Cubic Splines

Formulas for the cubic spline interpolating the data $\{y_i\}$ at the points $\{x_i\}$ $(i = 0, \ldots, n)$, $x_0 < \ldots < x_n$. On the interval $[x_{i-1}, x_i]$

$$f(x) = \frac{x - x_{i-1}}{x_i - x_{i-1}} y_i + \frac{x - x_i}{x_{i-1} - x_i} y_{i-1} + \frac{(x - x_i)(x - x_{i-1})}{6(x_i - x_{i-1})} \left((x + x_i - 2x_{i-1})C_i - (x + x_{i-1} - 2x_i)C_{i-1}\right)$$

The coefficients C_i are determined by the requirements

$$(x_i - x_{i-1})C_{i-1} + 2(x_{i+1} - x_{i-1})C_i + (x_{i+1} - x_i)C_{i+1} = 6\left(\frac{y_{i+1} - y_i}{x_{i+1} - x_i} - \frac{y_i - y_{i-1}}{x_i - x_{i-1}}\right) \qquad i = 1, \ldots n-1$$

along with the boundary conditions $C_0 = C_n = 0$ (for the natural spline).

Sample Problem. Interpolate tha data

x_i	0	3	4	5	6	8
y_i	0	1.2	1.4	1.3	1.3	0

Equations for C_i:

$$\begin{aligned} 8C_1 + C_2 &= -1.2 \\ C_1 + 4C_2 + C_3 &= -1.8 \\ C_2 + 4C_3 + C_4 &= 0.6 \\ C_3 + 6C_4 &= -3.9 \end{aligned}$$

Solution (4 s.f.):

$C_1 = -0.08165 \qquad C_2 = -0.5468 \qquad C_3 = 0.4687 \qquad C_4 = -0.7281$

The spline function (10 s.f.):

$$f(x) = \begin{cases} -0.004536340853\,x^3 + 0.4408270677\,x & 0 \le x \le 3 \\ -0.07751879702\,x^3 + 0.6568421055\,x^2 - 1.529699249\,x + 1.970526317 & 3 \le x \le 4 \\ 0.1692481204\,x^3 - 2.304360904\,x^2 + 10.31511279\,x - 13.82255640 & 4 \le x \le 5 \\ -0.1994736842\,x^3 + 3.226466167\,x^2 - 17.33902255\,x + 32.26766919 & 5 \le x \le 6 \\ 0.06067669173\,x^3 - 1.456240602\,x^2 + 10.75721804\,x - 23.92481203 & 6 \le x \le 8 \end{cases}$$

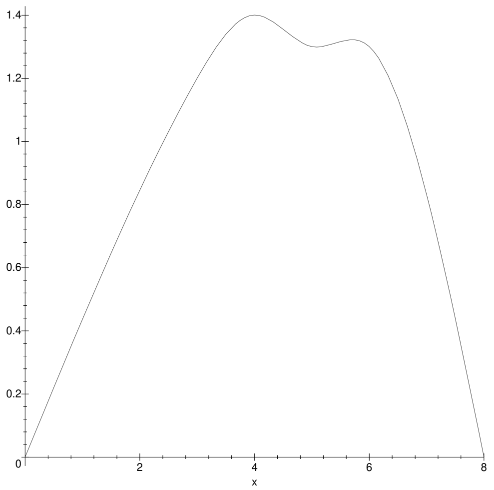

Figure 1: Plot of $f(x)$

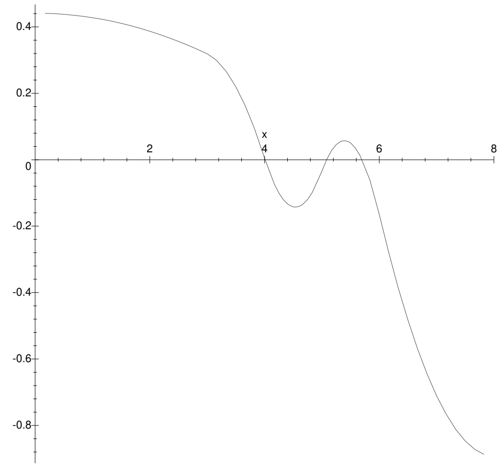

Figure 2: Plot of $f'(x)$

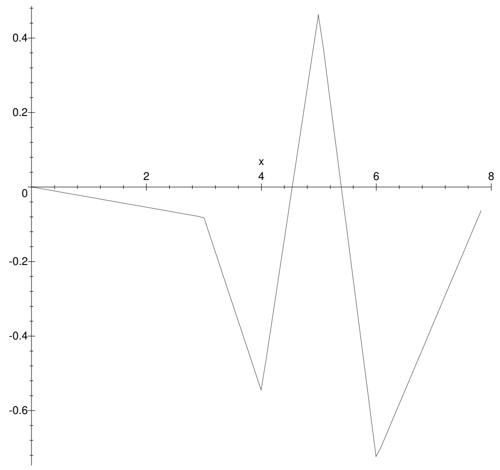

Figure 3: Plot of $f''(x)$