Polynomial Interpolation: Example

Problem: find a quartic polynomial interpolating the following data:

x	-2	-1	0	1	2
y	0	$\frac{1}{4}$	1	$\frac{3}{4}$	$\frac{2}{5}$

Direct approach: write the polynomial $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$;
to find $a_0, \ldots a_4$ solve the system

$$\begin{pmatrix} 1 & -2 & 4 & -8 & 16 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{4} \\ 1 \\ \frac{3}{4} \\ \frac{2}{5} \end{pmatrix}.$$

This gives

$$p(x) = 1 + \tfrac{3}{10}x - \tfrac{3}{5}x^2 - \tfrac{1}{20}x^3 + \tfrac{1}{10}x^4$$

Lot of work to get, but easy to use in the end.

Lagrange method:

$$\begin{aligned} p(x) &= 0 \cdot \frac{(x+1)x(x-1)(x-2)}{(-2+1)(-2)(-2-1)(-2-2)} + \frac{1}{4} \cdot \frac{(x+2)x(x-1)(x-2)}{(-1+2)(-1)(-1-1)(-1-2)} \\ &+ 1 \cdot \frac{(x+2)(x+1)(x-1)(x-2)}{(0+2)(0+1)(0-1)(0-2)} + \frac{3}{4} \cdot \frac{(x+2)(x+1)x(x-2)}{(1+2)(1+1)(1)(1-2)} \\ &+ \frac{2}{5} \cdot \frac{(x+2)(x+1)x(x-1)}{(2+2)(2+1)(2)(2-1)} \end{aligned}$$

This is the same as above! Easy to get, but a lot of work to use.

Newton method: form table of divided differences

-2	0				
-1	$\frac{1}{4}$	$\frac{1}{4}$			
0	1	$\frac{3}{4}$	$\frac{1}{4}$		
1	$\frac{3}{4}$	$-\frac{1}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	
2	$\frac{2}{5}$	$-\frac{7}{20}$	$-\frac{1}{20}$	$\frac{3}{20}$	$\frac{1}{10}$

So

$$p(x) = 0 + \tfrac{1}{4}(x+2) + \tfrac{1}{4}(x+2)(x+1) - \tfrac{1}{4}(x+2)(x+1)x + \tfrac{1}{10}(x+2)(x+1)x(x-1)$$

or in nested form

$$p(x) = (x+2)\left(\tfrac{1}{4} + (x+1)\left(\tfrac{1}{4} + x\left(-\tfrac{1}{4} + (x-1)\left(\tfrac{1}{10}\right)\right)\right)\right)$$

Moderately easy to get, easy to use; also easy to add points.