Efficient covering of convex domains by congruent discs

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Abstract. In this paper, we consider the problem of covering a plane region with unit discs. We present improved upper and lower bounds on the number of discs needed for such a covering, depending on the area and perimeter of the region. We provide algorithms for efficient covering of convex polygonal regions using unit discs. We show that the computational complexity of the algorithms is polynomial in the size of the input and the output. We also show that these algorithms provide a constant factor approximation of the optimal covering of the region, and that the approximation becomes asymptotically optimal for fat convex regions.

1 Introduction

In discrete geometry, an efficient covering with unit circles for a given domain is a well known problem, with various applications such as facility location and cellular network design. The question of the optimal covering of a region is also a fundamental question in discrete geometry related to many deep questions on the nature of Euclidean space. In this manuscript, we introduce an algorithm which determines the locations of unit discs that induce an efficient covering of a given polygonal domain. Our approach is based on the properties of the hexagonal regular lattice (the honeycomb), which is the optimal lattice among all lattices in the plane in the density of the covering for a given radius of the enclosing circle (see [12]). The presented algorithms are based on placing the centers of the discs at lattice points, where the location and orientation of the lattice relative to the covered region is optimally selected by the algorithms. We describe three algorithms, from the simplest one, achieving good results for convex polygons with low computational complexity, to a more complex one, which guarantees obtaining the optimal lattice-based covering for any convex polygon. It should be noted that the presented algorithms are presented for obtaining a covering based on the hexagonal lattice, since it is asymptotically optimal for a fat region, as will be proven below. However, they can be adapted to every given lattice, which may be desirable in some cases. We show that the presented algorithms have polynomial complexity in the combined size of input and output.

Blaschke, Tóth and Hardwiger [9] showed that a convex domain with area A and perimeter L can be covered by at most

$$\left\lfloor \frac{2}{3\sqrt{3}}A + \frac{2}{\pi\sqrt{3}}L + 1 \right\rfloor \approx \lfloor 0.384 \cdot A + 0.367 \cdot L + 1 \rfloor \tag{1}$$

unit discs. Their result, using the probabilistic method, is based on estimating the expected number of hexagons in a hexagonal lattice that intersect a domain placed in a random orientation and location (see section 6.1). This result is existential and not constructive, and does not provide the desired locations of the discs that give such a covering. The algorithms presented in this paper produce the list of discs location that guarantees a covering which achieves this bound. In fact, the second algorithm briefly described guarantees the optimal covering among all coverings based on the hexagonal lattice. We also provide an improved formula for this upper bound on the number of needed discs, which improves Eq. (1).

We also provide a bound on the approximation ratio between the number of discs in the optimal covering and the number of discs required by the presented algorithms for any region. Taking an asymptotic approach, and defining fat regions as a sequence of regions for which $L = o(A)$, we obtain an optimal approximation ratio. That is, the ratio between the number of discs required by the algorithm to the minimum number of discs required for any covering approaches 1 when the covered region becomes large.

2 Related work

[3, 4] described a mechanism for the special case which locates n discs with given radius r to cover a maximum fraction of the area of a unit disc. The goal of the facility location problem is to locate a minimal number of facilities such that a set of points (or possibly an entire domain) is covered. The first studies of this subject focused on methods of Integral Geometry [9, 6, 12, 13]. With the advances in computer science, new algorithms and approaches have been developed for the facility location problem, and to the closely related k-centers and P-centers problems.

Megiddo and Supowit showed that the P-centers is NP-hard [8]. In [10], a heuristic upper bound to the optimal solution was described in a square. Hwang and Lee [7] showed that time complexity of the most efficient algorithm is $O(n^{O(\sqrt{P})})$. In [11], a lower bound for the facility location problem was obtained and an algorithm achieving a constant approximation ratio was presented. In [5], a restricted version of the facility location problem was studied and a constant factor approximation was presented. In [2], a learning mechanism was proposed to solve the P-centre problem for a continuous area.

In [1], an algorithm for approximating the non-uniform minimum-cost multi-cover problem was described by studying the example of matching clients to servers.

3 Preliminaries

Definition 1. *The Minkowski sum of any two sets $A, B \subset \mathbb{R}^2$ is defined to be $A \boxplus B := \{x + y : x \in A, y \in B\}$. For $s \geq 0$, the Minkowski dilation by factor s is defined to be $sA = \{sx : x \in A\}$.*

Definition 2. *For every $\theta \in [0, 2\pi)$, the support function of a domain Ω, $h(\theta)$, is a function that maps every θ to the largest p such that $L(p, \theta) \cap \Omega \neq \emptyset$, where $L(p, \theta)$ is the line $\{(x, y) | \cos(\theta)x + \sin(\theta)y = p\}$. The width is $w(\theta) = h(\theta) + h(\theta + \pi)$.*

Denote the diameter of a given domain Ω by D. We will define $\Delta := [-D - 3, D + 3] \times [-D - 3, D + 3] \subset \mathbb{R}^2$, i.e., there exists a rigid motion such that $\Omega \subseteq \Delta$.

Definition 3. *Denote the n lattice points in Δ by $\overline{x}_{mn} := m \cdot (\sqrt{3}, 0) + n \cdot (\frac{\sqrt{3}}{2}, \frac{3}{2})$ $m, n \in \mathbb{Z}$, which are contained in the hexagonal regular lattice Λ_h. If $m = n = 0$, the respective hexagon defined by $\hexagon_0$.*

We state some standard theorems from integral geometry. Proofs can be found in standard textbooks. e.g. [9].

Theorem 1 (Cauchy's Formula). *Let Ω be a bounded convex domain*

$$L(\partial\Omega) = \int_0^{2\pi} h(\theta)d\theta = \int_0^{\pi} w(\theta)d\theta \ ,$$

where h is the support function of Ω and w is the width.

Theorem 2. *Suppose Ω is a compact, convex domain with a $C^2(\mathbb{R})$ boundary (C^2 in the sense that the derivative f' and f'' exist and continuous). Then*

$$A(\Omega) = \frac{1}{2}\int_0^{2\pi} (h^2 - h'^2)d\theta \ . \tag{2}$$

Definition 4. *Let $T_{ab} := \{(x, y) \mid (R(\pi) \cdot \Omega) \boxplus \hexagon_{ab}\}$ where $R(\pi)$ denotes the rotation of Ω by an angle of π, and (a, b) is the center of the unit hexagon)).*

4 Lower bounds for covering

In this section we will give a lower bound based on the properties of multi-graphs. For a convex region Ω (not necessarily a polygon) with area A and perimeter L we will prove the following

Theorem 3. *Let n_{opt} be the number of unit discs in a minimum covering of Ω, then $n_{\text{opt}} \geq \max\left\{\frac{2A}{3\sqrt{3}}, \frac{L}{4}\right\} - C$, for some constant C.*

We will prove the theorem in two parts, first proving the area term and then the perimeter term.

Let V_i denote the Voronoi cell of the center of the ith disc, and let Ω^c be the complement of Ω. These regions define a graph G where adjacency in the graph reflects shared edges. We will distinguish between two Voronoi cells types: inner cells are the Voronoi cells that do not intersect the boundary of Ω (except possibly at discrete points) and exterior cells are all the cells that do intersect the boundary of Ω. Each of the edges between Voronoi cells is arbitrarily ascribed to only one of the cells, so the cells are disjoint and their union is the entire plane. Assume there are a total of n cells, of which m are exterior.

We will use the following technical result

Lemma 1. *Let $i_1, i_2, i_3, \ldots, i_\ell$ be the sequence of indices of cells visited by touring the boundary counterclockwise starting at an arbitrary point. We have $\ell \leq 2m$.*

Proof. The sequence cannot contain a subsequence of the form $i, \ldots, j, \ldots, i, \ldots, j$ since the line segment between two points on the boundary pertaining to the two visits in the ith cell bisects Ω into two disjoint regions, and thus the line segment between two points on the boundary pertaining to the two visits in the jth cell, who belong to the two different regions, must intersect it. However, by convexity of Voronoi cells, the intersection point must belong to both V_i and V_j, that are disjoint, leading to a contradiction.

Thus, the graph of adjacency of the exterior cells is a cactus graph, which can be constructed by starting with a single vertex i, and using the following basic steps:

1. Adding and edge, i.e., replacing some i in the sequence by i, j, i, where j does not appear anywhere else in the sequence.
2. adding a triangle, i.e., replacing i in the sequence by i, j, k, i, where j and k do not appear anywhere else in the sequence.
3. extending an edge, i.e., replacing i, j in the sequence with i, k, j where k does not appear anywhere else in the sequence.
4. extending a bidirectional edge, i.e., replacing $i, j, \ldots, j, i$ in the sequence with $i, k, j, \ldots, j, k, i$ where k does not appear anywhere else in the sequence.

It can be seen that each of these steps increases m by at least one, and none of the steps increases t by more than two. Thus $t \leq 2m$. □

Lemma 2. *For any covering of Ω by n unit discs, $n \geq \frac{2(A-C)}{3\sqrt{3}}$, where C is a constant.*

Proof. We define the Voronoi polygons U_i to be V_i if it is internal, and if it is external we define it to be the convex polygon (convex hull) formed by the edges between the Voronoi cell V_i and neighboring Voronoi celles and by the points of intersection of V_i with the boundary of Ω. Denote by A_i the area of U_i.

If U_i is Voronoi polygon with d_i edges, then its area is maximized if it is a regular polygon inscribed in the unit disc, thus $A_i \leq \frac{d_i}{2} \sin \frac{2\pi}{d_i}$. By the convexity of the function $x \sin(2\pi/x)$ and using Jensen's inequality we get

$$\sum_{i=1}^{n} A_i \leq \sum_{i=1}^{n} \frac{d_i}{2} \sin \frac{2\pi}{d_i} \leq n\frac{\overline{d}}{2} \cdot \sin \frac{2\pi}{\overline{d}} .$$

Consider the graph of adjacency between the V_is and Ω^c. This is a planar graph with $n+1$ vertices, and thus the sum of degrees is at most $6(n+1) - 12 = 6n - 6$. m of these edges are incident with Ω^c. Thus the sum of degrees of the Voronoi cells is at most $6n - 6 - m$. However, each of the exterior cells may have more than one edge shared with Ω^c. The excess degrees obtained by this is at most $t - n \leq n$ (by Lemma 1), which leads to $\sum_{i=1}^{m} d_i \leq 6m - 6 - n + n = 6m - 6$, or $\overline{d} < 6$. Thus,

$$\sum_{i=1}^{n} A_i < n\frac{6}{2} \cdot \sin \frac{2\pi}{6} = \frac{3\sqrt{3}}{2} .$$

Now let $C = A - \sum_{i=1}^{n} A_i$ be the area of Ω not included in any of the Voronoi polygons. Let e_i, $i = 1, \ldots, t$ be the exterior edges of the exterior Voronoi polygons. Let l_i be the lengths of the boundary of Ω that are external to their respective e_i. by the isoperimetric inequality we have that the maximum area between each l_i and its e_i is obtained when l_i is a circular arc and the area is a circular segment. The area of the segment is given by $C_i = (l_i r_i - r_i^2 \sin(l_i/r_i))/2$, where r_i is the radius of curvature. Since the boundary of Ω is a Jordan curve we have that the total curvature is $\sum_{i=1}^{t} l_i/r_i = 2\pi$. We also have that since each l_i resides in a unit disc, its length is at most $l_i \leq 2\pi$. Thus, the total area outside the Voronoi polygons is thus

$$C \leq \sum_{i=1}^{t} \frac{l_i r_i - r_i^2 \sin(l_i/r_i)}{2} \leq \sum_{i=1}^{t} \frac{l_i^3/r_i}{12} \leq (2\pi)^2 \sum_{i=1}^{t} \frac{l_i/r_i}{12} = \frac{2\pi^3}{3} ,$$

using the fact that $\sin x \geq x - x^3/6$ for $0 \leq x \leq \pi$. Therefore, C is bounded by a constant. □

For the perimeter term we have the following:

Lemma 3. *For any covering of Ω by n unit discs, $n \geq \frac{L}{4} - C$.*

Proof. Consider the boundary of $U := \cup_i U_i$. U is a polygon, which has at most $t \leq 2m \leq 2n$ edges by Lemma 1. For each disc, i, having D_i exterior edges we get that the longest total length of the external edges is obtained when the external edges form a regular polygon inscribed in the disc. This polygon has perimeter 2 for $D_i = 1$ and perimeter $2D_i \sin(\pi/D_i)$ for $D_i \geq 2$. Assume k cells have $D_i = 1$ and the rest of the exterior cellas have $D_i \geq 2$. The total length of the external edges is thus at most

$$2k+\sum_{i=1}^{m} 2D_i \sin\frac{\pi}{D_i} \leq 2k+2(m-k)\overline{D}\sin\frac{\pi}{\overline{D}} \leq 2k+2(m-k)\frac{2m-k}{n-k}\sin\frac{\pi(m-k)}{2m-k} \leq 4m \leq 4n \;,$$

where $\overline{D}$ is the average degree of the cells having at least two exterior edges, and we have used the Jensen inequality and Lemma 1.

Now each point on the boundary of Ω has a point at a distance at most 2 on inside U, since they fit in the same disc. Thus, the support function $h_\Omega(\theta)$ of Ω relative to some point inside U satisfies $h_\Omega(\theta) \leq h_U(\theta) + 2$. By Theorem 1 $L(\partial\Omega) \leq L(\partial U) + 4\pi$. □

It should be noted that both the area and perimeter terms are asymptotically sharp, as can be seen in the case of covering a large fat region by discs arranged in a hexagonal lattice configuration (see Section 6.3), and by covering a long narrow (width zero) rectangle by a line of kissing discs, respectively. However, it may be possible to improve Theorem 3 by using some combination of the area and perimeter terms which is not the maximum.

5 Algorithms for covering

We would like to obtain a covering by unit circles. Since every regular hexagon is bounded by a unit circle, the covering by a set of unit circles can be represent by the hexagonal regular lattice which has been proven by Tóth in the optimal lattice in the plane. We would like to find a special point for locating the domain Ω leading to a minimal number of regions in the hexagonal regular lattice which intersects the domain Ω. This number of intersections leads to the number of lattice points (in the hexagonal regular lattice) which cover the convex domain. Obviously, the domain Ω can be placed on the lattice in different positions, every position leads to a different number of intersections. We will show that due to the group of rigid motion, we can shift and rotate the given domain Ω such that the number of intersections can be minimized, i.e., find the appropriate location.

Its turn out (see Theorem (4)) that the Minkowski sum is a tool for defining a covering for a given lattice which leads to the number of hits of a given domain and the number of fundamental regions in the given lattice.

5.1 2D Algorithm for a given θ

We will take in consideration that these two variables are independent. Let the optimal orientation θ be given. We will describe below an algorithm which determines the exact position for a given θ. For each of the lattice points we will find the respective Minkowski sum

$$T^{\Omega}_{\overline{x}_{mn}}(\theta) := (R(\pi+\theta)\cdot\Omega) \boxplus \hexagon_{\overline{x}_{mn}},$$

where θ is the optimal orientation of Ω.

Notice that if $\overline{x}_{mn} \notin \Delta$, then $\hexagon_0 \cap T_{\overline{x}_{mn}}(\theta) = \emptyset$. Therefore, we can concentrate only on the points $\overline{x}_{mn} \in \Delta$.

Since the lattice is periodic, we can limit our discussion to the hexagon $\hexagon_0$ (it is sufficient to do the shifting in $(R(\pi+\theta)\cdot\Omega) \boxplus \hexagon_{(0+\varepsilon)}$, where $\varepsilon \in \hexagon_0$).

Algorithm 1: The 2D Algorithm

Result: List of centers of unit discs covering Ω.

Given a convex polygon Ω
  if *Diameter of minimum enclosing circle of* $\Omega \leq 1$ **then**
    return center of minimum enclosing circle
end
Calculate the width function w_0
  Find the diameter of the given domain
  Calculate the Minkowski sum of $(R(\pi+\theta)\cdot\Omega) \boxplus \hexagon_0$
Find all the lattice points $\overline{x}_{mn}$ which are contained in Δ
Find the optimal angle θ by minimizing:

$$\theta_0 = \arg\min_{\theta}\left\{ w_0(\theta) + w_0\left(\theta+\frac{\pi}{3}\right) + w_0\left(\theta+\frac{2\pi}{3}\right) \quad (0<\theta\leq\pi)\right\}$$

  Place Ω in orientation with θ_0
$points = \emptyset$
for $\overline{x}_{mn} \in \Delta$ **do**
    Find $T^{\Omega}_{\overline{x}_{mn}}(\theta)$
end
for $i = 1{:}|\overline{x}_{mn}|$ **do**
    for $j = i+1{:}|\overline{x}_{mn}|$ **do**
        for each intersection point $y \in \partial T_i \cap \partial T_j \cap \hexagon_0$ **do**
            $points = [points, y_i]$
        end
    end
end
for *j=1:*$|points|$ **do**
    for *i=1:*$|\overline{x}_{mn}|$ **do**
        Index(j)=0
        if $points_i \in IntT^{\Omega}_i(\theta)$ **then**
            $Index_j = Index_j + 1$
            $\min(Index_j)$
        end
    end
end
$point^* = points(\arg\_\min\{Index(j)\})$
return list of all lattice points contained in the Minkowski sum

5.2 3D Algorithm

The algorithm we presented in Section (5.1) finds the optimal orientation for a random location and finds the optimal location for this orientation. However, this does not guarantee that the combination of orientation and location is optimal. We now present an algorithm that determines this optimal combination.

We define the Minkowski sum as follows:

$$T_{ab}(\theta) := \left\{(x,y,\theta) \mid (x,y) \in \left(R(\pi+\theta)\cdot \Omega \boxplus \hexagon_{ab}\right), \theta \in \left[0, \frac{\pi}{3}\right]\right\} \ , \tag{3}$$

where $R(\pi+\theta)$ is a rotation.

We define the Minkowski sum T for each of the n lattice points in the ball B_{D+3} (where $D+3$ is the radius of the ball), which is the sequence

$$\{T^{\Omega}_{\overline{x}_{mn}}(\theta) \ |\overline{x}_{mn} \in B_{D+3} \cap \Lambda_h\} \ .$$

In order to find the optimal placement and orientation of the convex region Ω, we suggest the following algorithm:
Each point (x,y,θ) represents a placement of Ω at the location (x,y) and orientation θ. Notice that a point (x,y,θ) is in the interior of a domain $T^{\Omega}_{\overline{x}_{mn}}(\theta)$ if and only if placing Ω at a location (x,y) and orientation θ intersect with the hexagon that is centered at $\overline{x}_{mn}$. In this case every Minkowski sum, respective to a a lattice point $\overline{x}_{mn}$, generates a 3D body by rotating the convex domain continuously $\frac{\pi}{3}$ radians and adding the hexagon. We will examine the domains inside $P_0 := \hexagon_0 \times [0, \frac{\pi}{3}]$ (hexagonal prism).

The hexagonal prism will be divided into different regions each of which is the intersection of a different number of $T^{\Omega}_{\overline{x}_{mn}}(\theta)$'s for lattice points $\overline{x}_{mn} \in B_{(D+3)}$. As for the translation optimizing version, it can be shown that by finding the intersections of the surfaces formed by the boundaries of the Minkowski sums, the optimal placement and angle can now be determined. The intersections can be found by solving systems of polynomial equations. An extension can also be constructed for non-convex polygons by representing them as unions of convex polygons. For brevity we skip the details.

6 Correctness and performance bounds

6.1 Upper bounds using integral geometry

The following two theorems are of classical results presented before by Blaschke. They are brought here for completeness.

Theorem 4. *Given domains Ω_0 and Ω_1, where Ω_1 is of the form $\Omega_1 = \{(\alpha,\beta,0)\} + \Omega'$, where (α,β) is chosen randomly by the uniform distribution in a ball with radius r, then $P(\Omega_0 \cap \Omega_1 \neq \emptyset) = \frac{\mathrm{Area}(\Omega_0 + R(\pi)\cdot\Omega_1)}{\pi r^2}$.*

Proof. (Based on [9]) Denote by b_0 and b_1 the centers of Ω_0 and Ω_1, respectively. The vector z_0 is the vector which determines Ω_0. In a similar way, we will define for Ω_1. Given $\Omega_0 \cap \Omega_1 \neq \emptyset$, there exists a point such that $b_0 + z_0 = b_1 + z_1$, i.e., $z_1 = z_0 + b_0 + (-b_1)$). Without loss of generality, take $b_0 = 0$. If $b_1 \in \Omega_1$ then $-b_1 \in R(\pi)\cdot\Omega_1$, so $z_1 \in K_0 \boxplus R(\pi)\cdot\Omega_1$, which is the area intersection between Ω_0 and Ω_1 in $\mathbb{R}^2$. So the probability in $\mathbb{R}^2$ for the desired intersection is

$$\frac{\text{Area}(\Omega_0 + R(\pi)\cdot\Omega_1)}{\pi r^2}\ ,$$

where $r \gg 1$. $\square$

Theorem 5. *Consider A_1 as the area of a finite convex region Ω_1 whose perimeter is L_1, then the region can be covered using*

$$\lfloor \frac{2}{3\sqrt{3}}A_1 + \frac{2}{\pi\sqrt{3}}L_1 + 1 \rfloor$$

unit circles, where the square brackets indicate the floor function.

Proof. As a consequence of the above theorem, the mean value of the number of pieces in which a domain Ω_1, limited by a single curve of length L_1, is divided when it is put at random on a lattice whose fundamental domains have area A_0 and contour of length L_0, is

$$\overline{\nu} = \frac{2\pi(A_0 + A_1) + L_0 L_1}{2\pi A_0}\ . \tag{4}$$

The number N of fundamental domains which have a common point with Ω_1 is always $N \leq \nu$. Consequently, $\overline{N} \leq \nu$ and we get: Any domain Ω_1 of area A_1, limited by a single curve of length L_1, can be covered by a number μ of fundamental domains of area A_0 and contour L_0 which satisfies the inequality $\mu \leq \overline{\nu}$, where $\overline{\nu}$ has the value (4).

If the lattice is that of regular hexagons (which is the optimal lattice) of side a, we find that every Ω_1 can be covered by a number of hexagons not exceeding

$$\left[1 + \frac{2L_1}{\sqrt{3}a\pi} + \frac{2A_1}{2\sqrt{3}a^2}\right]\ . \tag{5}$$

Considering the circles circumscribed about the regular hexagons of side a, we obtain the result that every Ω_1 can be covered by this number of circles of radius a. Choose $a = 1$ and we have the desired result. $\square$

Definition 5. *The support function of the hexagon will be denoted by $h_{\hexagon}(\varphi)$. The support function of the convex domain Ω will be denoted by $h_0(\theta)$.*

The canonical hexagon is the diamter 2 regular hexagon having the center of gravity at the origin and two vertices on the y axis. Directly calculating the support function of the canonical hexagon, one obtains $h_{\hexagon}(\varphi)\max_{n\in\mathbb{Z}}\cos\left(\varphi - \frac{n\pi}{3}\right)$.

Lemma 4. *If we place Ω in orientation θ such that*

$$w_0(\theta) + w_0\left(\theta + \frac{\pi}{3}\right) + w_0\left(\theta + \frac{2\pi}{3}\right) \quad (0 < \theta \leq \pi)$$

is minimal, where w_0 is the width of Ω, then the Minkowski sum $(R(\pi)\cdot\Omega)\boxplus\hexagon$ is minimized, and the expected number of covering hexagons is

$$\left\lfloor \frac{2}{3\sqrt{3}}A + \frac{2}{3\sqrt{3}}\left(w_0(\theta) + w_0\left(\theta + \frac{\pi}{3}\right) + w_0\left(\theta + \frac{2\pi}{3}\right)\right) + 1 \right\rfloor .$$

Proof. We will define the Minkowski sum of $(R(\pi) \cdot \Omega) \boxplus \hexagon$. The hexagon will be defined by (**??**). The convex domain Ω will be placed such that the hexagon and Ω intersect. We will denote the center of gravity of Ω by r. Take an intersection point of Ω and $\hexagon$. Then

$$r + z_0 = z_1 + 0 \Longleftrightarrow r = z_1 + (-z_0) \ ,$$

where $z_0 \in \Omega$ and $z_1 \in \hexagon$. Denote the respective support function of z_0 by h_1 and in a similar way for z_1 by $h_{\hexagon}$. So, we will define the support function of the Minkowski sum as

$$\hat{h} = h_{\hexagon}(\varphi) + h_0(\varphi + \theta + \pi) \, .$$

By (2) the area of the Minkowski sum is:

$$\begin{aligned} A(\hexagon \boxplus R(\pi+\theta)\cdot\Omega) &= \frac{1}{2}\int_0^{2\pi} \Big((h_{\hexagon}(\varphi) + (h_0(\varphi+\pi+\theta))^2 - \big(h'_{\hexagon}(\varphi) + h'_0(\varphi+\pi+\theta)\big)^2 \Big) \, d\varphi \\ &= A + \frac{\sqrt{27}}{2} + \int_0^{2\pi} \big(h_0(\varphi+\pi+\theta) h_{\hexagon}(\varphi) - h'_0(\varphi+\pi+\theta) h'_{\hexagon}(\varphi) \big) \, d\varphi \quad (6) \end{aligned}$$

Denote the shifting factor in (**??**) for every interval by c. Thus we will get

$$\begin{aligned} &\int_0^{2\pi} (h_0(\varphi+\pi+\theta)\cdot\cos(\varphi+c) + h'_0(\varphi+\pi+\theta)\sin(\varphi+c)) \, d\varphi \\ &\qquad = \textstyle\int_0^{2\pi} (h_0(\varphi+\pi+\theta)\cdot\sin(\varphi+c))' \, d\varphi \ . \end{aligned} \qquad (7)$$

The solution for (7) where $-\frac{\pi}{6} < \varphi \leq \frac{\pi}{6}$ gives the support function $h_{\hexagon} = \cos(\varphi)$, is

$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (h_0(\varphi+\pi+\theta)\cdot\sin(\varphi))' \, d\varphi = \frac{1}{2}\cdot h_0\left(\frac{7\pi}{6}+\theta\right) + \frac{1}{2} h_0\left(\frac{5\pi}{6}+\theta\right) \ .$$

Similar calculations can be conducted for the other five regimes.

Merging all the results gives:

$$\int_0^{2\pi} (h_0(\varphi+\pi+\theta)\cdot\sin(\varphi+c))' \, d\varphi = \sum_{n=0}^{5} h_0\left(\frac{(2n+1)\pi}{6}+\theta\right) \ , \qquad (8)$$

leading to

$$f(\theta) = w_0(\theta) + w_0\left(\theta + \frac{\pi}{3}\right) + w_0\left(\theta + \frac{2\pi}{3}\right) \ . \qquad (9)$$

Since we would like to determine the orientation of Ω such that the Minkowski sum is minimal, it is sufficient to minimize f. Thus the points that should be inspected are the critical points of (9) which are either zeroes or discontinuities of the derivative of f. □

Theorem 6. *For any convex domain, Ω, there exists a covering of Ω with at most*

$$\left\lfloor \frac{2}{3\sqrt{3}} A + \frac{2}{3\sqrt{3}} \left(w_0(\theta) + w_0\left(\theta + \frac{\pi}{3}\right) + w_0\left(\theta + \frac{2\pi}{3}\right) \right) + 1 \right\rfloor$$

unit discs, for any θ.

Proof. There exists a cover with at most the integer part of the expected cover size, where the expected size is given in Lemma 4. □

The definition of f is valid for every $0 < \theta \leq \pi$. Obviously, if the width is constant a minimization to θ is not possible. In this case we will get that $f(\theta) = \frac{3L}{\pi}$. Since the area of the fundamental region in the hexagonal lattice is $\frac{\sqrt{27}}{2}$, the ratio $\frac{A\left((R(\pi)\cdot\Omega)\boxplus\hexagon_0\right)}{\sqrt{27}/2}$ determines the number of hexagonal lattice points which covers the convex domain Ω.

6.2 Correctness of the algorithm

The hexagon will be divided into different regions each of which is the intersection of a different number of elements in the series $(R(\pi+\theta)\cdot\Omega)\boxplus\hexagon_{\bar{x}_{mn}}$. In order to examine all of these regions, we will choose for each region special points on its boundary, which will be defined as follows:

Definition 6. *Denote* $Q(x,y) = \left\{T^{\Omega}_{\bar{x}_{mn}}(\theta) | (x,y) \in \int T^{\Omega}_{\bar{x}_{mn}}\right\}$.

That is, $Q(x,y)$ is the set of all regions including the point (x,y) in their interior.

Definition 7. *Denote* $C(x,y) = \bigcap_{T\in Q(x,y)} T$.

That is, $C(x,y)$ is the intersection of all regions in $Q(x,y)$. Notice that $\{C(x,y)|(x,y) \in \hexagon_0\}$ is a partition of the unit hexagon into equivalence classes of points, which are convex.

Definition 8. *Denote* $N(x,y) = |Q(x,y)|$*, i.e., the number of domains whose interior contains* (x,y)*.*

The index of the intersections boundaries of the Minkowski sums are convex domains and finite, thus the intersection points can be easily calculated.

Theorem 7. *Let* $(x,y) \in \hexagon_0$ *then* $\exists(\tilde{x},\tilde{y}) \in \hexagon_0$ *such that* $N(\tilde{x},\tilde{y}) \leq N(x,y)$ *and* $(\tilde{x},\tilde{y})$ *is either*

(i) one of the intersection points of the pair of $T^{\Omega}_{\bar{x}_{mn}}$*, whose boundaries lie inside the domain* $\hexagon_0$*; or the intersection of a domain* $T^{\Omega}_{\bar{x}_{mn}}$ *with the edges of the hexagon;*
(ii) a corner point of $T^{\Omega}_{\bar{x}_{mn}}$ *(which is contained in* $\hexagon_0$*) or a corner point of* $\hexagon_0$*.*

Proof. Take a point $(x,y) \in \hexagon_0$. Now, let (x',y') be an arbitrary point in $\partial C(x,y)$. If $T \in Q(x',y')$ then $\int T \cap C(x,y) \neq \emptyset$, since $\int T$ is an open set and thus $T \in Q(x,y)$. Thus $Q(x',y') \subseteq Q(x,y)$, and therefore $N(x',y') \leq N(x,y)$.

Now consider $C(x',y')$. Every point in $C(x',y')$ belongs to some edge of some $T^{\Omega}_{\bar{x}_{mn}}$ or of $\hexagon_0$. If the closure $\overline{C}(x',y')$ contains a corner of a polygon or of the unit hexagon, $(\tilde{x},\tilde{y})$, we are done, as every $T^{\Omega}_{\bar{x}_{mn}}$ covering $(\tilde{x},\tilde{y})$ must also cover the interior of $C(x',y')$. Otherwise, since $\overline{C}(x',y')$ is a closed set, consisting of the intersection of boundaries of polygons and the unit hexagon, it must contain at least one point, $(\tilde{x},\tilde{y})$, of the intersection between two $T^{\Omega}_{\bar{x}_{mn}}$ or an intersection of one of the $T^{\Omega}_{\bar{x}_{mn}}$ and the unit hexagon. By the same argument, $N(\tilde{x},\tilde{y}) \leq N(x',y')$. □

Theorem 8. *The 2D Algorithm gives the optimal placement.*

Proof. The algorithm checks all points which are of one of the types mentioned in the statement of Theorem 7. The correctness of the theorem follows immediately from the algorithm. □

So finally we will shift Ω to the point $(\tilde{x}, \tilde{y})$ and get the desired location.

6.3 Performance bounds

Theorem 9. *Let n_{opt} be the minimum number of unit discs necessary to cover a convex domain Ω with area A and circumference L, then the algorithms give an approximation ratio of* $1 + \frac{8}{\pi\sqrt{3}} + o(1)$.

Proof. The algorithms find an optimal (placement or angle and placement, respectively) covering using the hexagonal lattice. Thus, from Theorem 5, it follows that either algorithms give a covering using $n \le \frac{2}{3\sqrt{3}}A + \frac{2}{\pi\sqrt{3}}L + 1$. Therefore, by Theorem 3,

$$\frac{n}{n_{\text{opt}}} \le \frac{2A/\sqrt{27} + 2L/(\pi\sqrt{3}) + 1}{\max\{2A/\sqrt{27}, L/4\} - C} \le 1 + \frac{8}{\pi\sqrt{3}} + o(1) ,$$

and the theorem follows. □

If we consider fat regions, having $L = o(A)$ we have

Conclusion 10. *Let Ω be a fat convex polygon. The algorithm is asymptotically optimal. That is, the covering it produces uses $n = (1 + o(1))n_{\text{opt}}$ discs where n_{opt} is the minimum number of unit discs needed to cover Ω.*

7 Computational Complexity

Theorem 11. *The optimal location and translation of Ω can be found in $O(D^3N^2)$ operations, where D is the diameter of Ω and N is the number of sides of Ω.*

Proof. The number of domains is the number of lattice points in Δ. Due to the properties of the Minkowski sum with the hexagon, the number of edges of each polygon is at most $N + 6$. Thus, the number of intersection points between polygons is at most $O(D^2N)$. For each such intersection point, one needs to examine how many other polygons contain it, requiring $O(ND)$ operations. Thus, the total time complexity is $O(D^3N^2)$. □

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