ARTICLE IN PRESS IPL:106039

Information Processing Letters ••• (••••) ••••••

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Information Processing Letters

www.elsevier.com/locate/ipl

Isomorphism for random k-uniform hypergraphs

Debsoumya Chakraborti[a], Alan Frieze[a,*,1], Simi Haber[b,2], Mihir Hasabnis[a]

[a] Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213, USA
[b] Department of Mathematics, Bar-Ilan University, Israel

ARTICLE INFO

Article history:
Received 14 February 2020
Received in revised form 7 October 2020
Accepted 7 October 2020
Available online xxxx
Communicated by Marek Chrobak

Keywords:
Analysis of algorithms
Random hypergraph
Isomorphism

ABSTRACT

We study the isomorphism problem for random hypergraphs. We show that it is solvable in polynomial time for the binomial random k-uniform hypergraph $H_{n,p;k}$, for a wide range of p. We also show that it is solvable w.h.p. for random r-regular, k-uniform hypergraphs $H_{n,r;k}, r = O(1)$.

1. Introduction

In this note we study the isomorphism problem for two models of random k-uniform hypergraphs, $k \geq 3$. A hypergraph is k-uniform if all of its edges are of size k. The graph isomorphism problem for random graphs is well understood and in this note we extend some of the ideas to hypergraphs.

The first paper to study graph isomorphism in this context was that of Babai, Erdős and Selkow [8]. They considered the model $G_{n,p}$ where p is a constant independent of n. They showed that w.h.p.[3] a *canonical labelling* of $G = G_{n,p}$ can be constructed in $O(n^2)$ time. In a canonical labelling we assign a unique label to each vertex of a graph such that labels are invariant under isomorphism. It follows that two graphs with the same vertex set are isomorphic, if and only if the labels coincide. (This includes the case where one graph has a unique labelling and the other does not. In which case the two graphs are not isomorphic.) The failure probability for their algorithm was bounded by $O(n^{-1/7})$. Karp [5], Lipton [7] and Babai and Kucera [3] reduced the failure probability to $O(c^n), c < 1$. These papers consider p to be constant and the paper of Czajka and Pandurangan [9] allows $p = p(n) = o(1)$. We use the following result from [9]: the notation $A_n \gg B_n$ means that $A_n/B_n \to \infty$ as $n \to \infty$.

Theorem 1. *Suppose that $p \gg \frac{\log^4 n}{n \log\log n}$ and $p \leq \frac{1}{2}$. Then there is a polynomial time algorithm that finds a canonical labelling q.s.*[4] *for $G_{n,p}$. In fact the running time of the algorithm is $O(n^2 p)$ q.s.*

Our first result concerns the random hypergraph $H_{n,p;k}$, the random k-uniform hypergraph on vertex set $[n]$ in which each of the possible edges in $\binom{[n]}{k}$ occurs independently with probability p. We say that two k-uniform hypergraphs H_1, H_2 are isomorphic if there is a bijection

* Corresponding author.
E-mail address: alan@random.math.cmu.edu (A. Frieze).

1 Research supported in part by NSF grant DMS1661063.

2 This research was partially supported by the BIU Center for Research in Applied Cryptography and Cyber Security in conjunction with the Israel National Directorate in the Prime Minister's office.

3 A sequence of events $\mathcal{E}_n, n \geq 1$ occurs with high probability (w.h.p.) if $\lim_{n\to\infty} \mathbb{P}(\mathcal{E}_n) = 1$.

4 A sequence of events $\mathcal{E}_n, n \geq 1$ occurs quite surely (q.s.) if $\mathbb{P}(\mathcal{E}_n) = 1 - O(n^{-K})$ for any positive constant K.

https://doi.org/10.1016/j.ipl.2020.106039
0020-0190/

$f: V(H_1) \to V(H_2)$ such that $\{x_1, x_2, \dots, x_k\}$ is an edge of H_1 if and only if $\{f(x_1), f(x_2), \dots, f(x_k)\}$ is an edge of H_2.

Theorem 2. *Suppose that $k \geq 3$ and $p, 1-p \gg n^{-(k-2)} \log n$ then there exists an $O(n^{2k})$ time algorithm that finds a canonical labelling for $H_{n,p;k}$ w.h.p.*

Bollobás [1] and Kucera [6] proved that random regular graphs have canonical labellings w.h.p. We extend the argument of [1] to regular hypergraphs. A hypergraph is regular of degree r if every vertex is in exactly r edges. We denote a random r-regular, k-uniform hypergraph on vertex set $[n]$ by $H_{n,r;k}$.

Theorem 3. *Suppose that r, k are constants. Then there is an $O(n^{8/5})$ time algorithm that finds a canonical labelling for $H_{n,r;k}$ w.h.p.*

2. Proof of Theorem 2

Given $H = H_{n,p;k}$ we let H_i denote the $(k-1)$-uniform hypergraph with vertex set $[n] \setminus \{i\}$ and edges $\{e \setminus \{i\} : i \in e \in E(H)\}$. H_i is known as the *link* associated with vertex i. Let $\mathcal{E}_k$ denote the event $\left\{\nexists i, j : H_i \cong H_j\right\}$.

Lemma 4. *Suppose that $k \geq 3$ and $\omega \to \infty$ and $p, 1-p \geq \omega n^{-(k-2)} \log n$. Then $\mathcal{E}_k$ occurs q.s.*

Proof.

$$\begin{aligned}\mathbb{P}(\exists i, j : H_i \cong H_j) &\leq n^4 n! (p^2 + (1-p)^2)^{\binom{n-4}{k-1}} \\ &\leq 3n^{9/2} \left(\frac{n}{e}\right)^n (p^2 + (1-p)^2)^{\binom{n-4}{k-1}p} \\ &\leq n^{-\omega/k!}.\end{aligned}$$

Explanation: There are $\binom{n}{2}$ choices for i, j. There are at most n^2 choices for $y = f(i), x = f^{-1}(j)$ in an isomorphism f between H_i and H_j. This accounts for the n^4 term. There are $(n-3)! < n!$ possible isomorphisms between $H_i - \{y, j\}$ and $H_j - \{x, i\}$. Then for every $(k-1)$-set of vertices S that includes none of i, j, x, y, the probability for there to be an edge or non-edge in both H_i and H_j is given by the expression $p^2 + (1-p)^2$.

The above estimation shows that even disregarding edges containing i, j, x or y, w.h.p. there are no i, j with $H_i \cong H_j$. $\square$

Let $\mathcal{G}_k$ be the event that a canonical labelling for $H_{n,p;k}$ can be constructed in $O(n^{2k})$ time. Now assume inductively that

$$\mathbb{P}(H_{n,p;k} \notin \mathcal{G}_k) \leq n^{-\omega/(k+1)!}. \tag{1}$$

The base case, $k = 2$, for (1) is given by the result of [7], although in addition [5], [9] can be used for constant p. Let $\mathcal{B}_i$ be the event that $H_i \notin \mathcal{G}_{k-1}$. Then

$$\mathbb{P}(H_{n,p;k} \notin \mathcal{G}_k) \leq (1 - \mathbb{P}(\mathcal{E}_k)) + \sum_{i=1}^{n} \mathbb{P}(\mathcal{B}_i). \tag{2}$$

Indeed, if none of the events in (2) occur then in time $O(n^2 \times n^{2(k-1)}) = O(n^{2k})$ we can by induction uniquely label each vertex via the labelled link. After this we can confirm that $\mathcal{E}_k$ has occurred. This confirms the claimed time complexity. Given that $\mathcal{E}_k$ has occured, this will determine the only possible isomorphism ϕ between H and any other k-uniform hypergraph H' on vertex set $[n]$. We determine ϕ by comparing the links of H, H', using induction to see if they are isomorphic. We can see if there is a mapping ϕ such that the link of i in H is isomorphic to the link of $\phi(i)$ in H' then and then we check to see whether or not ϕ is actually an isomorphism.

Going back to (2) we see by induction that

$$\begin{aligned}\mathbb{P}(H_{n,p;k} \notin \mathcal{G}_k) &\leq n^{-\omega/k!} + n^2 \times (k-1) n^{2k-2} n^{-\omega/k!} \\ &\leq n^{-\omega/(k+1)!}.\end{aligned}$$

This completes the proof of Theorem 2.

3. Proof of Theorem 3

We extend the analysis of Bollobás [1] to hypergraphs. For a vertex v, we let $d_\ell(v)$ denote the number of vertices at hypergraph distance ℓ from v in $H = H_{n,r;k}$. We show that if

$$\ell^* = \left\lceil \frac{3}{5} \log_\rho n \right\rceil \text{ where } \rho = (r-1)(k-1).$$

then w.h.p. no two vertices have the same sequence $(d_\ell(v), \ell = 1, 2, \dots, \ell^*)$. By doing a breadth first search from each vertex of H we can therefore w.h.p. distinctly label each vertex within $O(n\rho^{\ell^*}) = O(n^{8/5})$ steps.

We use the configuration model for hypergraphs, which is a simple generalisation of the model in Bollobás [2]. We let $W = [rn]$ where $m = rn/k$ is an integer. Assume that it is partitioned into sets $W_1, W_2, \dots, W_n$ of size r. We define $f : W \to [n]$ by $f(w) = i$ if $w \in W_i$. A configuration F is a partition of W into sets $F_1, F_2, \dots, F_m$ of size k. Given F we obtain the (multi)hypergraph $\gamma(F)$ where $F_i = \{w_1, w_2, \dots, w_k\}$ gives rise to the edge $\{f(w_1), f(w_2), \dots, f(w_k)\}$ for $i = 1, 2, \dots, m$. Configurations can contain multiple edges and loops. Nevertheless, it is known that if $\gamma(F)$ has a hypergraph property w.h.p. then $H_{n,r;k}$ will also have this property w.h.p., see for example [4].

In the following $H = H_{n,r;k}$. For a set $S \subseteq [n]$, we let $e_H(S)$ denote the number of edges of H that are contained in S.

Lemma 5. *Let $\ell_0 = \lceil 100 \log_\rho \log n \rceil$. Then w.h.p., $e_H(S) < \frac{|S|+1}{k-1}$ for all $S \subseteq [n]$, $|S| \leq 10\ell_0$.*

Proof. We have that

$$\mathbb{P}\left(\exists S : |S| \leq 10\ell_0, e_H(S) \geq \frac{|S|+1}{k-1}\right)$$

$$\leq \sum_{s=4}^{10\ell_0} \binom{n}{s} \binom{sr}{\frac{s+1}{k-1}} \left(\frac{\binom{sr}{k-1}}{\binom{km - 10k\ell_0}{k-1}}\right)^{\frac{s+1}{k-1}} \tag{3}$$

$$\le \sum_{s=4}^{10\ell_0} \left(\frac{ne}{s}\right)^s (er(k-1))^{\frac{s+1}{k-1}} \left(\frac{rs}{rn-o(n)}\right)^{s+1}$$
$$\le \frac{1}{n^{1-o(1)}} \sum_{s=4}^{10\ell_0} se^s\,(e(k-1)r)^{\frac{s+1}{k-1}} = o(1).$$

Explanation for (3): we choose a set S and then a set X of $(s+1)/(k-1)$ members of $W_S = \bigcup_{i\in S} W_i$. We then estimate the probability that each member of X is paired in F with $k-1$ members of $W_S \setminus X$. For each $x \in X$, given some previous choices, there are at most $\binom{sr}{k-1}$ choices contained in W_S, out of at least $\binom{km-10k\ell_0}{k-1}$ choices overall. □

Let $\mathcal{E}$ denote the high probability event in Lemma 5. We will condition on the occurrence of $\mathcal{E}$.

Now for $v \in [n]$, let $S_\ell(v)$ denote the set of vertices at distance ℓ from v and let $S_{\le \ell}(v) = \bigcup_{j\le \ell} S_j(v)$. Here the distance between vertices u, v is the minimum length of a path/sequence of edges $e_1, e_2, \dots, e_k$ such that $u \in e_1$, $v \in e_k$ and $e_i \cap e_{i+1} \neq \emptyset$ for $1 \le i < k$. We note that

$$|S_\ell(v)| \le (k-1)r\rho^{\ell-1} \text{ for all } v \in [n], \ell \ge 1. \tag{4}$$

Furthermore, Lemma 5 implies that there exist $b_{r,k} < a_{r,k} < (k-1)r$ such that w.h.p., we have for all $v, w \in [n]$, $1 \le \ell \le \ell_0$,

$$|S_\ell(v)| \ge a_{r,k}\rho^{\ell-1}. \tag{5}$$
$$|S_\ell(v) \setminus S_\ell(w)| \ge b_{r,k}\rho^{\ell-1}. \tag{6}$$

To see this, observe that $|S_{\ell+1}| = \rho|S_\ell|$ unless there are two vertices $x, y \in S_\ell$ and either (i) an edge e of $\gamma(F)$ such that $e \supseteq \{x, y\}$ or (ii) a vertex $z \in S_{\ell+1}$ and edges e, f of $\gamma(F)$ such that $e \supseteq \{x, z\}$ and $f \supseteq \{y, z\}$. Lemma 5 implies that w.h.p. there is at most one such case of (i) or (ii) for $1 \le \ell \le \ell_0$. Suppose that there are two distinct edges $e_i, i = 1, 2$ that cause (i) at levels ℓ_1, ℓ_2 and suppose that $\{x_i, y_i\}$ corresponds to $e_i, i = 1, 2$. Each $x \in S_\ell$ lies in the final edge of an ℓ-length path P_u from v to x. Now $\mathcal{P} = P_{x_1}, P_{x_2}, P_{y_1}, P_{y_2}$ spans $\Pi \le 2(\ell_1 + \ell_2) \le 2\ell_0$ edges and we can choose these paths to not contain e_1 or e_2. Furthermore, $\mathcal{P}$ spans at most $1 + (k-1)\Pi$ vertices, since adding a new edge to a connected set of vertices adds at most $k-1$ new vertices. If we add e_1, e_2 to these Π edges then we have at most $1 + (k-1)\Pi + 2(k-2)$ vertices spanning $\Pi + 2$ edges and this contradicts Lemma 5. The remaining two cases (2 times (ii) or (i) and (ii)) can be argued similarly. So, typically adding an edge in the construction of $S_{\ell+1}$ adds $k-1$ new vertices. W.h.p., there is one case and this only adds $k-2$ vertices. This explains (5).

A similar argument yields (6). Having constructed $S_\ell(w)$, we see that typically adding an edge in the construction of $S_\ell(v)$ adds $k-1$ new vertices to the union $S_\ell(v) \cup S_{\ell_0}(w)$ and that w.h.p. it adds at least $k-2$ vertices.

Now consider $\ell > \ell_0$. Consider doing breadth first search from v or v, w exposing the configuration pairing as we go. Let an edge be *dispensable* if it contains two vertices already known to be in $S_{\le \ell}$. The argument above implies that w.h.p. there is at most one dispensable edge in $S_{\le \ell_0}$.

Lemma 6. *With probability $1 - o(n^{-2})$, (i) at most 20 of the first $n^{\frac{2}{5}}$ exposed edges are dispensable and (ii) at most $n^{1/4}$ of the first $n^{\frac{3}{5}}$ exposed edges are dispensable.*

Proof. The probability that the σth edge is dispensable is at most $\frac{r((\sigma-1)(k-1)+1)(k-1)}{rn-k\sigma}$, independent of the history of the process. (Knowing one vertex of this edge and choosing the rest of it, there are at most $r((\sigma-1)(k-1)+1)(k-1)$ choices out of at least $rn - k\sigma$ that will lead to this edge being dispensable.) Hence,

$$\mathbb{P}(\exists\, 20 \text{ dispensable edges in the first } n^{2/5}) \le \binom{n^{2/5}}{20}\left(\frac{rk^2n^{2/5}}{rn-o(n)}\right)^{20} = o(n^{-2}).$$
$$\mathbb{P}(\exists\, n^{1/4} \text{ dispensable edges in first } n^{3/5}) \le \binom{n^{3/5}}{n^{1/4}}\left(\frac{rk^2n^{3/5}}{rn-o(n)}\right)^{n^{1/4}} = o(n^{-2}). \quad \square$$

Now let $\ell_1 = \lceil \log_{r-1} n^{2/5} \rceil$ and $\ell_2 = \lceil \log_{r-1} n^{3/5} \rceil$. Then, we have that, conditional on $\mathcal{E}$, with probability $1 - o(n^{-2})$,

$$|S_\ell(v)| \ge (a_{r,k}\rho^{\ell_0-1} - 40)\rho^{\ell-\ell_0} : \ \ell_0 < \ell \le \ell_1.$$
$$|S_\ell(v)| \ge (a_{r,k}\rho^{\ell_1-1} - 40\rho^{\ell_1-\ell_0} - 2n^{1/4})\rho^{\ell-\ell_1}; \quad \ell_1 < \ell \le \ell_2.$$
$$|S_\ell(w) \setminus S_\ell(v)| \ge (b_{r,k}\rho^{\ell_0-1} - 40)\rho^{\ell-\ell_0} : \ \ell_0 < \ell \le \ell_1.$$
$$|S_\ell(w) \setminus S_\ell(v)| \ge (b_{r,k}\rho^{\ell_1-1} - 40\rho^{\ell_1-\ell_0} - 2n^{1/4})\rho^{\ell-\ell_1}; \quad \ell_1 < \ell \le \ell_2.$$

We deduce from this that if $\ell_3 = \lceil \log_{r-1} n^{4/7} \rceil$ and $\ell = \ell_3 + a, a = O(1)$ then with probability $1 - o(n^{-2})$,

$$|S_\ell(w)| \ge (a_{r,k} - o(1))\rho^{\ell-1} \approx a_{r,k}\rho^{a-1}n^{4/7}.$$
$$|S_\ell(w) \setminus S_\ell(v)| \ge (b_{r,k} - o(1))\rho^{\ell-1} \approx b_{r,k}\rho^{a-1}n^{4/7}.$$

Suppose now that we consider the execution of breadth first search up until we have exposed $S_{\ell+1}(v) \cup S_{\ell+1}(w)$ and the edges $\widehat{E}$ defining this set. We let $U = W \setminus \widehat{E}$. We will show that we can find a position in the process so that conditioning up to this point, in order to have $|S_{\ell+1}(v)| = |S_{\ell+1}(w)|$ there will have to be a prescribed, but unlikely, outcome for a large number of edge selections.

Our conditioning includes all the choices of $e \in F$ that are necessary to construct $S_{\ell+1}(v) \cup S_\ell(w)$. We refer to a choice of e as an *edge-selection*. After an edge-selection e, we update $U \leftarrow U \setminus \{e\}$. Consider the edge-selections involving $W_x, x \in S_\ell(w) \setminus S_{\ell+1}(v)$. Now at most $n^{1/4}$ of these edge-selections involve vertices in $S_{\le \ell+1}(v) \cup S_{\le \ell}(w)$. Condition on these as well. There must now be $\lambda = \Theta(n^{4/7})$ further edge-selections containing elements of $W_x, x \in S_\ell(w) \setminus S_{\ell+1}(v)$ and $W_y, y \notin S_{\ell+1}(v) \cup S_\ell(w)$. Let Z denote the vertices in $S_\ell(w)$ involved in these λ edge-selections. Furthermore, to have $|S_{\ell+1}(v)| = |S_{\ell+1}(w)|$ these λ selections must involve exactly t of the sets $W_y, y \notin S_{\ell+1}(v) \cup S_\ell(w)$. Here t is the unique value that will ensure that

$|S_{\ell+1}(w)| = |S_{\ell+1}(v)|$. The important point is that t is determined *before* the making of these λ edge-selections. Let $R = \bigcup_{y \notin S_{\ell+1}(v) \cup S_\ell(w)} W_y$ at the point immediately prior to the λ edge-selections. Let $S = R \cap \bigcup_{e: e \cap Z \neq \emptyset} e$ and note that S is a random s-subset of R for some $s = \Theta(n^{4/7})$.

The following lemma will easily show that w.h.p. H has a canonical labelling defined by the values of $|S_\ell(v)|, 1 \le \ell \le \ell^*, v \in [n]$.

Lemma 7. *Let $R = \bigcup_{i=1}^{\mu} R_i$ be a partitioning of an $r\mu$ set R into μ subsets of size r. Suppose that S is a random s-subset of R, where $\mu^{5/9} < s < \mu^{3/5}$. Let X_S denote the number of sets R_i intersected by S. Then*

$$\max_j \mathbb{P}(X_S = j) \le \frac{c_0 \mu^{1/2}}{s},$$

for some constant c_0.

Proof. The probability that S has at least 3 elements in some set R_i is at most

$$\frac{\mu \binom{r}{3} \binom{r\mu - 3}{s-3}}{\binom{r\mu}{s}} \le \frac{s^3}{6\mu^2} \le \frac{\mu^{1/2}}{6s}.$$

But

$$\mathbb{P}(X_S = j) \le \mathbb{P}\left(\max_i |S \cap R_i| \ge 3\right) + \mathbb{P}\left(X_S = j \text{ and } \max_i |S \cap R_i| \le 2\right).$$

So the lemma will follow if we prove that for every j,

$$P_j = \mathbb{P}\left(X_S = j \text{ and } \max_i |S \cap R_i| \le 2\right) \le \frac{c_1 \mu^{1/2}}{s}, \qquad (7)$$

for some constant c_1.

Clearly, $P_j = 0$ if $j < s/2$ and otherwise

$$P_j = \frac{\binom{\mu}{j}\binom{j}{s-j} r^{2j-s} \binom{r}{2}^{s-j}}{\binom{r\mu}{s}}. \qquad (8)$$

Now for $s/2 \le j < s$ we have

$$\frac{P_{j+1}}{P_j} = \frac{(\mu - j)(s-j)}{(2j+2-s)(2j+1-s)} \frac{2r}{r-1}. \qquad (9)$$

We note that if $s - j \ge \frac{10s^2}{\mu}$ then $\frac{P_{j+1}}{P_j} \ge \frac{10r}{3(r-1)} \ge 2$ and so the j maximising P_j is of the form $s - \frac{\alpha s^2}{\mu}$ where $\alpha \le 10$. If we substitute $j = s - \frac{\alpha s^2}{\mu}$ into (9) then we see that

$$\frac{P_{j+1}}{P_j} \in \frac{2\alpha r}{r-1}\left[1 \pm c_2 \frac{s}{\mu}\right]$$

for some absolute constant $c_2 > 0$.

It follows that if j_0 is the index maximising P_j then

$$\left| j_0 - \left(s - \frac{(r-1)s^2}{2r\mu}\right)\right| \le 1.$$

Furthermore, if $j_1 = j_0 - \frac{s}{\mu^{1/2}}$ then

$$\frac{P_{j+1}}{P_j} \le 1 + c_3 \frac{\mu^{1/2}}{s} \text{ for } j_1 \le j \le j_0,$$

for some absolute constant $c_3 > 0$.

This implies that for all $j_1 \le j \le j_0$,

$$P_j \ge P_{j_0}\left(1 + c_3 \frac{\mu^{1/2}}{s}\right)^{-(j_0 - j_1)}$$
$$= P_{j_0} \exp\left\{-(j_0 - j_1)\left(c_3 \frac{\mu^{1/2}}{s} + O\left(\frac{\mu}{s^2}\right)\right)\right\} \ge P_{j_0} e^{-2c_3}.$$

It follows from this that

$$P_{j_0} \le e^{2c_3} \min_{j \in [j_1, j_0]} P_j \le \frac{e^{2c_3}}{j_0 - j_1} \sum_{j \in [j_1, j_0]} P_j$$
$$\le \frac{e^{2c_3} \mu^{1/2}}{s}. \quad \square$$

We apply Lemma 7 with $\mu = n - o(n), s = \Theta(n^{4/7})$ to show that

$$\mathbb{P}((|S_\ell(v)| = |S_\ell(w)|, \ell \in [\ell_3, \ell_3 + 14]) \le \left(\frac{c_0 n^{1/2}}{n^{4/7}}\right)^{15}$$
$$= o(n^{-2}).$$

This completes the proof of Theorem 3.

Declaration of competing interest

No conflicts.

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