88-914 Solitons

Admin:

1. Course Time: Thursday 14:00-16:00
2. My office hours Tuesday 14:00-16:00, can also speak to me after class
3. Grade on basis of combination of exercise work and exam
4. Prerequisites: Undergrad courses in complex variables, ODE, PDE and numerical methods
5. Syllabus: See first lecture.

Lecture 1: Physical Phenomena in Simple PDEs; Content of this Course

Almost throughout the course: we will look at simple PDEs for a single function $u(x,t)$ of one space variable x and one time variable t. You should think of the PDEs as describing the time evolution of a function u(x).

1. **The Wave Equation and the Transport Equation.** The wave equation:

$$u_{tt} = c^2 u_{xx} \ , \qquad c > 0.$$

D'Alembert solution:

$$u(x,t) = f(x+ct) + g(x-ct)$$

f, g arbitrary functions. f a *left-moving* wave, g a *right-moving* wave, see illustrations 1 and 2. A *transport phenomenon*, the initial form/shape does not change with time (except for mixing of f and g).
The pure transport equation:

$$u_t = cu_x$$

General solution $u = f(x+ct)$. $c > 0$ left moving only, $c < 0$ right moving only.

2. **The Heat/Diffusion Equation.**

$$u_t = \kappa u_{xx} \ , \qquad \kappa > 0$$

(we assume inifnite spatial interval, can also look at finite interval with different boundary conditions). Assuming L^1 initial condition $u(x,0) = f(x)$, the solution is

$$u(x,t) = \int_{-\infty}^{\infty} G(x-y,t) f(y) dy$$

where the *Green's function* $G(x,t)$ is

$$G(x,t) = \frac{1}{2\sqrt{\pi\kappa t}} \ e^{-x^2/4\kappa t} \ .$$

The Green's function is the solution for initial condition $f(x) = \delta(x)$. See illustrations 3 and 4. A *dissipation phenomenon*: solution tends to zero (with only a moderate amount of spreading) as t increases.

3. **A third order linear PDE.** Let us now look at the third order equation

$$u_t = au_x + bu_{xxx} \ ,$$

where a, b are constants with $b > 0$, and we assume an L^1 initial condtion $u(x,0) = f(x)$. We solve using the Fourier transform:

$$\begin{aligned} U(p,t) &= \int_{-\infty}^{\infty} u(x,t)e^{ipx}\, dx \ , \\ u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} U(p,t)e^{-ipx}\, dp \ . \end{aligned}$$

Taking the FT of the equation we have

$$U_t = (-iap + ibp^3)U \ ,$$

and thus

$$U(p,t) = e^{it(bp^3-ap)}F(p),$$

where F is the FT of the initial data $f(x)$. Doing the inverse transform we have

$$u(x,t) = \int_{-\infty}^{\infty} G(x-y,t)f(y)dy$$

where the *Green's function* $G(x,t)$ is

$$\begin{aligned} G(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it(bp^3-ap)}e^{-ipx}\, dp \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(bp^3t-p(x+at))}\, dp \\ &= \frac{1}{2\pi(3bt)^{1/3}}\int_{-\infty}^{\infty} e^{i(q^3/3-q(x+at)/(3bt)^{1/3})}\, dq \qquad \text{(substitute } p = q/(3bt)^{1/3}\text{)} \\ &= \frac{1}{(3bt)^{1/3}}\mathrm{Ai}\left(-\frac{x+at}{(3bt)^{1/3}}\right) \ , \end{aligned}$$

where

$$\mathrm{Ai}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(q^3/3+zq)}dq = \frac{1}{\pi}\int_{-\infty}^{\infty} \cos\left(\frac{q^3}{3}+zq\right)dq$$

is the *Airy function*. (Note: the integral converges for all real z.) Without loss of generality take $b = \frac{1}{3}$ (otherwise rescale time). a plays the role of the speed (no surprise), and the solution looks different for a positive and negative. Note (easy to check) that the Airy function $\mathrm{Ai}(z)$ satisfies the differential equation

$$\frac{d^2y}{dz^2} = zy \ .$$

A plot of the Airy function (AiryAi(x) in Maple) appears in illustration 5. And an animation of the solution $G(x,t)$ of the PDE, for $b = \frac{1}{3}$, $a = 1$, appears in illustration 6. There are 3 effects you should notice: there is some overall dissipation, and there is transport to the left with speed about 1, but there is a new effect, a wave spreading to the right. This new effect is called *dispersion* and is familiar as the ripple spreading out on a pond when I throw in a stone.

You should note that dispersion is different from dissipation and transport (consider the equation $u_t = u_{xx} + u_x$ for the latter). We could have demonstrated dispersion on the simpler equation $u_t = u_{xxx}$, but there is little lost in adding the extra u_x term.

4. **A nonlinear equation.** (Sometimes called: Burger's equation, Euler's equation in 1d, the shock equation, the nonlinear transport equation etc.) Consider the first-order nonlinear (quasilinear) PDE

$$u_t = uu_x \ .$$

You can solve this by the method of characteristics, the general solution is given by a solution of the equation

$$u(x,t) = f(x + u(x,t)t) \ .$$

(Check: Differentiating the above we have

$$\begin{aligned} u_t &= f'(u_t t + u) \\ u_x &= f'(1 + u_x t) \end{aligned}$$

So

$$u_t - uu_x = f'\left[(u_t t + u) - u(1 + u_x t)\right] = f't(u_t - uu_x) \ ,$$

which implies $u_t - uu_x = 0$ as it is not possible that $f't = 1$ for all x and t.) Note that $u(x,0) = f(x)$, so $f(x)$ is the initial condition. We see in the above that u plays the role of the speed in this nonlinear wave. As a result of this, if we start with an initial condition with $u > 0$, at the places where u is higher the wave will move faster. This leads to steepening of the wave, and after a certain critical time the function $u(x,t)$ will not be uniquely defined. See illustration 7 for the case $f(x) = \text{sech}^2(x)$. You can confirm the picture by considering (analytically) for what t's the equation $u = \text{sech}^2(x + ut)$ has a unique solution (for all x).

This new phenomenon we are seeing here — in which u_x becomes infinite in a finite time — is called *blow–up.*

Summary of phenomena: transport, dissipation, dispersion, blow–up.

PDEs with combinations of terms (and other terms we have not yet seen) can have a great variety of different behaviors. This course will concern itself with one particular interesting phenomenon that occurs in a variety of nonlinear PDEs, that of *solitary waves* or *solitons*. A soliton solution of a PDE is a solution that (for any fixed t) is *of finite spatial extent* in the sense that $u \to 0$ (sufficiently quickly) as $|x| \to \infty$, and that is transported (not necessarilly with fixed speed) without change of shape as t increases.

If we are dealing with an equation that is *translation invariant* (x does not appear explicitly in the equation, implying that if $u(x,t)$ is a solution, then so is $u(x-a,t)$ for any a), then we can consider a solution of the PDE that starts from an initial condition that looks like 2 (or more) well-separated solitons. The question is: what happens to this solutions as t increases? There are two possibilities, either that the solitons remain intact for all t, or that (due to collisions) at some time they lose their integrity. In the former case we say we are dealing with *integrable solitons*, in the latter case *nonintegrable solitons*. Note: some authors require integrability as a part of the definition of solitons as opposed to more general solitary waves. We will try to be specific when necessary.

This course concerns itself with integrable soliton equations. In fact we will focus just on 2 equations the *Korteweg-de Vries equation* (KdV) and the *Camassa-Holm equation* (CH). We shall learn two methods frequently applied to equations that support integrable solitons, the *inverse scattering transform* (IST) and *loop group methods*. As such we shall only be covering a small corner of the theory of soliton equations, which itself is only a small corner of the theory (and applications) of nonlinear PDE. But this corner of a corner contains a lot of material, as we shall see.

The book by Drazin and Johnson on Solitons is strongly recommended.