(ii) for any node $p$ the maximal connected component $C$ of the graph $\Gamma(\Sigma(p))$ such that $p \in C$ has a unique maximal state.

The time complexity of the algorithm is $O(|\Gamma|^2 g)$. The space complexity of the algorithm is $O(n)$. The algorithm used linear test of acyclicity of the graph. The test can be executed separately. The considered algorithm essentially improves similar algorithm to verify piecewise testability of DFA of order $O(|\Gamma|^3 g)$ described by Stern (1985) and implemented by Caron (1998). All considered algorithms are based on some modifications of results from the paper (Simon 1975).

Not complicated algorithm for the transition semigroup of an automaton verifies piecewise testability of the semigroup in $O(n^2)$ time and space. The algorithm uses the following theorem (recall that $x^\omega$ denotes an idempotent power of the element $x$).
Theorem 2.9 (Simon 1975) Finite semigroup \( S \) is piecewise testable iff \( S \) is aperiodic and for any two elements \( x, y \in S \) holds

\[(xy)\omega x = y(xy)\omega = (xy)\omega\]

REFERENCES


