FRAMINGS AND PROJECTIVE FRAMINGS FOR 3-MANIFOLDS

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ABSTRACT. We give an elementary and easily visualizable proof of the fact that any orientable 3-manifold admits a framing (i.e. is parallelizable) and any non-orientable 3-manifold admits a projective framing. The proof uses only basic facts about immersions of surfaces in 3-space.

1. INTRODUCTION

A framing for a smooth n-manifold M is a smooth choice of ordered basis $(v_1, \dots, v_n)$ for the tangent space at each point of M. A *projective* framing is a smooth choice of a *pair* of ordered bases of the form $\{(v_1, \dots, v_n), (-v_1, \dots, -v_n)\}$. It has long been established that any orientable 3-manifold admits a framing ([S]) and any non-orientable 3-manifold admits a projective framing ([HH]). The original proofs rely on the notion of characteristic classes. We present a proof for compact M, which will only use basic facts about immersions of surfaces in 3-space.

We now present the facts on immersions that we will need. Denote $\mathcal{H} = (\frac{1}{2}\mathbb{Z})/(2\mathbb{Z})$, which is a cyclic group of order 4. Let U denote an annulus or Mobius band. There are two regular homotopy classes of immersions of U into $\mathbb{R}^3$. To each such class we attach a value in $\mathcal{H}$ as follows: For U an annulus, the regular homotopy class of immersions which includes an embedding whose image is $S^1 \times [0,1] \subseteq \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$, will have value $0 \in \mathcal{H}$. The other class, (containing an embedding differing from the previous embedding by one full twist) will have value $1 \in \mathcal{H}$. For U a Mobius band we attach once and for all the value $\frac{1}{2} \in \mathcal{H}$ to one of the classes and the value $-\frac{1}{2} \in \mathcal{H}$ to the other (which again differ by one full twist). Now let F be a closed surface and $i : F \to \mathbb{R}^3$ an immersion. We define a map $g^i : H_1(F, \mathbb{Z}/2) \to \mathcal{H}$ as follows: Given $x \in H_1(F, \mathbb{Z}/2)$ let $c \subseteq F$ be an embedded circle which represents x. Let U be a thin neighborhood of c in F, then U is an annulus or Mobius band. We define $g^i(x)$ to be the value in $\mathcal{H}$ attached above to the immersion $i|_U$. It has been shown in [P] that g^i is

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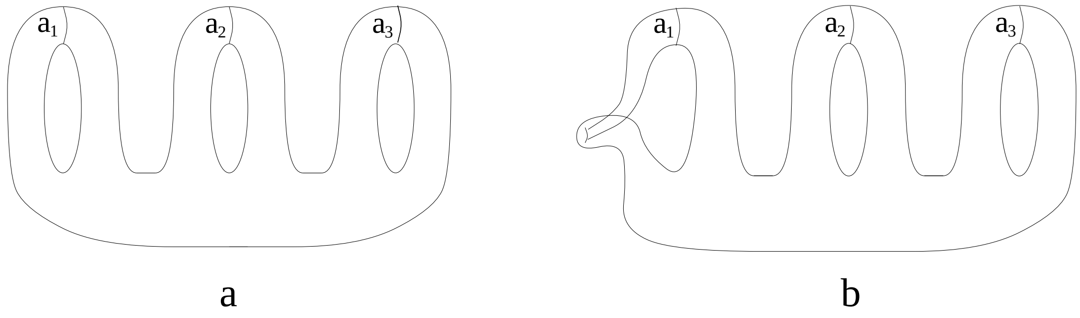

FIGURE 1

indeed well defined on $H_1(F, \mathbb{Z}/2)$ and satisfies the following property:

$$g^i(x+y) = g^i(x) + g^i(y) + x \cdot y \tag{1}$$

where $x \cdot y$ denotes the intersection form on $H_1(F, \mathbb{Z}/2)$. Note $x \cdot y \in \mathbb{Z}/2 \subseteq \mathcal{H}$.

We remark that the notation in [P] differs from ours in that $\mathcal{H}$ is taken there to be $\mathbb{Z}/4\mathbb{Z}$ rather than $(\frac{1}{2}\mathbb{Z})/2\mathbb{Z}$. And so the numerical value of g^i appearing there is twice the value here, and our property (1) appearing above is replaced there by $g^i(x+y) = g^i(x) + g^i(y) + 2(x \cdot y)$.

We point out that the above facts about immersions are indeed basic, in the sense that they may be obtained without knowledge of the Smale-Hirsch Theorem. E.g. for the classification of regular homotopy classes of immersions of U into $\mathbb{R}^3$ where U is an annulus or Mobius band, one needs to show that the easily constructed map into $\pi_1(SO_3)$ is bijective. In general one would use the Smale-Hirsch Theorem, but in this case surjectivity follows by direct construction of the two immersions, and injectivity follows by familiarity with the "belt trick".

2. Proof of Theorem

We will prove the statement for M a closed 3-manifold. The statement for M with boundary will follow by restricting a framing from the double of M. So let M be a closed 3-manifold, and let $F \subseteq M$ be a Heegard surface, i.e. F splits M into two handlebodies A, B (orientable or non-orientable) with common boundary F. Let $a_1, \dots, a_n \subseteq F$ be a system of disjoint circles which may be compressed in A, reducing A to a ball, and let $b_1, \dots, b_n \subseteq F$ be such system compressible in B. Then a thin neighborhood in F of each of the a_ks and b_ks is an annulus. We fix some immersion $i : F \to \mathbb{R}^3$ satisfying $g^i(a_k) = 0$ for all $1 \leq k \leq n$, e.g. we may take the immersion depicted in Figures 1a and 1b for the case where F is orientable or non-orientable, respectively. (We allow a_k to denote both the circle in F and its homology class in $H_1(F,\mathbb{Z}/2)$.)

Lemma 2.1. *There exists an $h : F \to F$ which is a composition of Dehn twists along some of the a_ks, such that $g^{i\circ h}(b_k) = 0$ for all $1 \le k \le n$.*

Proof. For any diffeomorphism $h : F \to F$, define $\phi_h : H_1(F,\mathbb{Z}/2) \to \mathbb{Z}/2$ by $\phi_h(x) = g^i(h_*(x)) - g^i(x)$, where $h_* : H_1(F,\mathbb{Z}/2) \to H_1(F,\mathbb{Z}/2)$ is the induced homomorphism. Note that indeed $\phi_h(x) \in \mathbb{Z}/2$ (since if c is an embedded circle in F then c and $h(c)$ either both have annulus neighborhood or both have Mobius band neighborhood), and by (1), ϕ_h is a linear functional. Let $h_k : F \to F$ denote the Dehn twist along a_k, then h_{k*} is given by $h_{k*}(x) = x + (x \cdot a_k)a_k$. We get (by (1)) for all $x \in H_1(F,\mathbb{Z}/2)$:

$$\phi_{h_k}(x) = g^i(x + (x\cdot a_k)a_k) - g^i(x) = g^i((x\cdot a_k)a_k) + (x\cdot a_k)^2 = x\cdot a_k. \tag{2}$$

(Indeed it is visually clear in Figure 1, that if c is an embedded circle in F and U a neighborhood of c, then a Dehn twist along a_k will add one twist to $i|_U$ for each intersection of c with a_k.)

Let $V_A, V_B \subseteq H_1(F,\mathbb{Z}/2)$ be the subspaces spanned by the a_ks and b_ks respectively, then the intersection form vanishes on V_A and V_B. It follows by (1) that $g^i|_{V_A}$ is identically 0 and $g^i|_{V_B}$ is a $\mathbb{Z}/2$ valued linear functional. By (2) each ϕ_{h_k} vanishes on V_A, and since the intersection form on $H_1(F,\mathbb{Z}/2)$ is non-degenerate, $\phi_{h_1},\dots,\phi_{h_n}$ span all linear functionals on $H_1(F,\mathbb{Z}/2)$ which vanish on V_A. In particular, they span a linear functional ψ vanishing on V_A and satisfying $\psi|_{V_B} = g^i|_{V_B}$. Now if $\psi = \phi_{h_{j_1}} + \phi_{h_{j_2}} + \cdots + \phi_{h_{j_r}}$ $(1 \le j_1 < j_2 < \cdots < j_r \le n)$, then let $h = h_{j_1}\circ h_{j_2}\circ\cdots\circ h_{j_r}$. We have $\phi_h = \psi$ (both are given by $x \mapsto x\cdot(a_{j_1}+a_{j_2}+\cdots+a_{j_r})$), and so for all $1 \le k \le n$: $g^{i\circ h}(b_k) = g^i(h_*(b_k)) = g^i(b_k) + \phi_h(b_k) = 2g^i(b_k) = 0$. □

We replace our chosen immersion i with $i\circ h$, where $h : F \to F$ is the map given by Lemma 2.1, and name it i again. So now we have $g^i(a_k) = 0$ and $g^i(b_k) = 0$, for all $1 \le k \le n$.

We will first construct a (projective) framing on $TM|_F$. Let $TF_p \subseteq TM_p$ denote the tangent spaces of F and M at $p \in F$. Let $n_p \in TM_p$ $(p \in F)$ be a smooth choice of nonzero vector which is pointing into B. If F is orientable then there exists a smooth choice of unit normal u_p for $i(F) \subseteq \mathbb{R}^3$. And so the differential of i together with the choice of n_p and u_p determine an isomorphism from TM_p to $T\mathbb{R}^3{}_{i(p)}$, with which we pull back the standard framing of $T\mathbb{R}^3{}_{i(p)}$ to TM_p. If F is non-orientable then a normal to $i(F)$ may not be continuously chosen. Let u_p be one of the two unit normals at $i(p)$. We define the isomorphism from TM_p to $T\mathbb{R}^3{}_{i(p)}$ as the composition of the isomorphism which uses u_p as before, with a $\frac{\pi}{2}$ rotation around u_p in the positive sense defined by the direction of u_p. When choosing the opposite u_p, then this isomorphism will differ by the reflection defined by u_p, composed with a π rotation around u_p, that is, they will differ by $-\mathrm{Id}$. So such pair of pullbacks define a projective framing on $TM|_F$.

The next step is to extend the framing to a thin neighborhood of the compressing discs of the a_ks and b_ks in A, B. Let d be one of the circles a_k or b_k. Let E be a thin neighborhood of the compressing disc for d, then $E = D^2 \times [0,1]$, and $U = \partial D^2 \times [0,1] \subseteq E$ is a thin neighborhood of d in F. For orientable F we now look at the restriction to U of the framing we have chosen on F. If F is non-orientable, then since U is orientable, there exists a continuous choice of unit normal u_p for $i(U)$, which selects a proper (i.e. non-projective) framing from the projective framing that has been restricted from F. We will now show that such proper framing on U may be extended to E. This follows from the following four simple facts: *Fact 1:* If two framings on U are homotopic, and one of them is extendible to E, then the other is extendible as well. *Fact 2:* The framing constructed in the non-orientable case by pulling back via an isomorphism which included a $\frac{\pi}{2}$ rotation, is homotopic to that pulled back without the $\frac{\pi}{2}$ rotation. *Fact 3:* Since $g^i(d) = 0$, $i|_U$ is regularly homotopic to an embedding which can be extended to an embedding $e : E \to \mathbb{R}^3$. If the choice of u_p is carried continuously along the regular homotopy, then the regular homotopy induces a homotopy of the pulled back framings. Furthermore, we may choose the regular homotopy so that the final u_p will point inward to $e(E)$, and we may similarly choose the regular homotopy so that the final u_p will point outward. *Fact 4:* For embedding $e : E \to \mathbb{R}^3$, let n_p be a smooth choice on U of vector pointing inward to E, and u_p the unit normal to $e(U)$ pointing inward to $e(E)$. Then the framing on U pulled back using n_p, u_p, is homotopic to the restriction of the framing on the whole E which is the pullback via e of the standard framing of $\mathbb{R}^3$. The same is true if both n_p and u_p are chosen to point outward. (Note that it was necessary to have both the inward and outward cases, since our n_p points outward for the a_ks and inward for the b_ks.)

It remains to extend the framing to the remaining 3-balls in A, B. Since a ball D is orientable, we may select a proper framing from the perhaps projective framing we already have on ∂D. This may always be extended to a framing on D, e.g. since $\pi_2(SO_3) = 0$.

References

[HH] J. Hass, J. Hughes: "Immersions of Surfaces in 3-Manifolds." *Topology* 24 (1985) No.1, 97–112.

[P] U. Pinkall: "Regular Homotopy Classes of Immersed Surfaces." *Topology* 24 (1985) No.4, 421–434.

[S] E. Stiefel: "Richtungsfelder und Fernparallelismus in Mannigfaltigkeiten" *Comm. Math. Helv.* 8 (1936) 3–51.

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