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Enkay Pairs: Solutions of an Inequality Arising in a Comparison of Graphs

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Abstract

We define an *enkay pair* to be a pair of positive integers (n, k) such that $(n-1)(n-1)! < 2^k < n!$. This inequality arises when comparing a hypercube with the smallest larger star graph. We list the 15 enkay pairs with $n \leq 10^6$, and speculate on whether the number of enkay pairs is finite or infinite.

An ancient, and apparently everlasting, pursuit of mathematicians is the definition of special kinds of numbers (such as pefect numbers or Mersenne primes), or maybe small sets of numbers (such as amicable pairs or Pythagorean triples), and the subsequent investigation of their properties. In this paper we introduce what we call — for want of a better name — *enkay pairs*, and present the little we know about them. We are not currently aware of any possible interest in enkay pairs beyond mathemtical curiosity, but since we originally arrived at the notion in a (subsequently aborted) comparison of two kinds of communication networks, we present them in this context.

The *star graph* [1] and the *hypercube* are two popular, frequently compared [2], models for communication or processor interconnection networks. The star graph S_n is the graph whose nodes are permutations of the first n naturals; two nodes are joined by an edge if they differ in their action, as permutations, on just two of the first n naturals, one of which is 1 (thus S_n has $n!$ nodes, each with $n-1$ nearest neighbors). The hypercube H_k is the graph whose nodes are points of $(\mathbf{Z}_2)^k$, i.e. $k-$tuples of zeros and ones; two nodes are joined by an edge if, as $k-$tuples, they differ in just one place (thus H_k has 2^k nodes, each with k nearest neighbors).

Let a hypercube H_k be given, and suppose a certain task can be accomplished *either* using the smallest star graph S_n with more nodes than H_k, *or* using the hypercube H_k and the star graph S_{n-1}. Which involves less nodes? Evidently, using the star graph S_n will be preferable if $n! < 2^k + (n-1)!$, or, equivalently, $(n-1)(n-1)! < 2^k$. These considerations motivate the following definition:

Definition. An *enkay pair* is a pair of positive integers (n, k) such that

$$(n-1)(n-1)! \ < \ 2^k \ < \ n! \ . \tag{1}$$

Proposition 1. (a) For any positive integer n there exists at most one positive integer k such that (n, k) is an enkay pair. (b) For any positive integer k there exists at most one positive integer n such that (n, k) is an enkay pair.

Proof. (a) Without loss of generality we can assume $n \geq 2$, as $(1, k)$ is not an enkay pair for any positive integer k. For $n \geq 2$ let k be the smallest positive integer such that (n, k) is an enkay pair, if any exist. Then for all integers $k' > k$

$$2^{k'} \ \geq \ 2.2^k \ > \ 2(n-1)(n-1)! \ > \ n! \ ,$$

so (n, k') is not an enkay pair. That is, for a given n, if a k exists such that (n, k) is an enkay pair, then it is unique. (b) Similarly, given a positive integer k, let n be the smallest positive integer such that (n, k) is an enkay pair, if any exist. Then for all integers $n' > n$

$$(n'-1)(n'-1)! \ \geq \ n! \ > \ 2^k \ ,$$

so (n', k) is not an enkay pair. •

Proposition 2. Given a hypercube H_k, let S_n be the smallest star graph with more nodes than H_k. S_n has less nodes than the union of H_k and S_{n-1} if and only if (n, k) is an enkay pair.

Proof. The star graph S_n has more nodes than the hypercube H_k if and only if $2^k < n!$. S_n has less nodes than the union of H_k and S_{n-1} if and only if $(n-1)(n-1)! < 2^k$, as explained before. It remains to show that if (n, k) is an enkay pair then S_n is the *smallest* star graph larger than H_k. Suppose S_{n-1} is also larger than H_k. Then $(n-1)! > 2^k > (n-1)(n-1)!$, so $n = 1$, a contradiction. •

Note. The same problem with the roles of the hypercube and the star graph reversed is trivial. Given a star graph S_n, let H_k be the smallest hypercube with more nodes than S_n.

If $n = 1, 2$ then H_k has the same number of nodes as the union of S_n and H_{k-1}. Otherwise, H_k always has less nodes (as S_n has more nodes than H_{k-1}, which has half as many nodes as H_k).

Examples. The following is a list of all the enkay pairs we know:

n	k
24	79
62	284
65	302
146	844
176	1064
254	1668
257	1692
841	6964
959	8122
5707	62989
10710	127928
11370	136792
95868	1448198
235928	3870478
837752	15275136

We believe this is a comprehensive list of all enkay pairs (n, k) with $n \leq 10^6$.

The next two propositions, which are a little technical, will be used in the subsequent discussion to formulate a conjecture about the number of enkay pairs. The basic idea is that we would like to replace the inequality (1) for 2^k by an inequality for k, which will be easier to interpret.

Proposition 3. There exists a positive, decreasing function h, with $\lim_{N\to\infty} h(N) = 0$, such that if (n, k) is an enkay pair with $n \geq N$ (N a positive integer), then

$$(n+\tfrac{1}{2})\log n - n + \tfrac{1}{2}\log 2\pi - \frac{11+h(N)}{12n} < k\log 2 < (n+\tfrac{1}{2})\log n - n + \tfrac{1}{2}\log 2\pi + \frac{1}{12n}\ . \quad (2)$$

Proof. Stirling's series for $\log\Gamma(x)$ gives

$$(n+\tfrac{1}{2})\log n - n + \tfrac{1}{2}\log 2\pi + \frac{1}{12n} - \frac{1}{360n^3} < \log n! < (n+\tfrac{1}{2})\log n - n + \tfrac{1}{2}\log 2\pi + \frac{1}{12n} \quad (3)$$

for all positive integer n [3]. The right hand inequality in the proposition can be obtained by taking the logarithm of the right hand side of (1) and applying the right hand inequality in (3). To obtain the left hand inequality in the proposition we take the logarithm of the left hand side of of (1), apply the left hand inequality in (3) with n replaced by $(n-1)$ (as explained above, we can assume $n \geq 2$), and after some elementary manipulations the resulting inequality can be written in the form

$$k \log 2 \;>\; (n+\tfrac{1}{2}) \log n - n + \tfrac{1}{2} \log 2\pi - \frac{11 + h(n)}{12n} \;, \tag{4}$$

$$\text{where} \quad h(n) \;=\; -12n(n+\tfrac{1}{2}) \left(\log(1-\tfrac{1}{n}) + \frac{1}{n} + \frac{1}{2n^2} \right) + \frac{2n-3}{n(n-1)} + \frac{1}{30n^2(1-\frac{1}{n})^3} \;.$$

It is straightforward to check that each of the three terms of $h(n)$ is a positive, decreasing function of n for $n \geq 3$ (we can assume this, as there are no enkay pairs with $n = 2$). Furthermore each one tends to zero as $n \to \infty$. Thus for $n \geq N$ we obtain the LHS inequality in the proposition. •

Note. In fact $h(n)$ as defined in the above proof has a Taylor series in powers of $1/n$ convergent for $1/n < 1$. Explicitly $h(n) = \frac{6}{n} + \frac{121}{30n^2} + \frac{3}{n^3} + \ldots$. $h(n)$ is already small for moderate n, and in particular $h(100) < 0.1$. We used the above proposition with $N = 100$ and $h(N)$ replaced by 0.1 to search for candidate enkay pairs.

Proposition 4. Suppose that $n \geq N$ (N a positive integer) and

$$(n+\tfrac{1}{2}) \log n - n + \tfrac{1}{2} \log 2\pi - \frac{11}{12n} < k \log 2 < (n+\tfrac{1}{2}) \log n - n + \tfrac{1}{2} \log 2\pi + \frac{1 - \frac{1}{30N^2}}{12n} \;. \tag{5}$$

Then (n, k) is an enkay pair.

Proof. The LHS of (3) implies

$$(n+\tfrac{1}{2}) \log n - n + \tfrac{1}{2} \log 2\pi < \log n! - \frac{1}{12n} + \frac{1}{360n^3} \;.$$

Using this and the RHS of the inequality in the proposition gives

$$k \log 2 < \log n! - \frac{1}{360n} \left(\frac{1}{N^2} - \frac{1}{n^2} \right) < \log n! \;, \qquad \text{since} \;\; n \geq N \;. \tag{6}$$

Taking the RHS of (3) with n replaced by $(n-1)$, adding $\log(n-1)$ to both sides and rearranging gives the inequality

$$(n+\tfrac{1}{2}) \log n - n + \tfrac{1}{2} \log 2\pi > \log(n-1)! + \log(n-1) - 1 - \frac{1}{12(n-1)} - (n+\tfrac{1}{2}) \log(1-\tfrac{1}{n}) \;.$$

Using this and the LHS of the inequality in the proposition gives

$$k \log 2 \;>\; \log(n-1)! + \log(n-1) + \frac{g(n)}{12n} \;, \tag{7}$$

$$\text{where} \quad g(n) \;=\; -12n(n+\tfrac{1}{2}) \left(\log(1-\tfrac{1}{n}) + \frac{1}{n} + \frac{2}{n^2} \right) + \frac{2n-3}{n(n-1)} \;.$$

For $n > 1$ both terms in $g(n)$ are positive. Combining (6) and (7) we have

$$\log(n-1)! + \log(n-1) < k \log 2 < \log n! \ ,$$

implying (n,k) is an enkay pair. •

Discussion. The meaning of Propositions 3 and 4 is as follows. It would be nice if we could show that (n,k) is an enkay pair if and only if $k\log 2$ lies in the interval $(s(n)-11/12n, s(n)+1/12n)$ where $s(n) = (n+\frac{1}{2})\log n - n + \frac{1}{2}\log 2\pi$. Unfortunately we cannot prove this, but Proposition 3 states that if (n,k) is an enkay pair then $k\log 2$ must lie in an interval just slightly larger than $(s(n)-11/12n, s(n)+1/12n)$, and Proposition 4 states that if $k\log 2$ lies in an interval just slightly smaller than $(s(n)-11/12n, s(n)+1/12n)$, then (n,k) is an enkay pair. The modifications necessary to the interval $(s(n)-11/12n, s(n)+1/12n)$ become smaller as we restrict to larger n, so we can think of being an enkay pair as being "equivalent in the limit of large n" to $k\log 2$ lying in the interval $(s(n)-11/12n, s(n)+1/12n)$.

This has ramifications for the statistics of enkay pairs. If we assume that for large n the fractional parts of the numbers $s(n)$ are independent and uniformly distributed on the interval $[0,1)$, then the probability of n having an associated enkay pair must behave as $1/n\log 2$. Thus for large N the expectation of the number of enkay pairs with $N \le n \le 2N$ is approximately 1, or equivalently there should be about $\log_2 N$ enkay pairs with $n < N$. This is certainly consistent with the list of enkay pairs given above. In addition, we expect there to exist an infinite number of enkay pairs. A proof of this would be interesting.

The notion of enkay pairs arose from considering when S_n, the smallest star graph with more nodes than a given hypercube H_k, has less nodes than the union of H_k with S_{n-1}. If (n,k) is an enkay pair we can ask if S_n also has less nodes than the union of H_k and S_{n-2}, i.e. whether $n! - (n-2)! < 2^k < n!$. We have yet to find a pair of positive integers satisfying this inequality. Considerations similar to those given for enkay pairs imply we expect only a finite number of such pairs, and it would be interesting to know if there are any at all.

References.

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[2] K.Day and A.Tripathi, *A Comparative Study of Topological Properties of Hypercubes and Star Graphs*, *IEEE Trans. Parallel and Distributed Systems*, **5** (1994) 31-38;
G.De Marco and U.Vaccaro, *Broadcasting in hypercubes and star graphs with dynamic faults*, *Inf.Proc.Lett.* **66** (1998) 321-326.

[3] See, for example, *Handbook of Mathematical Functions*, ed. M.Abramowitz and I.A.Stegun, Dover (1965).