HexaKdV

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Abstract

An analog of the lattice KdV equation of Nijhoff *et al.* is constructed on a hexagonal lattice. The resulting system of difference equations exhibits soliton solutions with interesting local structure: there is a nontrivial phase shift on moving between adjacent lattice sites, with the magnitude of the shift tending to zero in the continuum limit.

Bobenko, Hoffmann and Suris [1] have recently proposed that some effort should be made to understand integrable systems of difference equations on lattices other than $\mathbf{Z}^2$. In addition to being of theoretical interest, it is possible that in certain cases this could have physical significance [2, 3]. This letter examines the analog of Nijhoff *et al.*'s lattice KdV equation [4, 5] on a hexagonal lattice, which we call "hexaKdV" for short. The soliton solutions of hexaKdV are constructed. HexaKdV solitons undergo a nontrivial phase shift on moving between adjacent lattice sites, with the magnitude of the shift tending to zero in the continuum limit.

Nijhoff *et al.*'s lattice KdV equation involves a single field b defined at the vertices of the standard lattice $\mathbf{Z}^2$. The equation gives a relation between the values of the field at the 4 vertices of each fundamental lattice plaquette. Writing b_1 for $b_{n,m}$, b_2 for $b_{n+1,m}$, b_3 for $b_{n+1,m+1}$ and b_4 for $b_{n,m+1}$ (so b_1, b_2, b_3, b_4 are values of the field as we go around a fundamental plaquette), the equation takes the form [5]

$$\frac{b_1 - b_2 - b_3 + b_4}{h} + \frac{-b_2 + b_3 + b_4 - b_1}{k} - b_1 b_2 + b_2 b_3 - b_3 b_4 + b_4 b_1 = 0 \ . \tag{1}$$

Two features of this equation are important in the sequel. The first is that the equation is unchanged on replacing b_1 by b_2, b_2 by b_3, b_3 by b_4, b_4 by b_1, h by k and k by $-h$. This symmetry is associated with rotating the lattice through 90°. Indeed, the equation can be written in the evidently symmetric, and rather more compact, form

$$\left(I - R + R^2 - R^3\right)\left(\frac{b_1 - b_2}{h} - b_1 b_2\right) = 0 \ , \tag{2}$$

where R is the rotation (or replacement) operator. The second important feature to note about standard lattice KdV is that the number of variables and the number of equations is properly balanced. Each equation relates between 4 different values of the field, but each value of the field appears in 4 different equations (as each vertex of the lattice belongs to 4 fundamental plaquettes). Thus lattice KdV is a set of equations that can be expected to yield a solution, given suitable initial/boundary data.

The hexaKdV system which will shortly be constructed also displays a rotation symmetry. Consistent with this, each equation in hexaKdV relates the 6 values of the field at the vertices of each fundamental hexagonal plaquette. But now a problem arises: Each vertex of the lattice evidently belongs to only 3 fundamental plaquettes, so it seems there are twice as many variables as equations (each equation calls 6 variables, but each variable only appears in 3 equations). The simplest imaginable solution to this problem is that there should be *two* equations associated with each plaquette. In fact there is a system of four equations associated with each plaquette, but they are degenerate and should be considered like two.

The hexaKdV system will be constructed on the hexagonal lattice with vertices

$$\{n_1 q + n_2 h\omega + n_3 k\omega^2 \ : \ n_1, n_2, n_3 \in \mathbf{Z}, \ n_1 + n_2 + n_3 = 0 \text{ or } 1 \ \} \ , \tag{3}$$

where $\omega = e^{2i\pi/3}$, and q, h, k are arbitrary positive reals. See figure 1. The vertex $n_1 q + n_2 h\omega + n_3 k\omega^2$ will be referred to in the sequel simply as "the vertex n_1, n_2, n_3". There are 3 kinds of edges, parallel to $1, \omega, \omega^2$ respectively. Following the ideas of [1] and [4], we look for a $GL(2)$-valued function $\Psi_{n_1,n_2,n_3}(\lambda)$ defined on the vertices of the lattice and dependent on the spectral parameter λ, satisfying the following equations, one for each edge in the lattice:

$$\begin{aligned}
\Psi_{n_1+1,n_2,n_3}(\lambda) &= \begin{pmatrix} 1 - q b_{n_1+1,n_2,n_3} & q \\ q\lambda + b_{n_1,n_2,n_3} - b_{n_1+1,n_2,n_3} - q b_{n_1,n_2,n_3} b_{n_1+1,n_2,n_3} & 1 + q b_{n_1,n_2,n_3} \end{pmatrix} \Psi_{n_1,n_2,n_3}(\lambda), \\
\Psi_{n_1,n_2+1,n_3}(\lambda) &= \begin{pmatrix} 1 - h b_{n_1,n_2+1,n_3} & h \\ h\lambda + b_{n_1,n_2,n_3} - b_{n_1,n_2+1,n_3} - h b_{n_1,n_2,n_3} b_{n_1+1,n_2,n_3} & 1 + h b_{n_1,n_2,n_3} \end{pmatrix} \Psi_{n_1,n_2,n_3}(\lambda), \\
\Psi_{n_1,n_2,n_3-1}(\lambda) &= \begin{pmatrix} 1 - k b_{n_1,n_2,n_3-1} & k \\ k\lambda + b_{n_1,n_2,n_3} - b_{n_1,n_2,n_3-1} - k b_{n_1,n_2,n_3} b_{n_1,n_2,n_3-1} & 1 + k b_{n_1,n_2,n_3} \end{pmatrix} \Psi_{n_1,n_2,n_3}(\lambda).
\end{aligned} \tag{4}$$

The consistency condition arising from the two different ways to go around a fundamental plaquette ($n_1, n_2, n_3 \to n_1, n_2 + 1, n_3 \to n_1, n_2 + 1, n_3 - 1 \to n_1 + 1, n_2 + 1, n_3 - 1$ or $n_1, n_2, n_3 \to n_1 + 1, n_2, n_3 \to$

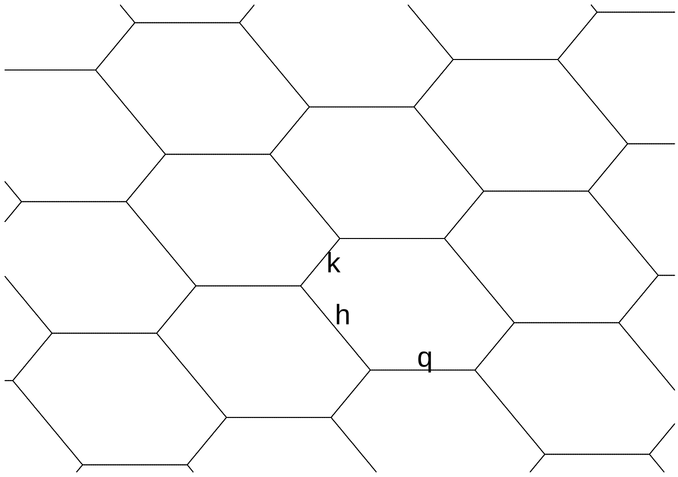

Figure 1: The hexagonal lattice

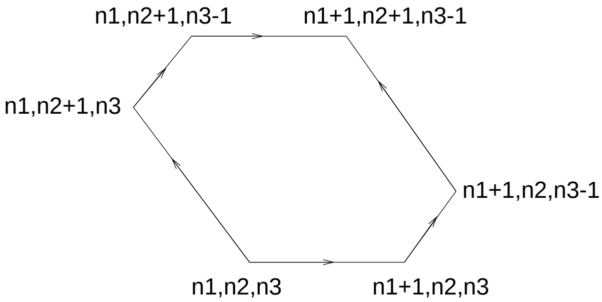

Figure 2: Two ways around a fundamental plaquette

$n_1+1, n_2, n_3-1 \to n_1+1, n_2+1, n_3-1$, see figure 2) can be written

$$\begin{pmatrix} 1-qb_4 & q \\ q\lambda+b_3-b_4-qb_3b_4 & 1+qb_3 \end{pmatrix}\begin{pmatrix} 1-kb_3 & k \\ k\lambda+b_2-b_3-kb_2b_3 & 1+kb_2 \end{pmatrix}\begin{pmatrix} 1-hb_2 & h \\ h\lambda+b_1-b_2-hb_1b_2 & 1+hb_1 \end{pmatrix}$$
$$= \tag{5}$$
$$\begin{pmatrix} 1-hb_4 & h \\ h\lambda+b_5-b_4-hb_5b_4 & 1+hb_5 \end{pmatrix}\begin{pmatrix} 1-kb_5 & k \\ k\lambda+b_6-b_5-kb_6b_5 & 1+kb_6 \end{pmatrix}\begin{pmatrix} 1-qb_6 & q \\ q\lambda+b_1-b_6-qb_1b_6 & 1+qb_1 \end{pmatrix}$$

where here for brevity b_1 stands for b_{n_1,n_2,n_3}, b_2 for b_{n_1,n_2+1,n_3}, b_3 for b_{n_1,n_2+1,n_3-1}, b_4 for b_{n_1+1,n_2+1,n_3-1}, b_5 for b_{n_1+1,n_2,n_3-1} and b_6 for b_{n_1+1,n_2,n_3}, these being the 6 values of b at vertices of a fundamental plaquette.

A symbolic manipulator was used to multiply out (5). Four equations emerge. Since full rotational symmetry has been broken by the choice of a starting and ending point in the fundamental plaquette, the equations are not rotation invariant, but this can be rectified by taking suitable linear combinations and just the final symmetric form of the equations will be presented here. Writing R for the 60°rotation operator, implemented by replacing b_1 by b_2, b_2 by b_3, b_3 by b_4, b_4 by b_5, b_5 by b_6, b_6 by b_1, h by k and k by q and q by $-h$, the equations are:

$$\left(I - R + R^2 - R^3 + R^4 - R^5\right)\left(\frac{b_1 - b_2}{h} - b_1 b_2\right) = 0 \,, \tag{6}$$

$$\left(I + R + R^2 + R^3 + R^4 + R^5\right)\left(\frac{(b_1 - b_2)^2}{h} - b_1 b_2 (b_1 - b_2)\right) = 0 \,, \tag{7}$$

$$\begin{aligned}\left(I - R + R^2 - R^3 + R^4 - R^5\right)\Bigg(\frac{2(b_1 + b_2)}{qk} &- b_1 b_4 (b_2 + b_3) \\ + \frac{b_1 b_3 + b_1 b_4 + b_2 b_3 - b_2 b_5 - b_3 b_4 - b_3 b_5}{h}\Bigg) &= 0 \,,\end{aligned} \tag{8}$$

$$\begin{aligned}&\left(I - R + R^2 - R^3 + R^4 - R^5\right)\Bigg(\frac{2(b_1 + b_2)^2}{qk} + b_1 b_2 (b_1^2 + b_2^2 - b_1 b_2 - 2 b_3 b_4 - b_4 b_1 - b_2 b_5) \\ &- \frac{b_1^3 - b_2^3 + b_2^2 (2b_1 + b_5) - b_1^2 (b_4 + 2b_2) + 2b_3^2 (b_4 - b_2) + 2(b_1 b_2 b_6 + b_2 b_3 b_5 - b_1 b_2 b_3 - b_1 b_3 b_4)}{h}\Bigg) = 0 \,.\end{aligned} \tag{9}$$

Note the similarity of the first equation, (6), with the standard lattice KdV equation (2). An analysis of the hexaKdV system (6)-(9) is quite straightforward with a symbolic manipulator. The first equation, (6), is linear in b_1, and can be used to eliminate b_1 provided its coefficient, $b_6 - b_2 + \frac{1}{h} - \frac{1}{q}$, does not vanish. Proceeding along these lines it can be shown that the solution set consists of two components, characterized by whether or not $b_6 - b_2 + \frac{1}{h} - \frac{1}{q}$ vanishes. One component is a dimension 3 set of solutions given by solutions of the 3 linear equations

$$\left.\begin{aligned} b_6 - b_2 + \tfrac{1}{h} - \tfrac{1}{q} &= 0 \\ b_2 - b_4 + \tfrac{1}{q} + \tfrac{1}{k} &= 0 \\ \left(\tfrac{1}{h^2} - \tfrac{1}{q^2}\right) b_1 + \left(\tfrac{1}{q^2} - \tfrac{1}{k^2}\right) b_3 + \left(\tfrac{1}{k^2} - \tfrac{1}{h^2}\right) b_5 &= \left(\tfrac{1}{q} - \tfrac{1}{h}\right)\left(\tfrac{1}{k} + \tfrac{1}{q}\right)\left(\tfrac{1}{h} + \tfrac{1}{k}\right) \end{aligned}\right\} . \tag{10}$$

The other component is a dimension 4 set of solutions in which b_1 is determined by

$$b_1 = \frac{b_3\, b_4 - b_3\, b_2 - b_5\, b_4 + b_5\, b_6 + \frac{b_5 - b_4 + b_2}{h} + \frac{-b_3 + b_4 - b_6}{q} + \frac{-b_3 + b_5 - b_6 + b_2}{k}}{b_6 - b_2 + \frac{1}{h} - \frac{1}{q}} \,, \tag{11}$$

and b_2, b_3, b_4, b_5, b_6 satisfy the single constraint

$$\begin{aligned} 0 \;=\; & b_2\, b_5\, b_6 + b_6\, b_3\, b_4 - b_2\, b_5\, b_4 - b_6\, b_3\, b_2 + b_5\, {b_4}^2 - b_3\, {b_4}^2 + b_3\, b_4\, b_2 - b_5\, b_4\, b_6 \\ & + \frac{b_4\, b_6 - {b_4}^2 - b_6\, b_2 + b_5\, b_6 - b_3\, b_6 + b_4\, b_2 + b_3\, b_4 - b_5\, b_4}{q} + \frac{b_6 + b_5 - b_3 - 2\, b_4 + b_2}{h\, k} \end{aligned}$$

$$+\frac{-b_3\,b_6+b_5\,b_6-2\,b_5\,b_4+2\,b_3\,b_4-b_3\,b_2+b_2\,b_5}{k}+\frac{-b_2-b_3+b_6+b_5}{h\,q} \tag{12}$$
$$+\frac{b_3\,b_4+b_2\,b_5+{b_4}^2-b_5\,b_4-b_4\,b_2-b_3\,b_2-b_4\,b_6+b_6\,b_2}{h}+\frac{-b_6+2\,b_4-b_2-b_3+b_5}{q\,k}$$
$$+\frac{-b_3+b_5-b_6+b_2}{k^2}\ .$$

It can be checked that the 4 dimensional set of solutions includes the 3 dimensional set obtained by rotating the solution of (10), i.e. the solutions of

$$\left.\begin{array}{rcl} b_1-b_3+\frac{1}{k}+\frac{1}{h} &=& 0\\ b_3-b_5-\frac{1}{h}+\frac{1}{q} &=& 0\\ \left(\frac{1}{k^2}-\frac{1}{h^2}\right)b_2+\left(\frac{1}{h^2}-\frac{1}{q^2}\right)b_4+\left(\frac{1}{q^2}-\frac{1}{k^2}\right)b_6 &=& \left(\frac{1}{h}+\frac{1}{k}\right)\left(-\frac{1}{q}+\frac{1}{h}\right)\left(\frac{1}{k}+\frac{1}{q}\right) \end{array}\right\}\ . \tag{13}$$

Thus, to summarize, we have arrived at the hexaKdV system. This consists of the system (6)-(9) of 4 polynomial equations in 6 variables on each plaquette of the lattice. Because the system (6)-(9) is degenerate, in the sense that it has a 4 dimensional solution set, it is more appropriate to think of hexaKdV as specifying 2 constraints for each plaquette. Thus, as explained above, there is a proper balance between the number of variables and the number of constraints in hexaKdV.

Soliton solutions. A direct computation shows that (6)-(9) has the following two parameter family of solutions:

$$\begin{array}{rcl} b_1 &=& C\tanh(z)\ ,\\ b_2 &=& C\tanh\left(z+\tanh^{-1}(hC)\right)\ ,\\ b_3 &=& C\tanh\left(z+\tanh^{-1}(hC)+\tanh^{-1}(kC)\right)\ ,\\ b_4 &=& C\tanh\left(z+\tanh^{-1}(hC)+\tanh^{-1}(kC)+\tanh^{-1}(qC)\right)\ ,\\ b_5 &=& C\tanh\left(z+\tanh^{-1}(kC)+\tanh^{-1}(qC)\right)\ ,\\ b_6 &=& C\tanh\left(z+\tanh^{-1}(qC)\right)\ . \end{array} \tag{14}$$

Here C, z are arbitrary. Such solutions on an individual plaquette can be pasted together to give a full solution of hexaKdV of form

$$b_{n_1,n_2,n_3}=C\tanh\left(n_1\tanh^{-1}(qC)+n_2\tanh^{-1}(hC)-n_3\tanh^{-1}(kC)+z\right)\ , \tag{15}$$

where again C, z are arbitrary constants. To get some understanding into the nature of the soliton solution, it is necessary to write it as a function of the standard Cartesian coordinates x and y of the vertex n_1, n_2, n_3. These are given by

$$x=n_1q-\frac{1}{2}(n_2h+n_3k)\ ,\qquad y=\frac{\sqrt{3}}{2}(n_2h-n_3k)\ . \tag{16}$$

Writing n_1, n_2, n_3 in terms of x, y and the quantity

$$s = n_1 + n_2 + n_3 \tag{17}$$

(which is 0 or 1), the soliton becomes

$$b(x, y, s) = C \tanh(D(x + cy) + Es + z) \ , \tag{18}$$

where

$$D = \frac{(k+h)\tanh^{-1}(qC) + h\tanh^{-1}(kC) - k\tanh^{-1} hC}{qk + qh + hk} \ , \tag{19}$$

$$c = \frac{(h-k)\tanh^{-1}(qC) + (h+2q)\tanh^{-1}(kC) + (k+2q)\tanh^{-1} hC}{\sqrt{3}\left((k+h)\tanh^{-1}(qC) + h\tanh^{-1}(kC) - k\tanh^{-1} hC\right)} \ , \tag{20}$$

$$E = \frac{kh\tanh^{-1}(qC) - qh\tanh^{-1}(kC) + kq\tanh^{-1} hC}{qk + qh + hk} \ . \tag{21}$$

Before discussing this result, note that for sufficiently small q, h, k (for which $\tanh^{-1}(qC) \approx qC$ etc)

$$D \approx \frac{q(k+h)}{qk + qh + hk} \ , \tag{22}$$

$$c \approx \frac{qk + 3qh + 2kh}{\sqrt{3}q(k+h)} \ , \tag{23}$$

$$E \approx \frac{qkh}{qk + qh + hk} \ . \tag{24}$$

In the form (18), the meaning of the soliton solution is quite clear. c is the speed of the soliton, C its amplitude, and equation (20) expresses the speed-amplitude relation. The dependence of D on C, through (19), shows that the width of the soliton also depends on the speed (or amplitude), a common phenomenon seen, for example, in the KdV equation. The novel feature of hexaKdV solitons is, however, the dependence of (18) on s. This means that *there is a phase shift between the soliton solution on "even" ($s = 0$) and "odd" ($s = 1$) sites of the lattice.* This interesting phenomenon distinguishes hexaKdV from standard lattice KdV. Equation (24) shows that as the continuum limit is approached (q, h, k tending to 0), the dependence on s becomes very weak, and indeed vanishes in the continuum. Equations (22) and (23) show that as q, h, k tend to 0 with constant ratios, the speed and width of the solitons tend to constant values, independent of amplitude.

The continuum limit. Is hexaKdV in any sense a discretization of a PDE? Consider making the following replacements in (6)-(9):

$$\begin{array}{lcl} b_1 & \rightarrow & b\left(x - \frac{1}{2}q + \frac{1}{4}h - \frac{1}{4}k, y - \frac{\sqrt{3}}{4}h - \frac{\sqrt{3}}{4}k\right) \ , \\ b_2 & \rightarrow & b\left(x - \frac{1}{2}q - \frac{1}{4}h - \frac{1}{4}k, y + \frac{\sqrt{3}}{4}h - \frac{\sqrt{3}}{4}k\right) \ , \\ b_3 & \rightarrow & b\left(x - \frac{1}{2}q - \frac{1}{4}h + \frac{1}{4}k, y + \frac{\sqrt{3}}{4}h + \frac{\sqrt{3}}{4}k\right) \ , \\ b_4 & \rightarrow & b\left(x + \frac{1}{2}q - \frac{1}{4}h + \frac{1}{4}k, y + \frac{\sqrt{3}}{4}h + \frac{\sqrt{3}}{4}k\right) \ , \\ b_5 & \rightarrow & b\left(x + \frac{1}{2}q + \frac{1}{4}h + \frac{1}{4}k, y - \frac{\sqrt{3}}{4}h + \frac{\sqrt{3}}{4}k\right) \ , \\ b_6 & \rightarrow & b\left(x + \frac{1}{2}q + \frac{1}{4}h - \frac{1}{4}k, y - \frac{\sqrt{3}}{4}h - \frac{\sqrt{3}}{4}k\right) \ . \end{array} \tag{25}$$

Expanding the 4 resulting equations in a Taylor series in q, h, k and retaining only leading orders (assuming h, q, k all to be of the same order) gives only 2 distinct equations. Equations (8) and (9) both give

$$b_y = \frac{qk + 3qh + 2kh}{\sqrt{3}q(k+h)} b_x \ . \tag{26}$$

This is consistent with the results on soliton solutions, as for small q, h, k the soliton speed is given by c as in (23), and this is precisely the factor that has just appeared in (26). However, both (6) and (7) reduce to another PDE, which is only consistent with (26) if $q = h$. So far we have no understanding of why it should be necessary to impose such a constraint for a consistent continuous limit, or whether there is maybe some reason to ignore it.

Dual Discretizations of PDEs. Although it is a digression from the main topic, since it has been shown that hexaKdV is a discretization of (26), at least in the case $h = q$, we briefly raise the question of how *ab initio* one might go about discretizing this (or another) PDE on a hexagonal lattice, with the intention of getting equations relating the 6 values of the field around each hexagonal plaquette.

The standard finite difference approach for discretizing a PDE involves writing a discrete equation to approximate the PDE at each vertex of the lattice being used. The discrete equation involves the values of the functions appearing in the PDE at the relevant vertex, as well as the values at neighboring vertices. The balance between the number of equations obtained and the number of variables is automatic, as both equal the number of lattice vertices in the relevant domain.

Here we follow a different "dual" approach. The aim is to approximate the PDE on a fundamental lattice plaquette (or, more precisely, at the center of gravity of a lattice plaquette), using the values of the function on lattice vertices. In the case of a hexagonal lattice, this means it is necessary to approximate derivatives of b at the point (x, y) using the 6 values of b that appeared in (25). This is straightforward. For example it can be shown that under the replacements (25)

$$\frac{b_4 - b_1 + b_6 - b_3}{2q} \quad = \quad b_x(x,y) + O(q^2, h^2, k^2, hk, qk, qh) \ , \tag{27}$$

$$\frac{1}{2\sqrt{3}}\left(\frac{b_5-b_2-b_6+b_3}{k}+\frac{b_4-b_1-b_5+b_2}{h}\right) \;=\; b_y(x,y)+O(q^2,h^2,k^2,hk,qk,qh)\ . \tag{28}$$

Thus the PDE $b_y = Mb_x/\sqrt{3}$ can apparently be discretized by the difference equation

$$\frac{b_5-b_2-b_6+b_3}{k}+\frac{b_4-b_1-b_5+b_2}{h}=M\ \frac{b_4-b_1+b_6-b_3}{q}\ . \tag{29}$$

However, there is now a counting problem of exactly the type mentioned before. The PDE has been replaced by a single difference equation for each plaquette, with the difference equation "calling" 6 values of the field. But, each value of the field only appears in 3 equations, so the number of variables is twice the number of equations. The resolution of this is that there is a constraint that needs to be imposed. Working to the same order as the approximations (27)-(28) it can be checked that

$$\left(\frac{1}{h}+\frac{1}{q}\right)(b_4-b_1)-\left(\frac{1}{k}+\frac{1}{h}\right)(b_5-b_2)+\left(\frac{1}{q}+\frac{1}{k}\right)(b_6-b_3)=0+O(q^2,h^2,k^2,hk,qk,qh)\ . \tag{30}$$

Thus equation (29) on each plaquette must be supplemented by the constraint

$$\left(\frac{1}{h}+\frac{1}{q}\right)(b_4-b_1)-\left(\frac{1}{k}+\frac{1}{h}\right)(b_5-b_2)+\left(\frac{1}{q}+\frac{1}{k}\right)(b_6-b_3)=0\ . \tag{31}$$

This resolves the counting problem.

Dual discretizations of PDEs may well be appropriate in a variety of settings. It is certainly no surprise that they arise in discretizations of equations in the KdV hierarchy, which are, in a natural way, zero curvature equations. In conclusion of this section, note that the constraint equation (31) looks very similar to what is obtained by applying all the necessary rotation operators just to the term $(b_1-b_2)/h$ in (6). But the differences turn out to be significant; for hexaKdV there does not seem to be a natural division of the equations into a constraint part and a dynamic part, as there is with the simple discretization comprised of (29) and (31).

OctaKdV and other lattices. The work presented in this letter can be extended to other lattices. We report briefly on the extension to a lattice of alternating octahedra and rectangles (with the octahedra having 4 pairs of parallel sides of equal length). On the rectangular plaquettes standard lattice KdV must hold. A simple counting argument then shows that there should be 3 equations (in 8 variables) associated to each octagonal plaquette. Looking at the analog of (5) with 4 matrices in each side gives a complicated system of 6 polynomial equations in 8 variables, with the first being an obvious extension of (2) and (6). We hypothesize that this system has a 5 parameter solution set, and thus should be thought of as just 3 constraints. As of yet we have not been able to verify this in general, the memory requirements for the algebra being substantial. We have, however, looked at a number of reductions, in which one variable is assumed to vanish; in such reductions there are 4 parameter solution sets, which is consistent with the hypothesis.

It may be possible to obtain some general results on the algebraic system obtained from the generalization of (5) with n matrices on each side, relevant to a plaquette with $2n$ sides. Presumably soliton solutions on more general lattices will also exhibit "small scale" phase shifts as we move round a plaquette, as there are for hexaKdV. These phase shifts may well be indicative of lattice structure.

It remains a mystery whether there is a way to formulate integrable systems of difference equations on lattices with plaquettes with odd numbers of sides, particularly triangular lattices.

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