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Notable trends concerning the synchronization of graphs and automata

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A word w is called synchronizing (recurrent, reset, directed) word of a deterministic finite automaton (DFA) if w sends all states of the automaton on a unique state. Jan Černy had found in 1964 a sequence of n-state complete DFA with shortest synchronizing word of length $(n-1)^2$. He had conjectured that it is an upper bound for the length of the shortest synchronizing word for any n-state complete DFA. This simply looking conjecture is now one of the most longstanding open problems in the theory of finite automata.
Moreover, the examples of DFA with shortest synchronizing word of length $(n-1)^2$ are relatively rare. To the Černy sequence were added in all examples of Černy, Piricka and Rosenauerova (1971), of Kari (2001) and of Roman (2004).
An effective program for search of automata with minimal reset word of relatively great length has studied all automata of size $n \leq 10$ for $q = 2$ (q - size of alphabet) and of size $n \leq 7$ for $q \leq 4$. There are no contradictory examples for the Černy conjecture in this class of automata. Moreover, the program does not find new examples of DFA with reset word of length $(n-1)^2$ for automata with $n > 4$ as well as for $q > 3$. New examples of DFA with shortest synchronizing word of length $(n-1)^2$ were found only for $n = 3, 4$ and for $q = 2, 3$.
And what is more, the examples with minimal length of reset word disappear even for values near the Černy bound $(n-1)^2$ with growth of the size of the automaton as well as of the size of the alphabet. The gap between $(n-1)^2$ and the nearest of the minimal lengths of reset word appears for $n = 6$.
The following table displays this interesting trend for the length of minimal reset words less than $(n-1)^2$.

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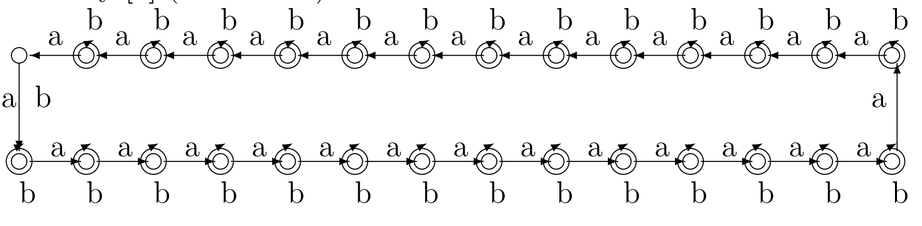
size	n=5 $q < 5$	n=6 $q < 5$	n=7 $q < 5$	n=8 q=2	n=9 q=2	n=10 q=2
$(n-1)^2$	16	25	36	49	64	81
maxlength	15	23	32	44	58	74

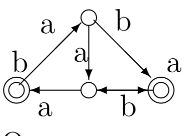
This growing gap supports the following funny

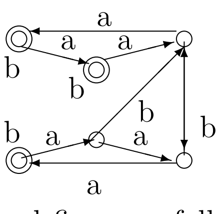
Conjecture 1 *The set of n-state DFA ($n > 2$) with minimal reset word of length $(n-1)^2$ contains only the sequence of Černy and the eight automata mentioned above, three of size 3, three of size 4, one of size 5 and one of size 6.*

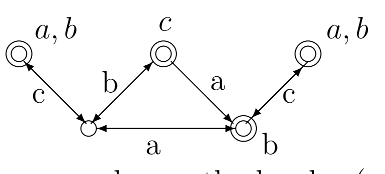
The testing of synchronizing automata is an indispensable part of investigation in this area. The best known to date algorithm of Eppstein [3], finds a synchronizing word for n-state DFA in $O(n^3 + n^2q)$ time. Our algorithm has $O(n^2q)$ time complexity for clear majority of automata and plays the central role in the program for search of DFA with minimal reset word. Algorithm is implemented in the package TESTAS
http://www.cs.biu.ac.il/∼trakht/Testas.html
The results altogether correspond to the Černy conjecture. The known n-state automata with minimal reset word of length $(n-1)^2$ are presented by sequence of Černy [1] (here n=28):



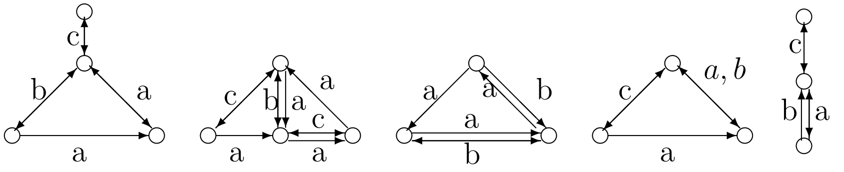
by automata supposed by Černy, Piricka and Rosenauerova [2], by Kari [4] and Roman [5].







Our program has found five new following examples on the border $(n-1)^2$. The loops of the complete graphs are omitted here for simplicity.



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