

AN APPLICATION OF GROTHENDIECK THEOREM TO THE THEORY OF MULTICORRELATION SEQUENCES, MULTIPLE RECURRENCE AND PARTITION REGULARITY OF QUADRATIC EQUATIONS.

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ABSTRACT. We use Grothendieck theorem to prove a structure theorem for multicorrelation sequences associated with two (not necessarily commuting) measure preserving actions on a probability space. We use this to deduce a multiple recurrence result concerning products of linear terms, and a partition regularity result of certain systems of quadratic equations, building on the work of Frantzikinakis and Host [8].

1. INTRODUCTION

The goal of this paper is to introduce a new application of Grothendieck theorem to the field of multicorrelation sequences and multiple recurrence in ergodic theory. More specifically, we prove a structure theorem for the correlation sequences of length two associated with (not necessarily) commuting actions on a probability spaces (Theorem 1.2), and as a corollary we obtain a generalization (Theorem 3.5 and Theorem 3.4) of the partition regularity and multiple recurrence result proved by Frantzikinakis and Host in [7].

1.1. Ergodic theory and multicorrelation sequences. Let Γ be a countable abelian group (e.g. $\Gamma = \mathbb{Z}$). A probability measure preserving Γ -system (or a Γ -system for short) is a quadruple $X = (X, \mathcal{B}, \mu, T)$ where (X, \mathcal{B}, μ) is a probability system and $T : \Gamma \rightarrow \text{Aut}(X, \mathcal{B}, \mu)$ is an action of Γ on (X, \mathcal{B}, μ) by measure preserving transformations. Namely, for every $\gamma \in \Gamma$, $T_\gamma : X \rightarrow X$ is a measurable map satisfying $\mu(T_\gamma^{-1}A) = \mu(A)$ for all $A \in \mathcal{B}$ and $T_{\gamma+\gamma'} = T_\gamma \circ T_{\gamma'}$ for all $\gamma, \gamma' \in \Gamma$. We abuse notation and denote the

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Koopman operator associated with T_γ by $T_\gamma(f) = f \circ T_\gamma$. The homomorphism $\gamma \mapsto T_\gamma$ is then a unitary representation of Γ on $L^2(X)$.

The study of multicorrelation sequences goes back to Furstenberg [11], who gave an ergodic theoretical proof to Szemerédi's theorem [20] about the existence of arbitrary long arithmetic progressions in dense subsets of the integers. In that work, Furstenberg studied the limit (\liminf) of the average

$$\frac{1}{N} \sum_{n=1}^N \int_X \prod_{i=0}^k T^{in} f_i(x) d\mu(x),$$

where f_0, \dots, f_k are non-negative bounded functions on a measure-preserving \mathbb{Z} -system. The term inside the average is called a *multicorrelation sequence*. More generally,

Definition 1.1. Let Γ be a countable abelian group and $X = (X, \mathcal{B}, \mu, T)$ be a Γ -system, and $k \geq 1$. A k -step multicorrelation sequence is a function $a : \Gamma \rightarrow \mathbb{C}$ of the form

$$(1) \quad a(\gamma) = \int_X \prod_{i=0}^k T^{i\gamma} f_i(x) d\mu(x),$$

where $f_0, \dots, f_k \in L^\infty(X)$ are bounded functions.

Classifying these sequences is a big open problem (see [6]). In [1], Bergelson, Host and Kra proved that for \mathbb{Z} -systems a correlation sequence is a sum of a nilsequence and a null-sequence (i.e. a sequence that tends to zero in uniform density)¹. This was then generalized in multiple directions by multiple authors (see e.g. [4], [5], [9], [13], [14], [15], [16], [17]). We note that when $k = 1$, and $\Gamma = \mathbb{Z}$, it is a consequence of the spectral theorem that every sequence of the form $a(n) = \int_X T^n f \cdot g d\mu$ can be written as $\int_{S^1} x^n d\nu_{f,g}(x)$ for some complex-valued measure $\nu_{f,g}$ on the torus. In [6], Frantzikinakis asks for a generalization of this formula for 2-step multicorrelation sequences. He also conjectured that one can obtain such formula as an integral over *generalized nilsequences* (as defined in [6, Section 2.4.2]), see also the work of Briët and Green [2] which implies the necessity of this

¹See Definition 1.7 and Definition 1.8 in their paper, we will not use these definitions here.

generalized notion. Below we give a different answer to a related question. More specifically, we prove the following result.

Theorem 1.2 (Structure theorem for 2-step correlation sequences for non-commutative transformations). *Let $X = (X, \mathcal{B}, \mu)$ be a probability space and let Γ be a countable abelian group. Let $T, S : \Gamma \rightarrow \text{Aut}(X)$ be two (not necessarily commuting) Γ -actions on X . Then for every $f, h \in L^2(X)$ and $g \in L^\infty(X)$, there exists a probability measure λ on the Pontryagin dual $\Sigma := \widehat{\Gamma}$ of Γ , and a continuous operator $G : L^2(\Sigma, \lambda) \rightarrow L^2(\Sigma, \lambda)$ such that*

$$(2) \quad \int_X T_\gamma S_{\gamma'} f \cdot T_\gamma g \cdot h \, d\mu = \int_\Sigma G(\xi_\gamma)(\chi) \cdot \xi_{\gamma'}(\chi) \, d\lambda(\chi),$$

where $\xi_\gamma : \Sigma \rightarrow S^1$ is the evaluation map $\xi_\gamma(\chi) = \chi(\gamma)$.

In section 3 we generalize a partition regularity result of Frantzikinakis and Host [7].

Remark 1.3. It may appear at first glance that this theorem is not really necessary as one can deal with the expression on the left hand side using the spectral theorem for T_γ or the spectral theorem for $S_{\gamma'}$, separately. However, this new structure theorem allows us to have a certain estimate involving both γ and γ' , simultaneously. More concretely, the Cauchy-Schwartz inequality gives that for any sequences $a, b : \Gamma \rightarrow \mathbb{C}$ for which the sums below are well defined, we have

$$\sum_{\gamma, \gamma' \in \Gamma} a_\gamma b_{\gamma'} \int_X T_\gamma S_{\gamma'} f \cdot T_\gamma g \cdot h \, d\mu \leq \|G\|_{op} \left\| \sum_\gamma a_\gamma \xi_\gamma \right\|_{L^2(\lambda)} \cdot \left\| \sum_\gamma b_\gamma \xi_\gamma \right\|_{L^2(\lambda)}.$$

This estimate plays an important role in our proof of the partition regularity result (Theorem (3.5)), see Equation (9)).

Remark 1.4. Let $\Gamma = \mathbb{Z}$. The set of all sequences $a(n) := \int_X T^n f S^n g \, d\mu$, where S and T are arbitrary (not necessarily commuting) measure-preserving transformations on some probability space $X = (X, \mathcal{B}, \mu)$ and $f, g \in L^\infty(X)$ consists of all bounded sequences (see [10, Corollary 4.2]). It is likely that a similar result is valid for our expressions as well and, in particular, we do not claim to give a satisfactory answer to [6, Problem 1].

The remark above leads to the following natural problem.

Problem 1.5. Determine for which λ and G the expression on the right hand side of (2) is a multikorrelation sequence for commuting T and S , or for $T = S$.

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2. THE SPECTRAL THEOREM AND GROTHENDIECK THEOREM

Let S be a compact Hausdorff space. Riesz-Markov-Kakutani theorem asserts that any continuous linear functional $\Phi : C(S) \rightarrow \mathbb{C}$ takes the form $\Phi(f) = \int f d\nu$ where ν is some complex-valued measure on S . This in particular implies that any continuous linear functional on $C(S)$ extends to a continuous linear functional on $L^2(S, \nu)$ (note that by the Cauchy-Schwartz inequality $|\Phi(f)| \leq \|f\|_{L^2(\nu)}$). The main new tool we use in this paper is a theorem of Grothendieck which classifies continuous bilinear forms.

Theorem 2.1 (Grothendieck). *Let T, S be compact Hausdorff spaces. Let $\Phi : C(T) \times C(S) \rightarrow \mathbb{C}$ be a bilinear map and suppose that*

$$|\Phi(\phi, \psi)| \leq \|\phi\|_\infty \cdot \|\psi\|_\infty.$$

Then there exists an absolute constant K , called the Grothendieck constant, and two Borel probability measures λ_1, λ_2 such that

$$|\Phi(\phi, \psi)| \leq K \|\phi\|_{L^2(\lambda_1)} \cdot \|\psi\|_{L^2(\lambda_2)}$$

for all $\phi \in C(T)$ and $\psi \in C(S)$. In particular, Φ extends to a unique bilinear map on $L^2(T, \lambda_1) \times L^2(S, \lambda_2)$.

The original proof of this result is due to Grothendieck and can be found in [12] (in French). A translation to English can be found in [18] or [19]. The latter also contains a long summary of the developments related to Grothendieck theorem. A satisfactory version of Grothendieck theorem for multilinear functionals would lead to a generalization of our result. However, even the trilinear Grothendieck theorem is a big problem (cf. [19]), only a few special cases of which are known, while most known results are in the negative direction (see e.g. [3]).

2.1. Gelfand theory and a spectral theorem for the multicorrelation sequences. In this section we prove Theorem 1.2. For the sake of generality, and the application given in the next section, we allow Γ to be an arbitrary countable abelian group, but the result is already new for $\Gamma = \mathbb{Z}$. Let Γ be a countable abelian group and let $T, S : \Gamma \rightarrow \text{Aut}(X)$ be two (not-necessarily commuting) Γ -actions on a probability space $X = (X, \mathcal{B}, \mu)$. Recall that we have unitary representations $T, S : \Gamma \rightarrow L^2(\mu)$. Let Σ denote the Pontryagin dual of Γ . Gelfand theory then gives rise to a \star -morphism $C(\Sigma) \rightarrow \mathcal{L}(L^2(X))$ sending any continuous function $\phi : \Sigma \rightarrow \mathbb{C}$ to a linear operator $T_\phi : L^2(X) \rightarrow L^2(X)$, with $\|T_\phi\|_{\text{op}} \leq \|\phi\|_\infty$. Fix $f, h \in L^2(X)$ and $g \in L^\infty(X)$. For every $\phi, \psi \in C(\Sigma)$ we have that $S_\psi f \in L^2(X)$ and therefore, $S_\psi f \cdot g$ and $T_\phi(S_\psi f \cdot g)$ are in $L^2(X)$. We conclude that the term

$$\Phi(\phi, \psi) := \int T_\phi(S_\psi f \cdot g) \cdot h \, d\mu$$

is well defined. Furthermore by the Cauchy-Schwartz inequality we have

$$\begin{aligned} |\Phi(\phi, \psi)| &\leq \|T_\phi(S_\psi f \cdot g)\|_{L^2(X)} \cdot \|h\|_{L^2(X)} \\ &\leq \|\phi\|_\infty \cdot \|S_\psi f \cdot g\|_{L^2(X)} \cdot \|h\|_{L^2(X)} \\ &\leq \|\phi\|_\infty \cdot \|S_\psi f\|_{L^2(X)} \cdot \|g\|_{L^\infty(X)} \cdot \|h\|_{L^2(X)} \\ &\leq \|\phi\|_\infty \cdot \|\psi\|_\infty \cdot \|f\|_{L^2(X)} \cdot \|g\|_{L^\infty(X)} \cdot \|h\|_{L^2(X)}. \end{aligned}$$

Therefore, Φ is a continuous bilinear map. By Theorem 2.1, we can find Borel probability measures λ_1, λ_2 (depending on f, g, h) so that Φ extends to a bilinear map on $L^2(\Sigma, \lambda_1) \times L^2(\Sigma, \lambda_2)$. Let $\lambda = \frac{1}{2}(\lambda_1 + \lambda_2)$. Since

$$2\|\phi\|_{L^2(\Sigma, \lambda)} \leq \min\{\|\phi\|_{L^2(\Sigma, \lambda_1)}, \|\phi\|_{L^2(\Sigma, \lambda_2)}\}$$

we can assume without loss of generality that $\lambda_1 = \lambda_2 = \lambda$. By Riesz representation theorem, any continuous bilinear map on a Hilbert space is associated with an operator. In other words, there exists an operator $G : L^2(\Sigma, \lambda) \rightarrow L^2(\Sigma, \lambda)$ so that

$$(3) \quad \Phi(\phi, \psi) = \int_X T_\phi(S_\psi f \cdot g) \cdot h \, d\mu = \int_\Sigma G(\phi) \cdot \psi \, d\lambda.$$

In particular, if $\xi_\gamma : \Sigma \rightarrow S^1$ is the evaluation by γ , then

$$(4) \quad \int_X T_\gamma S_{\gamma'} f \cdot T_{\gamma'} g \cdot h \, d\mu = \int_\Sigma G(\xi_\gamma) \cdot \xi_{\gamma'} \, d\lambda.$$

This completes the proof of Theorem 1.2. \square

3. MULTIPLE RECURRENCE FOR PRODUCTS OF LINEAR TERMS

Definition 3.1. The equation $p(x, y, n) = 0$ is called *partition regular* in \mathbb{N} if for any partition of \mathbb{N} into finitely many cells, for some $n \in \mathbb{N}$, one of the cells contains distinct x, y that satisfy the equation.

In [8], Frantziknakis and Host proved the following partition regularity result for certain quadratic equations.

Theorem 3.2 (The three squares theorem). *Let p be the quadratic form*

$$p(x, y, z) = ax^2 + by^2 + cz^2 + dxy + exz + fyz,$$

where a, b, c are non-zero and d, e, f are arbitrary integers. Suppose that all three forms $p(x, 0, z)$, $p(0, y, z)$, $p(x, x, z)$ have non-zero square discriminants. Then the equation $p(x, y, n) = 0$ is partition regular. The last hypothesis means that the three integers

$$\begin{aligned}\nabla_1 &:= e^2 - 4ac, \\ \nabla_2 &:= f^2 - 4bc, \\ \nabla_3 &:= (e + f)^2 - 4c(a + b + d)\end{aligned}$$

are non-zero squares.

The main key ingredient in the proof is the following multiple recurrence result [7].

Theorem 3.3. *Let l_1 be positive and l_2, l_3 non-negative integers with $l_2 \neq l_3$. Then for every set $E \subseteq \mathbb{N}$ of positive multiplicative density, there exist $m, n \in \mathbb{N}$ such that the integers $m \cdot (m + l_1n)$ and $(m + l_2n) \cdot (m + l_3n)$ are distinct and belong to E .*

In this paper we extend this result to three terms by proving the following theorem.

Theorem 3.4. *Let l_1, \dots, l_7 be non-negative, with $l_1 \neq 0$, $l_2 \neq l_3$, $l_4 \neq l_5$ and $l_6 \neq l_7$. Then for every set E of positive multiplicative density there exists $m, n, m', n' \in \mathbb{N}$ such that the integers*

$$\begin{aligned}m \cdot (m + l_1n) \cdot (m' + l_4n') \cdot (m' + l_5n'), \\ m \cdot (m + l_1n) \cdot (m' + l_6n') \cdot (m' + l_7n'),\end{aligned}$$

and

$$(m + l_2n) \cdot (m + l_3n) \cdot (m' + l_6n') \cdot (m' + l_7n')$$

are distinct and belong to E .

As a corollary we obtain the following *simultaneous partition regularity result* which generalizes Theorem 3.2.

Theorem 3.5. *Let p_1, p_2 be two quadratic forms, each satisfying the properties in Theorem 3.2. Then for any partition of \mathbb{N} into finitely many cells, then for some $n, n' \in \mathbb{N}$, there exists distinct $x, y, x', y', k \in \mathbb{N}$ so that $p_1(x, y, n) = 0$ and $p_2(x', y', n') = 0$ and $\frac{x \cdot x'}{k}, \frac{x \cdot y'}{k}, \frac{y \cdot y'}{k}$ are distinct integers which belong to the same cell.*

Given Theorem 3.4, the proof of Theorem 3.5 is an immediate corollary of [7, Proposition 1.4]. Therefore, in the following sections we focus on proving Theorem 3.4.

3.1. Frantzikinakis and Host decomposition of multiplicative functions.

We need some notations. Given a function $f : \mathbb{Z}/N\mathbb{Z} \rightarrow \mathbb{C}$, for some $N \in \mathbb{N}$ and a subset $A \subseteq \mathbb{Z}/N\mathbb{Z}$, we denote the average of f in A by $\mathbb{E}_{n \in A} f(n) = \frac{1}{|A|} \sum_{n \in A} f(n)$. The Gowers uniformity norms of f are defined as follows.

Definition 3.6. Let $f : \mathbb{Z}/N\mathbb{Z} \rightarrow \mathbb{C}$ be a bounded function and let C denote the complex conjugation. The Gowers d -norm² is defined by the formula

$$\|f\|_{U^d}^{2^d} = \mathbb{E}_{x, h_1, \dots, h_d \in \mathbb{Z}/N\mathbb{Z}} \prod_{\omega_1, \dots, \omega_d \in \{0, 1\}} C^{\omega_1 + \dots + \omega_d} f(x + h_1\omega_1 + \dots + h_d\omega_d) .$$

Throughout, $l = \sum_{i=1}^7 l_i$. Given $N \in \mathbb{N}$, we let \tilde{N} denote the smallest prime that is larger than $10l \cdot N$. A function $\chi : \mathbb{N} \rightarrow \mathbb{C}$ is called *multiplicative* if $\chi(n \cdot m) = \chi(n) \cdot \chi(m)$ for all $m, n \in \mathbb{N}$. For any such function and any $N \in \mathbb{N}$, we denote by $\chi_N : \mathbb{Z}/\tilde{N}\mathbb{Z}$ the map defined by

$$\chi_N(n) = \begin{cases} \chi(n) & n \leq N \\ 0 & \text{otherwise} \end{cases} .$$

Definition 3.7. A *kernel* on $\mathbb{Z}/N\mathbb{Z}$ is a non-negative function $\psi : \mathbb{Z}/N\mathbb{Z} \rightarrow \mathbb{R}_{\geq 0}$ with average 1.

² $\|\cdot\|_{U^d}$ is a seminorm when $d = 1$, and a norm for $d > 2$.

A key result in our proof is the following decomposition theorem of Frantzikinakis and Host [7, Theorem 1.6].

Theorem 3.8 (Structure theorem for multiplicative functions). *Let $s \geq 2$, $\varepsilon > 0$, λ be a probability measure on the set of all multiplicative functions \mathcal{M}^3 and $F : \mathbb{N} \times \mathbb{N} \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be arbitrary. Then there exist positive integers Q and R that are bounded by a constant which depends only on s, ε, F , such that the following holds: For every sufficiently large $N \in \mathbb{N}$ which depends only on s, ε, F , and for every $\chi \in \mathcal{M}$ the function $\chi_N = \chi \cdot 1_{[N]}$ admits the decomposition*

$$\chi_N(n) = \chi_{N,st}(n) + \chi_{N,un}(n) + \chi_{N,er}(n)$$

for every $n \in \mathbb{Z}/\tilde{N}\mathbb{Z}$, where $\chi_{N,st}, \chi_{N,un}, \chi_{N,er}$ satisfy the following properties:

- (i) $\chi_{N,st} = \chi_N * \psi_{N,1}$ and $\chi_{N,st} + \chi_{N,er} = \chi_N * \psi_{N,2}$, where $\psi_{N,1}, \psi_{N,2}$ are kernels on $\mathbb{Z}/\tilde{N}\mathbb{Z}$ that do not depend on f , and the convolution product is defined in $\mathbb{Z}/\tilde{N}\mathbb{Z}$. As a consequence, $\chi \mapsto \chi_{N,un}, \chi \mapsto \chi_{N,st}$ and $\chi \mapsto \chi_{N,er}$ are continuous, $|\chi_{N,st}| \leq 1$ and $|\chi_{N,un}|, |\chi_{N,er}| \leq 2$;
- (ii) $|\chi_{N,st}(n+Q) - \chi_{N,st}(n)| \leq \frac{R}{\tilde{N}}$ for every $n \in \mathbb{Z}/\tilde{N}\mathbb{Z}$, where $n+Q$ is taken mod \tilde{N} ;
- (iii) $\|\chi_{N,un}\|_{U^s(\mathbb{Z}/\tilde{N}\mathbb{Z})} \leq \frac{1}{F(Q,R,\varepsilon)}$;
- (iv)

$$\mathbb{E}_{n \in \mathbb{Z}/\tilde{N}\mathbb{Z}} \int_{\mathcal{M}} |\chi_{N,er}(n)| d\nu(\chi) \leq \varepsilon.$$

Remark 3.9. A version of this theorem for higher order uniformity norms was also established by Frantzikinakis and Host in [8]. Using this result and the same argument as in [7], they generalize Theorem 3.3, proving that for every $k \geq 2$, E contains $L_1(m, n), L_2(m, n)$ where each one of the L_i 's is a product of k linear terms. This procedure can also be applied here to generalize Theorem 3.4.

4. SPECTRAL REFORMULATING OF THEOREM 3.4

In this section we reformulate Theorem 3.4 in the language of correlation sequences. We follow closely the arguments in [7], where the main difference is that we apply Theorem 1.2 in place of the spectral theorem.

³Equipped with the pointwise multiplication and the compact-open topology.

Definition 4.1 (Multiplicative density). A *multiplicative Følner sequence* is an increasing family of finite subsets $(\Phi_N)_{N \in \mathbb{N}}$ of \mathbb{N} satisfying

$$\limsup_{N \rightarrow \infty} \frac{|a \cdot \Phi_N \Delta \Phi_N|}{|\Phi_N|} = 0$$

for all $a \in \mathbb{N}$. The *multiplicative density* $d_{\text{mult}}(E)$ of a subset $E \subseteq \mathbb{N}$, relatively to a Følner sequence Φ_N is defined by the formula

$$d_{\text{mult}}(E) := \limsup_{N \rightarrow \infty} \frac{|E \cap \Phi_N|}{|\Phi_N|}.$$

Throughout we fix some (any) multiplicative Følner sequence Φ_N , and we implicitly assume that d_{mult} is defined relatively to this sequence.

In [7], Frantzikinakis and Host use the term *action by dilation*, which in our language is simply a (\mathbb{Q}^+, \cdot) -system. Here, \mathbb{Q}^+ denote the set of all positive rational numbers. Note that the prime decomposition gives rise to an isomorphism $(\mathbb{Q}^+, \cdot) \cong \bigoplus_{i=1}^{\omega} \mathbb{Z}$. In particular, we see that the Pontryagin dual \mathcal{M} of (\mathbb{Q}^+, \cdot) , which consists of all multiplicative functions on \mathbb{N} and is equipped with pointwise multiplication and the compact open topology, is isomorphic as a topological space to the infinite dimensional torus.

The Furstenberg correspondence principle allows us to translate our combinatorial problem into a question about the multiple recurrence of a certain (\mathbb{Q}^+, \cdot) -system.

Proposition 4.2 (Furstenberg Correspondence Principle). *Let E be a subset of \mathbb{N} and let $\Gamma = (\mathbb{Q}^+, \cdot)$. There exists a Γ -system $X = (X, \mathcal{B}, \mu, T)$ and a set $A \in \mathcal{B}$ with $\mu(A) = d_{\text{mult}}(E)$, such that for every $k \in \mathbb{N}$ and $n_1, \dots, n_k \in \mathbb{N}$, we have*

$$(5) \quad d_{\text{mult}}(n_1^{-1}E \cap \dots \cap n_k^{-1}E) \geq \mu(T_{n_1}^{-1}A \cap \dots \cap T_{n_k}^{-1}A).$$

By Furstenberg correspondence principle we get the following ergodic theoretical reformulation of Theorem 3.4. Fix integers l_1, l_2, \dots, l_7 as in Theorem 3.4 and let $L_1(m, n) := m \cdot (m + l_1 n)$, $L_2(m, n) := (m + l_2 n) \cdot (m + l_3 n)$, $L'_1(m, n) := (m + l_4 n) \cdot (m + l_5 n)$ and $L'_2(m, n) := (m + l_6 n) \cdot (m + l_7 n)$.

Theorem 4.3 (Dynamical reformulation). *Let $X = (X, \mathcal{B}, \mu, T)$ be an action by dilation on X and let A be a measurable set with $\mu(A) > 0$ and let*

l_1, l_2, \dots, l_7 be integers as in Theorem 3.4. Then there exists infinitely many quadruples m, n, m', n' so that

$$\mu(T_{L_1(m,n) \cdot L'_1(m',n')}^{-1} A \cap T_{L_1(m,n) \cdot L'_2(m',n')}^{-1} A \cap T_{L_2(m,n) \cdot L'_2(m',n')}^{-1} A) > 0.$$

It is left to prove this theorem.

Let $R(m, n) := \frac{L_2(m,n)}{L_1(m,n)}$ and $R'(m, n) := \frac{L'_2(m',n')}{L'_1(m',n')}$. Since T is measure preserving, the theorem above is equivalent to proving that

$$\mu(A \cap T_{R'(m',n')}^{-1} A \cap T_{R(m,n) \cdot R'(m',n')}^{-1} A) > 0$$

for infinitely many quadruples $m, n, m', n' \in \mathbb{N}$. By (4) there exists a Borel probability measure $\lambda = \lambda_A$ on \mathcal{M} and an operator $G = G_A : L^2(\mathcal{M}, \lambda_A) \rightarrow L^2(\mathcal{M}, \lambda_A)$ so that

$$\mu(A \cap T_{R'(m',n')}^{-1} A \cap T_{R(m,n) \cdot R'(m',n')}^{-1} A) = \int_{\mathcal{M}} G(\xi_{R'(m',n')})(\chi) \cdot \xi_{R(m,n)}(\chi) d\lambda(\chi),$$

where for every $t \in \mathbb{N}$, $\xi_t(\chi) := \chi(t)$ is the evaluation map. Therefore, it suffices to prove that

$$(6) \quad \liminf_{N \rightarrow \infty} \liminf_{N' \rightarrow \infty} \mathbb{E}_{m,n \in \Theta_N} \mathbb{E}_{m',n' \in \Theta_{N'}} \int_{\mathcal{M}} G(\xi_{R'(m',n')})(\chi) \cdot \xi_{R(m,n)}(\chi) d\lambda(\chi) > 0$$

where $\Theta_N = \{(m, n) \in [N] \times [N] : 1 \leq m + l_i n \leq N \text{ for all } 1 \leq i \leq 3\}$, and $\Theta_{N'} = \{(m', n') \in [N'] \times [N'] : 1 \leq m + l_i n' \leq N' \text{ for all } 4 \leq i \leq 7\}$.

Lemma 4.4. *To prove (6), it suffices to show that*

$$\liminf_{N \rightarrow \infty} \liminf_{N' \rightarrow \infty} \mathbb{E}_{m,n \in \mathbb{Z}/\tilde{N}\mathbb{Z}} \mathbb{E}_{m',n' \in \mathbb{Z}/\tilde{N}'\mathbb{Z}} \int_{\mathcal{M}} G(1_{[N']}(n') \xi_{R'(m',n')})(\chi) \cdot 1_{[N]}(n) \xi_{R(m,n)}(\chi) d\lambda(x) > 0.$$

Proof. First, observe that

$$\begin{aligned} & \mathbb{E}_{m,n \in \Theta_N} \mathbb{E}_{m',n' \in \Theta_{N'}} \int_{\mathcal{M}} G(\xi_{R'(m',n')})(\chi) \cdot \xi_{R(m,n)}(\chi) d\lambda(x) = \\ & \frac{\tilde{N} \cdot \tilde{N}'}{|\Theta_N| \cdot |\Theta_{N'}|} \cdot \mathbb{E}_{m,n \in \mathbb{Z}/\tilde{N}\mathbb{Z}} \mathbb{E}_{m',n' \in \mathbb{Z}/\tilde{N}'\mathbb{Z}} \int_{\mathcal{M}} G(1_{[N']}(n') \xi_{R'(m',n')})(\chi_{N'}) \cdot 1_{[N]}(n) \xi_{R(m,n)}(\chi_N) d\lambda(x). \end{aligned}$$

Indeed, if $n > N$, then the term inside the average is zero because $1_{[N]}(n) = 0$. Moreover, $\xi_{R(m,n)}(\chi_N) = \overline{\chi_N(m)} \cdot \overline{\chi_N(m + l_1 n)} \cdot \chi_N(m + l_2 n) \cdot \chi_N(m + l_3 n)$ and so that term is equal to zero also when $m + l_i n > N$ for some $1 \leq i \leq 3$. Using the adjoining operator we can move G to the other side, then by the same argument as above the term from before is zero also when $m' > N$ or $n' > N$ or $m' + l_i n' > N'$ for some $4 \leq i \leq 7$. Now, since $cN^2 < |\Theta_N| < N^2$,

and $c' \cdot (N')^2 < |\Theta_{N'}| < (N')^2$, for some constants c, c' depending only on l and $\tilde{N}/N, \tilde{N}'/N' < 10 \cdot l$, we see that if the term in the Lemma is positive, (6) must also be positive (and vice versa). \square

The following estimate was established in [7, Lemma 2.7].

Lemma 4.5 (U^3 -estimate). *Let $a_i, i = 0, 1, 2, 3$, be functions on $\mathbb{Z}/\tilde{N}\mathbb{Z}$, with $\|a_i\|_{L^\infty(\mathbb{Z}/\tilde{N}\mathbb{Z})} \leq 1$ and $l_1, l_2, l_3 \in \mathbb{N}$ be distinct. Then there exists a constant c_2 depending only on $l = l_1 + l_2 + l_3$ such that*

$$\left| \mathbb{E}_{m,n \in \mathbb{Z}/\tilde{N}\mathbb{Z}} 1_{[N]}(n) \cdot a_0(m) \cdot a_1(m + l_1 n) \cdot a_2(m + l_2 n) \cdot a_3(m + l_3 n) \right| \leq c_2 \min_{0 \leq j \leq 3} \|a_j\|_{U^3(\mathbb{Z}/\tilde{N}\mathbb{Z})}^{\frac{1}{2}} + \frac{2}{\tilde{N}}.$$

We need the following non-negativity lemma.

Lemma 4.6. *Let $X = (X, \mathcal{B}, \mu, T)$ be an action by dilations, let A be a subset of X and let G and λ be as above. Let ψ, ψ' be non-negative functions defined on $\mathbb{Z}/\tilde{N}\mathbb{Z}$ and $\mathbb{Z}/\tilde{N}'\mathbb{Z}$, respectively. For every $n \in \mathbb{Z}/\tilde{N}\mathbb{Z}$ let $\xi_{\psi, N, n}$ denote the map on \mathcal{M} which sends χ to $\chi_N * \psi(n)$, and define $\xi_{\psi', N', n}$ similarly. Then*

$$\int_{\mathcal{M}} G(\xi_{\psi', N', n'_1} \cdot \xi_{\psi', N', n'_2} \cdot \xi_{\psi', N', n'_3} \cdot \xi_{\psi', N', n'_4})(\chi) \cdot \xi_{\psi, N, n_1} \cdot \xi_{\psi, N, n_2} \cdot \xi_{\psi, N, n_3} \cdot \xi_{\psi, N, n_4}(\chi) d\lambda(\chi) \geq 0$$

for every $n_1, n_2, n_3, n_4 \in \mathbb{Z}/\tilde{N}\mathbb{Z}$, $n'_1, n'_2, n'_3, n'_4 \in \mathbb{Z}/\tilde{N}'\mathbb{Z}$.

Proof. The convolution product $\chi_N * \psi$ is defined on the group $\mathbb{Z}/\tilde{N}\mathbb{Z}$ by the

$$\chi_N * \psi(n) = \mathbb{E}_{k \in \mathbb{Z}/\tilde{N}\mathbb{Z}} \psi(n - k) \chi_N(k).$$

It follows that for every $n \in [\tilde{N}]$ there exist sequences $(a_n(k))_{k \in \mathbb{Z}/\tilde{N}\mathbb{Z}}$, $(a'_n(k))_{k \in \mathbb{Z}/\tilde{N}'\mathbb{Z}}$ of non-negative numbers, such that $\xi_{N, \psi, n} = \mathbb{E}_{k \in \mathbb{Z}/\tilde{N}\mathbb{Z}} a_n(k) \xi_{N, k}$ and $\xi_{\psi', n} = \mathbb{E}_{k \in \mathbb{Z}/\tilde{N}'\mathbb{Z}} a'_n(k) \cdot \xi_k$. Therefore, the left hand side of the inequality we want to prove is equal to

$$\sum_{k_1, \dots, k_4 \in \mathbb{Z}/\tilde{N}\mathbb{Z}} \sum_{k'_1, \dots, k'_4 \in \mathbb{Z}/\tilde{N}'\mathbb{Z}} \prod_{i=1}^4 a_{n_i}(k_i) a'_{n'_i}(k'_i) \int_{\mathcal{M}} G(\xi_{k'_1} \cdot \dots \cdot \xi_{k'_4}) \cdot \xi_{k_1} \cdot \dots \cdot \xi_{k_4} d\lambda$$

By (3) this equals to

$$\sum_{k_1, \dots, k_4 \in \mathbb{Z}/\tilde{N}\mathbb{Z}} \sum_{k'_1, \dots, k'_4 \in \mathbb{Z}/\tilde{N}'\mathbb{Z}} \prod_{i=1}^4 a_{n_i}(k_i) a'_{n'_i}(k'_i) \mu(T_{\prod_{i=1}^4 k_i k'_i}^{-1} A \cap T_{\prod_{i=1}^4 k'_i}^{-1} A \cap A) \geq 0.$$

This completes the proof. \square

5. COMPLETING THE PROOF OF THEOREM 3.4

We proved in the previous section that in order to prove Theorem 3.4, it suffices to show that the term appearing in Lemma 4.4 is positive. The main new novelty here (compared to [7]) is the estimate (9), which is also where we use Grothendieck inequality. We let $f = 1_A$ denote the characteristic function of A , and set $\delta := \mu(A) = \int f d\mu$. Let $\varepsilon = c_3 \cdot \delta^4$ and $F(x, y, z) = c_4^2 \frac{x^4 y^4}{z^2}$ where c_3, c_4 are constants depending only on l_0, \dots, l_7 to be chosen later. Let

$$A(N, N') := \int \left(\mathbb{E}_{m', n' \in \mathbb{Z}/\tilde{N}'\mathbb{Z}} G(1_{[N']}(n) \cdot \xi_{N', R'(m', n')}) \right) \cdot \left(\mathbb{E}_{(m, n) \in \mathbb{Z}/\tilde{N}\mathbb{Z}} 1_{[N]}(n) \cdot \xi_{N, R(m, n)} \right) d\lambda.$$

We apply Theorem 3.8 to the ε, F, λ defined above. Let Q, R be as in the theorem, and write $\xi_{N, n} = \xi_{N, n}^u + \xi_{N, n}^s + \xi_{N, n}^e$ where $\xi_{N, n}^u(\chi) = \chi_{N, un}(n)$, $\xi_{N, n}^s(\chi) = \chi_{N, sn}(n)$ and $\xi_{N, n}^e(\chi) = \chi_{N, en}(n)$ satisfy the properties of the theorem. We also write $\xi_{N, n}^{s, e} = \xi_{N, n}^s + \xi_{N, n}^e$. Now look at

$$A_1(N, N') := \int \left(\mathbb{E}_{m', n' \in \mathbb{Z}/\tilde{N}'\mathbb{Z}} G(1_{[N']}(n) \cdot \xi_{N', R'(m', n')}^{s, e}) \right) \cdot \left(\mathbb{E}_{(m, n) \in \mathbb{Z}/\tilde{N}\mathbb{Z}} 1_{[N]}(n) \cdot \xi_{N, R(m, n)}^{s, e} \right) d\lambda.$$

Namely, the term obtained by replacing each instance of ξ with $\xi^{s, e}$. We bound $A(N, N') - A_1(N, N')$. To do so we introduce an intermediate term

$$B(N, N') := \int \left(\mathbb{E}_{m', n' \in \mathbb{Z}/\tilde{N}'\mathbb{Z}} G(1_{[N']}(n) \cdot \xi_{N', R'(m', n')}^{s, e}) \right) \cdot \left(\mathbb{E}_{(m, n) \in \mathbb{Z}/\tilde{N}\mathbb{Z}} 1_{[N]}(n) \cdot \xi_{N, R(m, n)} \right) d\lambda.$$

By Cauchy-Schawrtz, we have for all bounded ϕ, ψ that $|\int G(\phi) \cdot \psi| \leq \|G\phi\|_{L^2(\lambda)} \|\psi\|_{L^2(\lambda)} \leq \|G\| \cdot \|\phi\|_\infty \cdot \|\psi\|_\infty$. We deduce that

$$|B(N, N') - A_1(N, N')| \leq \|G\|_{op} \cdot \left\| \mathbb{E}_{(m, n) \in \mathbb{Z}/\tilde{N}\mathbb{Z}} 1_{[N]}(n) \cdot \left(\xi_{N, R(m, n)} - \xi_{N, R(m, n)}^{s, e} \right) \right\|_\infty.$$

Recall that for every χ , $\xi_{N, R(m, n)} = \overline{\xi_N(m) \cdot \xi_N(m + l_1 n) \cdot \xi_N(m + l_2 n) \cdot \xi_N(m + l_3 n)}$. Therefore, the average on the right hand side in the equation above can be written as a sum of 4 terms, each is a multiple of 4 terms, taking the same form as in Lemma 4.5. Moreover, each of these summands contains at least one multiple that has uniform norm $\leq \frac{1}{F(Q, R, \varepsilon)}$. Therefore, we deduce that

$$|B(N, N') - A_1(N, N')| \leq \|G\|_{op} \cdot \frac{4c_2}{F(Q, R, \varepsilon)^{\frac{1}{2}}} + \frac{8}{\tilde{N}}.$$

Using the adjoint, we can also write

$$B(N, N') = \int \left(\mathbb{E}_{m', n' \in \mathbb{Z}/\tilde{N}'\mathbb{Z}} 1_{[N']}(n) \cdot \xi_{N', R(m', n')}^{s, e} \right) \cdot \left(\mathbb{E}_{(m, n) \in \mathbb{Z}/\tilde{N}\mathbb{Z}} G^* (1_{[N]}(n) \cdot \xi_{N, R(m, n)}) \right) d\lambda$$

and obtain the estimate $|A(N, N') - B(N, N')| \leq \frac{4c_2 \cdot \|G\|_{op}}{F(Q, R, \varepsilon)^{\frac{1}{2}}} + \frac{8}{\tilde{N}'}$ using the exact same argument as above (and using the well known fact that $\|G\|_{op} = \|G^*\|_{op}$). By the triangle inequality we deduce that

$$(7) \quad |A(N, N') - A_1(N, N')| < \frac{8c_2 \cdot \|G\|_{op}}{F(Q, R, \varepsilon)^{\frac{1}{2}}} + \frac{8}{\tilde{N}'} + \frac{8}{\tilde{N}}.$$

We now work with $A_1(N, N')$. We want to eliminate the error term, but first, we have to pass to an average over a sub-progression related to the property of $\chi_{N, st}$ in Theorem 3.8. Let $\eta := \frac{\varepsilon}{QR}$, by Lemma 4.6, we have

$$\begin{aligned} & \sum_{m, n \in \mathbb{Z}/\tilde{N}\mathbb{Z}} \sum_{m', n' \in \mathbb{Z}/\tilde{N}'\mathbb{Z}} \int_{\mathcal{M}} G(1_{[N']}(n') \xi_{R'(m', n')}^{s, e}) \cdot 1_{[N]}(n) \xi_{R(m, n)}^{s, e} d\lambda \geq \\ & \sum_{m \in \mathbb{Z}/\tilde{N}\mathbb{Z}} \sum_{n=1}^{\lfloor \eta N \rfloor} \sum_{m' \in \mathbb{Z}/\tilde{N}'\mathbb{Z}} \sum_{n'=1}^{\lfloor \eta N' \rfloor} \int_{\mathcal{M}} G(\xi_{N', \tilde{R}'(m', n')}^{s, e}) \cdot \xi_{N, \tilde{R}(m, n)}^{s, e} d\lambda. \end{aligned}$$

Where $\tilde{R}(m, n) = \frac{(m+l_2 Qn)(m+l_3 Qn)}{m(m+l_1 Qn)}$ and $\tilde{R}'(m', n') = \frac{(m'+l_6 Qn')(m'+l_7 Qn')}{(m'+l_4 Qn')(m'+l_5 Qn')}$. Indeed, the summands associated with $n > N$ or $n' > N'$ in the first term are zero. For the rest of the terms we notice that in smaller term we have less summands and so the inequality follows from Lemma 4.6. We denote

$$A_2(N, N') := \mathbb{E}_{m \in \mathbb{Z}/\tilde{N}\mathbb{Z}} \mathbb{E}_{n \leq \lfloor \eta N \rfloor} \mathbb{E}_{m' \in \mathbb{Z}/\tilde{N}'\mathbb{Z}} \mathbb{E}_{n' \leq \lfloor \eta N' \rfloor} \int_{\mathcal{M}} G(\xi_{N', \tilde{R}'(m', n')}^{s, e}) \cdot \xi_{N, \tilde{R}(m, n)}^{s, e} d\lambda.$$

By the inequality above we have

$$(8) \quad \begin{aligned} A_1(N, N') & \geq \frac{\lfloor \eta N \rfloor}{\tilde{N}} \frac{\lfloor \eta N' \rfloor}{\tilde{N}'} \cdot A_2(N, N') \\ & \geq \frac{\eta^2}{160 \cdot l} \cdot A_2(N, N') \\ & = \frac{\varepsilon^2}{160 \cdot l \cdot Q^2 \cdot R^2} \cdot A_2(N, N'). \end{aligned}$$

We therefore work with $A_2(N, N')$ from now on. Let,

$$A_3(N, N') := \mathbb{E}_{m \in \mathbb{Z}/\tilde{N}\mathbb{Z}} \mathbb{E}_{n \leq \lfloor \eta N \rfloor} \mathbb{E}_{m' \in \mathbb{Z}/\tilde{N}'\mathbb{Z}} \mathbb{E}_{n' \leq \lfloor \eta N' \rfloor} \int_{\mathcal{M}} G(\xi_{N', \tilde{R}'(m', n')}^s) \cdot \xi_{N, \tilde{R}(m, n)}^s d\lambda.$$

Namely, $A_3(N, N')$ is obtained by replacing $\xi^{s, e}$ with ξ^s in $A_2(N, N')$. To estimate $|A_2(N, N') - A_3(N, N')|$ we use a similar argument as we used to get

(7), but here we have to rely on another general estimate involving the L^1 norm. By the Cauchy-Schwartz inequality we have

$$\left| \int_{\mathcal{M}} G(\phi)\psi d\lambda \right| \leq \|G\|_{op} \cdot \|\phi\|_{L^2(\lambda)} \|\psi\|_{L^2(\lambda)}.$$

Since the inequality $\|\cdot\|_{L^2} \leq \sqrt{\|\cdot\|_{L^1} \cdot \|\cdot\|_{L^\infty}}$ holds in all probability spaces we deduce that

$$(9) \quad \left| \int_{\mathcal{M}} G(\phi)\psi d\lambda \right| \leq \|G\|_{op} \cdot \sqrt{\|\phi\|_{L^1(\lambda)} \cdot \|\phi\|_{L^\infty(\lambda)} \cdot \|\psi\|_{L^1(\lambda)} \cdot \|\psi\|_{L^\infty(\lambda)}}.$$

Recall that by Theorem 3.8, $\|\xi_{N,n}^e\|_{L^1(\lambda)} < \varepsilon$. Again we introduce an intermediate term

$$B_2(N, N') := \mathbb{E}_{m \in \mathbb{Z}/\tilde{N}\mathbb{Z}} \mathbb{E}_{n \leq \lfloor \eta N \rfloor} \mathbb{E}_{m' \in \mathbb{Z}/\tilde{N}'\mathbb{Z}} \mathbb{E}_{n' \leq \lfloor \eta N' \rfloor} \int_{\mathcal{M}} G(\xi_{\tilde{R}'(m', n')}^{s,e}) \cdot \xi_{\tilde{R}(m,n)}^s d\lambda.$$

Since all the ξ 's are 1-bounded, (9) implies that

$$|A_2(N, N') - B_2(N, N')| \leq \|G\|_{op} \mathbb{E}_{m \in \mathbb{Z}/\tilde{N}\mathbb{Z}} \mathbb{E}_{n \leq \lfloor \eta N \rfloor} \sqrt{\|\xi_{\tilde{R}(m,n)}^{s,e} - \xi_{\tilde{R}(m,n)}^s\|_{L^1}}.$$

Now we can write $\xi_{\tilde{R}(m,n)}^{s,e} - \xi_{\tilde{R}(m,n)}^s$ as four summands each is a multiple of 4 terms, where all terms are 1-bounded in L^∞ norm, but at least one of them is bounded by ε in L^1 -norm. This gives the estimate

$$|A_2(N, N') - B_2(N, N')| \leq 4 \cdot \sqrt{\varepsilon} \cdot \|G\|_{op}.$$

Using the adjoint technique as in the previous argument, we also get the bound

$$|B_2(N, N') - A_3(N, N')| \leq 4 \sqrt{\varepsilon} \cdot \|G^*\|_{op},$$

and so by the triangle inequality we have

$$(10) \quad |A_2(N, N') - A_3(N, N')| < 8 \sqrt{\varepsilon} \cdot \|G\|_{op}.$$

It is left to estimate $A_3(N, N')$. Now that we are left with the structure term we can use the periodicity. Recall that $\xi_{N, \tilde{R}(m,n)}^s(\chi) = \overline{\chi_N^s(m)} \cdot \overline{\chi_N^s(m + l_1 Qn)} \cdot \chi^s(m + l_2 Qn) \cdot \chi^s(m + l_3 Qn)$. By the property of ξ^s in Theorem 3.8, we see that for every $1 \leq k \leq \lfloor \eta N \rfloor$,

$$\|\xi_{N, \tilde{R}(m,n)}^s - |\xi_{N, \tilde{R}(m,n)}^s|^4\|_\infty \leq 4 \cdot l \cdot k \cdot N \cdot \frac{R}{\tilde{N}} \leq 4 \cdot l \cdot \eta \cdot N \cdot \frac{R}{\tilde{N}} \leq \frac{4 \cdot \varepsilon}{Q}.$$

Here in the last estimate we used $\tilde{N} > l \cdot N$. Let $A_4(N, N') := \int_{\mathcal{M}} G(\mathbb{E}_{m' \in \mathbb{Z}/\tilde{N}'\mathbb{Z}} |\xi_{N, m'}^s|^4) \cdot \mathbb{E}_{m \in \mathbb{Z}/\tilde{N}\mathbb{Z}} |\xi_{N, m}^s|^4 d\lambda$, by the Cauchy-Schwartz inequality we have

$$(11) \quad |A_3(N, N') - A_4(N, N')| \leq 16 \cdot \|G\|_{op} \cdot \frac{\varepsilon}{Q} \leq 16 \cdot \varepsilon \cdot \|G\|_{op}.$$

By (3), $\int_{\Sigma} G(\phi) \cdot \psi d\lambda = \int_X T_{\phi}(T_{\psi} f \cdot f) \cdot f d\mu$, where $f = 1_A$ is non-negative. Therefore, the left hand side is positive whenever ϕ, ψ are positive. We deduce that G is a positive operator (sends non-negative functions to non-negative functions λ -a.e.). Thus, we have the inequality

$$(12) \quad A_4(N, N') \geq \mathbb{E}_{m \in \mathbb{Z}/\tilde{N}\mathbb{Z}} \mathbb{E}_{m' \in \mathbb{Z}/\tilde{N}'\mathbb{Z}} G(\mathbb{E}_{m' \in \mathbb{Z}/\tilde{N}'\mathbb{Z}} |\xi_{N, m'}^s|^4)(\mathbf{1}) \cdot |\mathbf{1}_{N, m}^s|^4 \cdot \lambda(\{\mathbf{1}\}),$$

where $\mathbf{1}$ denotes the multiplicative function that is equal to the constant 1. Recall that $\mathbf{1}_N^s$ is the convolution of $\mathbf{1}_N$ with a kernel. Therefore

$$\mathbb{E}_{m \in \mathbb{Z}/\tilde{N}\mathbb{Z}} \mathbf{1}_N^s(\chi) = \frac{N}{\tilde{N}}$$

for all $\chi \in \mathcal{M}$. Therefore, by Jensen's inequality,

$$(13) \quad \mathbb{E}_{m \in \mathbb{Z}/\tilde{N}\mathbb{Z}} |\mathbf{1}_{N, m}^s|^4 \geq |\mathbb{E}_{m \in \mathbb{Z}/\tilde{N}\mathbb{Z}} \mathbf{1}_{N, m}^s|^4 \geq \left(\frac{N}{\tilde{N}}\right)^4 \geq \frac{1}{20^4 \cdot l^4}.$$

For the sake of simplicity of notation we let $C := \frac{1}{20^4 \cdot l^4}$.

Recall that $\int_{\Sigma} G(\phi) \cdot \psi d\mu = \int_X T_{\phi}(T_{\psi} f \cdot f) \cdot f d\mu$, if we plug in $\psi = \delta_{\mathbf{1}}$, the indicator of $\{\mathbf{1}\}$, we get that $G(\phi)(\mathbf{1}) \cdot \lambda(\{\mathbf{1}\}) = \int_X f d\mu \cdot \int_X T_{\phi} f \cdot f d\mu = \mu(A) \cdot \int_X \phi d\nu_f$ where ν_f is the spectral measure for f . If we now take $\phi = \mathbb{E}_{m \in \mathbb{Z}/\tilde{N}'\mathbb{Z}} \xi_{N, m'}^s$ we get that

$$G(\phi)(\mathbf{1}) \cdot C \cdot \lambda(\{\mathbf{1}\}) = C \cdot \mu(A) \cdot \int_{\Sigma} \phi d\nu_A \geq C \cdot \mu(A) \phi(\mathbf{1}) \nu_A(\mathbf{1}) = C^2 \cdot \mu(A)^2 = C^2 \delta^2.$$

Combining this with (7), (8), (10) and (11) we get that $A(N, N')$ is bounded below by

$$\frac{\varepsilon}{160 \cdot 20^4 \cdot l^4 \cdot Q^2 \cdot R^2} \left(\delta^2 - 16 \cdot \|G\|_{op} \cdot \varepsilon - 8 \cdot \sqrt{\varepsilon} \|G\|_{op} \right) - \left(\frac{8c_2 \cdot \|G\|_{op}}{F(Q, R, \varepsilon)^{\frac{1}{2}}} + \frac{8}{\tilde{N}'} + \frac{8}{\tilde{N}} \right).$$

Recall that $\varepsilon = c_3 \cdot \delta^4$ for some positive constant that we did not chose yet.

Now take $c_3 < 1$ sufficinelty small, so that

$$\frac{\left(\delta^2 - 16 \cdot \|G\|_{op} \cdot \varepsilon - 8 \cdot \sqrt{\varepsilon} \|G\|_{op} \right)}{160 \cdot 20^4 \cdot l^4} > c_5 \delta^2$$

for some positive constant c_5 depending only on l . Therefore,

$$A(N, N') \geq \frac{c_5}{R^2 \cdot Q^2} \cdot \delta^4 - \left(\frac{8c_2 \cdot \|G\|_{op}}{F(Q, R, \varepsilon)^{\frac{1}{2}}} + \frac{8}{\tilde{N}'} + \frac{8}{\tilde{N}} \right).$$

Now, $F(Q, R, \varepsilon) = c_4^2 \cdot \frac{Q^4 R^4}{\varepsilon}$ where c_4 was not specified. Taking

$$c_4 := 8 \cdot \frac{c_2 \cdot c_3 \cdot \|G\|_{op}}{c_5},$$

we conclude that

$$A(N, N') \geq \frac{c_5}{R^4 \cdot Q^4} \cdot \delta^4 - \left(\frac{c_5 \delta^8}{Q^4 R^4} + \frac{8}{\tilde{N}'} + \frac{8}{\tilde{N}} \right) \geq \frac{c_5}{R^4 \cdot Q^4} (\delta^4 - \delta^8) - \frac{8}{\tilde{N}} - \frac{8}{\tilde{N}'}.$$

The last two terms go to zero as N, N' goes to infinity. On the other hand, the first term is bounded by some constant depending only on δ . This proves that the term appearing in Lemma 4.4 is positive and so the proof is now complete.

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