A PRIORI ANALYSIS OF INITIAL DATA FOR THE RICATTI EQUATION AND ASYMPTOTIC PROPERTIES OF ITS SOLUTIONS

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Abstract. We obtain two main results for the Cauchy problem

$$y'(x) + \frac{1}{r(x)}y^2 = q(x), \qquad y(x)\,\big|_{x=x_0} = y_0 \tag{1}$$

where $x_0, y_0 \in \mathbb{R}$, $r > 0$, $q \geq 0$, $\frac{1}{r} \in L_1^{\text{loc}}(\mathbb{R})$, $q \in L_1^{\text{loc}}(\mathbb{R})$ and

$$\int_{-\infty}^{x} \frac{1}{r(t)} \int_t^x q(\xi)d\xi dt = \int_x^{\infty} \frac{1}{r(t)} \int_x^t q(\xi)d\xi dt = \infty, \quad \forall x \in \mathbb{R}\ .$$

1) For given initial data x_0, y_0 and functions r and q, we give a condition that can be used to determine whether the solution of (1) can be continued to the whole of $\mathbb{R}$.
2) When the solution of (1) is defined on an infinite interval, we study its asymptotic properties as the argument tends to infinity.

1. Introduction

In the present paper, we consider the Cauchy problem for the Riccati equation

$$y'(x) + \frac{1}{r(x)}y^2 = q(x), \tag{1.1}$$

$$y(x)\,\big|_{x=x_0} = y_0 \tag{1.2}$$

where $x_0, y_0 \in \mathbb{R}$ and r and q satisfy the conditions

$$r > 0, \quad q \geq 0, \quad \frac{1}{r} \in L_1^{\text{loc}}(\mathbb{R}), \quad q \in L_1^{\text{loc}}(\mathbb{R}), \tag{1.3}$$

$$\int_{-\infty}^{x} q(t)dt > 0, \quad \int_x^{\infty} q(t)dt > 0 \quad \forall x \in \mathbb{R}. \tag{1.4}$$

Our general goal is to develop further the investigation started in [1]. In [1] we studied the problem of continuation of the solution of (1.1)–(1.2) to $\mathbb{R}$ in the case $r \equiv 1$. It arises because for every point $(x_0, y_0) \in \mathbb{R}^2$, the problem (1.1)–(1.2) has a unique solution in a neighborhood of this point, but in general this cannot be continued to the whole real axis. In [1], in the case $r \equiv 1$, we found unbounded domains P and Q such that

- the solution of (1.1)–(1.2) can be continued to $\mathbb{R}$ if $(x_0, y_0) \in P$;
- the solution of (1.1)–(1.2) cannot be continued to $\mathbb{R}$ if $(x_0, y_0) \in Q$.

The main goal of the current paper is to find domains $P, Q \subset \mathbb{R}^2$ with similar properties in the case $r \not\equiv 1$. The logic proceeds as follows (see §3 for precise statements): We show that (1.1) has two well-defined solutions $y_1 < 0$ and $y_2 > 0$ defined over all of $\mathbb{R}$, and that a solution y of the problem (1.1)–(1.2) can be continued to $\mathbb{R}$ if and only if

$$y_1(x_0) \leq y_0 \leq y_2(x_0). \tag{1.5}$$

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To make the implicit condition (1.5) more concrete, we introduce some requirements complementary to (1.3)–(1.4). Then certain sharp-by-order two-sided estimates are shown to hold for the solutions y_1, y_2. These estimates, together with (1.5), allow one to define domains P, Q with the desired properties.

From (1.5) it follows that the solution (1.1)–(1.2) for all x satisfies the inequalities

$$y_1(x) \le y(x) \le y_2(x). \tag{1.6}$$

Inequalities (1.6) lead to the question on the relationship between $y_1(x)$, $y_2(x)$ and $y(x)$ as $|x| \to \infty$. We show that

- If for some $x_0 \in \mathbb{R}$ the solution $y(x)$ of (1.1) is defined on $[x_0, \infty)$ and $y \ne y_1$, then $y(x)$ is equivalent to $y_2(x)$ as $x \to \infty$.
- If for some $x_0 \in \mathbb{R}$ the solution $y(x)$ of (1.1) is defined on $(-\infty, x_0]$ and $y \ne y_2$, then $y(x)$ is equivalent to $y_1(x)$ as $x \to -\infty$.

Thus asymptotic integration of (1.1) (as $|x| \to \infty$) reduces to asymptotic integration of $y_1(x)$ and $y_2(x)$ (in the case $r \equiv 1$ see [1],[2]).

Finally, we note that the proposed analysis of the Cauchy problem (1.1)–(1.2) is not complete because it does not include points $(x_0, y_0) \in \mathbb{R}^2 \setminus (P \cup Q)$. This problem arises because in condition (1.5) we replace the exact values $y_1(x_0), y_2(x_0)$, $x_0 \in \mathbb{R}$., with a priori estimates. According to Liouville's Theorem, one cannot in general find the exact values of $y_1(x_0), y_2(x_0)$. This problem can, however be studied numerically. We make some initial observations on this subject in the final section of this paper.

2. Preliminaries

Theorem 2.1. [3] *Under conditions* (1.3)–(1.4)*, the equation*

$$(r(x)z'(x))' = q(x)z(x), \quad x \in \mathbb{R} \tag{2.1}$$

has a fundamental system of solutions (FSS) $\{u, v\}$ *with the following properties:*

$$v(x) > 0, \quad u(x) > 0, \quad v'(x) \ge 0, \quad u'(x) \le 0, \quad x \in \mathbb{R}, \tag{2.2}$$

$$r(x)\left(v'(x)u(x) - u'(x)v(x)\right) = 1, \quad x \in \mathbb{R}, \tag{2.3}$$

$$\lim_{x\to-\infty} \frac{v(x)}{u(x)} = \lim_{x\to\infty} \frac{u(x)}{v(x)} = 0, \tag{2.4}$$

$$\int_{-\infty}^0 \frac{dt}{r(t)u^2(t)} < \infty, \quad \int_0^\infty \frac{dt}{r(t)v^2(t)} < \infty, \quad \int_{-\infty}^0 \frac{dt}{r(t)v^2(t)} = \int_0^\infty \frac{dt}{r(t)u^2(t)} = \infty. \tag{2.5}$$

Properties (2.2)–(2.5) determine the FSS $\{u, v\}$ up to constant positive factors inverse one to another.

Remark 2.2. The inequalities for u', v' in (2.2) can be strengthened, namely:

$$v'(x) > 0, \qquad u'(x) < 0 \qquad \text{for} \quad x \in \mathbb{R}. \tag{2.6}$$

Indeed $\int_{x_1}^x q(t)dt > 0$ for some $x_1 < x$ by (1.4). So from Theorem 2.1 it follows that

$$r(x)v'(x) = r(x_1)v'(x_1) + \int_{x_1}^x q(t)v(t)dt \ge v(x_1)\int_{x_1}^x q(t)dt > 0.$$

The second inequality in (2.6) can be checked in a similar way. From (2.2)–(2.5) it follows that

$$v(x)=u(x)\int_{-\infty}^{x}\frac{dt}{r(t)u^2(t)},\qquad u(x)=v(x)\int_{x}^{\infty}\frac{dt}{r(t)v^2(t)},\quad x\in\mathbb{R}. \tag{2.7}$$

By Theorem 2.1 and (2.7), we conclude that the function

$$\rho(x)\overset{\text{def}}{=}u(x)v(x)=v^2(x)\int_{x}^{\infty}\frac{dt}{r(t)v^2(t)}=u^2(x)\int_{-\infty}^{x}\frac{dt}{r(t)u^2(t)},\quad x\in\mathbb{R} \tag{2.8}$$

does not depend on the choice of a FSS of (2.1) and is uniquely determined by (2.1), i.e., by r and q.

Theorem 2.3. [6] *For all $x\in\mathbb{R}$ we have the relations*

$$\frac{r(x)u'(x)}{u(x)}=-\frac{1-r(x)\rho'(x)}{2\rho(x)},\qquad \frac{r(x)v'(x)}{v(x)}=\frac{1+r(x)\rho'(x)}{2\rho(x)},\qquad r(x)|\rho'(x)|<1\ . \tag{2.9}$$

Theorem 2.4. [5, Ch. III, §40] *The general solution of* (1.1) *is of the form*

$$y(x)=\frac{c_1r(x)u'(x)+c_2r(x)v'(x)}{c_1u(x)+c_2v(x)}. \tag{2.10}$$

Here $\{u,v\}$ is a FSS of (2.1) and c_1, c_2 are arbitrary constants with $|c_1|+|c_2|\neq 0$.

Remark 2.5. Theorem 2.4 is given in [5] for $r\equiv 1$. It can be extended to the case (1.3)–(1.4) without any difficulties using Theorem 2.1.

3. Statement of results

The proofs of the assertions below are given in §4.

Theorem 3.1. *Under condition* (1.3)*–*(1.4)*, equation* (1.1) *has solutions y_1 and y_2 where*

$$y_1(x)=\frac{r(x)u'(x)}{u(x)},\quad y_2(x)=\frac{r(x)v'(x)}{v(x)},\quad x\in\mathbb{R}\ ,\quad \{u,v\}\ \textit{is a FSS of}\ (2.1). \tag{3.1}$$

The solutions $y_1(x)$, $y_2(x)$ are defined for all $x\in\mathbb{R}$ and

$$y_1(x)<0,\qquad y_2(x)>0,\quad x\in\mathbb{R}. \tag{3.2}$$

A solution y of problem (1.1)*–*(1.2) *can be continued to $\mathbb{R}$ if and only if* (1.5) *holds.*

In the sequel (until Theorem 3.8), we assume that, together with (1.3), the following condition also holds:

$$\int_{-\infty}^{x}\frac{1}{r(t)}\int_{t}^{x}q(\xi)d\xi dt=\int_{x}^{\infty}\frac{1}{r(t)}\int_{x}^{t}q(\xi)d\xi dt=\infty,\quad \forall x\in\mathbb{R}. \tag{3.3}$$

Clearly (1.4) follows from (1.3) and (3.3). Such a strengthening of the requirements on r and q will be used for a more detailed study of (3.1).

The following lemma is useful in that it often simplifies checking (3.3).

Lemma 3.2. *Suppose* (1.3)*–*(1.4) *hold and, in addition,*

$$\int_{-\infty}^{0}\frac{dt}{r(t)}=\int_{0}^{\infty}\frac{dt}{r(t)}=\infty. \tag{3.4}$$

Then (3.3) *holds.*

Lemma 3.3. *Suppose* (1.3) *and* (3.3) *hold. Then for every* $x \in \mathbb{R}$, *each of the following equations in* d

$$\int_{x-d}^{x} \frac{1}{r(t)} \int_t^x q(\xi)d\xi dt = 1, \qquad \int_x^{x+d} \frac{1}{r(t)} \int_x^t q(\xi)d\xi dt = 1 \tag{3.5}$$

has a unique finite positive solution.

Denote the solutions of (3.5) by $d_1(x)$, $d_2(x)$, respectively. For $x \in \mathbb{R}$ let us introduce the functions

$$\varphi(x) = \int_{x-d_1(x)}^{x} \frac{dt}{r(t)}, \qquad \psi(x) = \int_x^{x+d_2(x)} \frac{dt}{r(t)}, \qquad h(x) = \frac{\varphi(x)\psi(x)}{\varphi(x)+\psi(x)}, \tag{3.6}$$

$$\theta_1(x) = y_2(x)\varphi(x) = \frac{r(x)v'(x)}{v(x)}\varphi(x), \tag{3.7}$$

$$\theta_2(x) = y_1(x)\psi(x) = \frac{r(x)|u'(x)|}{u(x)}\psi(x). \tag{3.8}$$

Theorem 3.4. *Suppose* (1.3) *and* (3.3) *hold. Then the functions* θ_1, θ_2 *are solutions to the following integral equations:*

$$\theta_1(x) = 1 + \int_{\Delta^-(x)}^{x} \frac{1}{r(t)} \int_t^x q(\xi)\mathcal{K}_1(x,\xi,\theta_1)d\xi dt, \quad x \in \mathbb{R}, \tag{3.9}$$

$$\theta_2(x) = 1 + \int_x^{\Delta^+(x)} \frac{1}{r(t)} \int_x^t q(\xi)\mathcal{K}_1(x,\xi,\theta_1)d\xi dt, \quad x \in \mathbb{R}. \tag{3.10}$$

Here $\Delta^-(x) = x - d_1(x)$, $\Delta^+(x) = x + d_2(x)$,

$$\mathcal{K}_1(x,\xi,\theta_1) = \exp\left(-\int_\xi^x \frac{\theta_1(s)ds}{r(s)\varphi(s)}\right) - \exp\left(-\int_{\Delta^-(x)}^x \frac{\theta_1(s)ds}{r(s)\varphi(s)}\right), \tag{3.11}$$

$$\mathcal{K}_2(x,\xi,\theta_2) = \exp\left(-\int_x^\xi \frac{\theta_2(s)ds}{r(s)\psi(s)}\right) - \exp\left(-\int_x^{\Delta^+(x)} \frac{\theta_2(s)ds}{r(s)\psi(s)}\right). \tag{3.12}$$

Corollary 3.5. *Under conditions* (1.3) *and* (3.3), *we have the inequalities*

$$1 \le \theta_1(x), \quad \theta_2(x) \le 2, \quad x \in \mathbb{R}, \tag{3.13}$$

$$1 \le y_1(x)\psi(x), \quad y_2(x)\varphi(x) \le 2, \quad x \in \mathbb{R}, \tag{3.14}$$

$$2^{-1}h(x) \le \rho(x) \le h(x), \quad x \in \mathbb{R}. \tag{3.15}$$

Note that inequalities of the form (3.15) are called Otelbaev inequalities, see [2].

Corollary 3.6. *Suppose* (1.3) *and* (3.3) *hold. If a solution* $y(x)$ *of problem* (1.1) *is defined for all* $x \in \mathbb{R}$, *then*

$$-2 \le y(x)h(x) \le 2, \qquad x \in \mathbb{R}. \tag{3.16}$$

We introduce domains P and Q on the plane $\mathbb{R}^2$ as follows:

$$P = \{(x,y) : y\varphi(x) \le 1\} \cap \{(x,y) : y\psi(x) \ge -1\}, \tag{3.17}$$

$$Q = \{(x,y) : y\varphi(x) \ge 2\} \cup \{(x,y) : y\psi(x) \le -2\}. \tag{3.18}$$

Theorem 3.7. *Under conditions* (1.3) *and* (3.3), *a solution* y *of problem* (1.1)–(1.2) *can be continued to* $\mathbb{R}$ *if* $(x_0, y_0) \in P$, *and cannot be continued to* $\mathbb{R}$ *if* $(x_0, y_0) \in Q$.

In the next assertion we establish a precise standard for the behavior of solutions of equation (1.1) at infinity.

Theorem 3.8. *Suppose* (1.3) *holds and, in addition,*

$$A\cdot B\overset{def}{=}\int_0^\infty\frac{dt}{r(t)}\cdot\int_0^\infty q(t)dt=\infty,\quad A_1\cdot B_1\overset{def}{=}\int_{-\infty}^0\frac{dt}{r(t)}\cdot\int_{-\infty}^0 q(t)dt=\infty. \tag{3.19}$$

Then the following assertions hold:

A) *If for some* x_0 *a solution* y *of equation* (1.1) *is defined on* $[x_0,\infty)$ *and does not coincide with* y_1*, then*
$$\lim_{x\to\infty}\frac{y(x)}{y_2(x)}=1. \tag{3.20}$$

B) *If for some* x_0 *a solution* y *of equation* (1.1) *is defined on* $(-\infty,x_0]$ *and does not coincide with* y_2*, then*
$$\lim_{x\to-\infty}\frac{y(x)}{y_1(x)}=1. \tag{3.21}$$

C) *If* (1.3) *and* (3.3) *hold, then* (3.19) *holds, too.*

4. Proofs

Proof of Theorem 3.1. From Theorem 2.1 it follows that the functions $y_1(x)$, $y_2(x)$ are defined for all $x\in\mathbb{R}$ and satisfy (1.1), and that (3.2) is a consequence of (2.6). Furthermore, suppose that y can be continued to $\mathbb{R}$. Clearly, the cases $y_0=y_1(x_0)$ and $y_0=y_2(x_0)$ are in one-to-one correspondence with the choice of $c_2=0$ and $c_1=0$ in (2.10). Therefore below we assume $c_1\cdot c_2\neq 0$. Then only one of the following 3 possibilities holds:

$$1)\ y_0>y_2(x_0);\quad 2)\ y_1(x_0)<y_0<y_2(x_0);\quad 3)\ y_0<y_1(x_0).$$

Let us show that $c=c_1\cdot c_2^{-1}<0$ in cases 1) and 3). In case 1) Theorems 2.1 and 2.4 imply

$$0<y_0-y_2(x_0)=\frac{r(x_0)v'(x_0)+cr(x_0)u'(x_0)}{v(x_0)+cu(x_0)}-\frac{r(x_0)v'(x_0)}{v(x_0)}=-\frac{c}{v(x_0)(v(x_0)+cu(x_0))}. \tag{4.1}$$

The assumption $c>0$ contradicts (4.1) and Theorem 2.1. Similarly, in case 3) we get

$$0<y_1(x_0)-y_0=\frac{r(x_0)u'(x_0)}{u(x_0)}-\frac{r(x_0)v'(x_0)+cr(x_0)u'(x_0)}{v(x_0)+cu(x_0)}=-\frac{1}{u(x_0)(v(x_0)+cu(x_0))}$$

implying $c<-v(x_0)/u(x_0)<0$. Thus $c<0$ in cases 1) and 3). Then there exists a point $x_1\in\mathbb{R}$ such that

$$v(x_1)+cu(x_1)=0. \tag{4.2}$$

Indeed, by Theorem 2.1 the function $f(x)=-v(x)/u(x)$, $x\in\mathbb{R}$, is continuous, negative for all $x\in\mathbb{R}$, and

$$f'(x)=-\frac{1}{r(x)u^2(x)}<0,\quad x\in\mathbb{R},\qquad f(x)\to\begin{cases}0 & \text{as}\quad x\to-\infty\\ -\infty & \text{as}\quad x\to+\infty\end{cases}.$$

Therefore, the equation $f(x)=c$ has a unique finite root x_1 which leads to (4.2). Then the solution y has a vertical asymptote at the point x_1, and it cannot be continued to $\mathbb{R}$. Hence we are in case 2), i.e., (1.5) holds. Conversely, suppose (1.5) holds. Then by Theorems 2.1

and 2.4, in some neighborhood of x_0 there exists a unique solution of (1.1)–(1.2), and it is of the form (2.10) with some $c_1 \neq 0$, $c_2 \neq 0$ (see above). By Theorem 2.1, this implies

$$\begin{aligned} 0 &< \frac{r(x_0)v'(x_0)+cr(x_0)u'(x_0)}{v(x_0)+cu(x_0)} - \frac{r(x_0)u'(x_0)}{u(x_0)} = \frac{1}{u(x_0)(v(x_0)+cu(x_0))}\ , \\ 0 &< \frac{r(x_0)v'(x_0)}{v(x_0)} - \frac{r(x_0)v'(x_0)+cr(x_0)u'(x_0)}{v(x_0)+cu(x_0)} = \frac{c}{v(x_0)(v(x_0)+cu(x_0))}\ . \end{aligned}$$

Hence $c > 0$. Then by Theorems 2.1 and 2.4, the solution $y(x)$ is defined for all $x \in \mathbb{R}$ as required. □

Proof of Lemma 3.2. By (1.4), for a given $x \in \mathbb{R}$ there exists $a < x$ and $b > x$ such that

$$\int_a^x q(t)dt > 0, \qquad \int_x^b q(t)dt > 0. \tag{4.3}$$

Then the statement of the lemma follows from (3.4) and (4.3):

$$\begin{aligned} \int_{-\infty}^x \frac{1}{r(t)} \cdot \int_t^x q(\xi)d\xi dt > \int_{-\infty}^a \frac{1}{r(t)} \int_t^x q(\xi)d\xi dt > \int_{-\infty}^a \frac{dt}{r(t)} \cdot \int_a^x q(\xi)d\xi &= \infty, \\ \int_x^\infty \frac{1}{r(t)} \cdot \int_x^t q(\xi)d\xi dt > \int_b^\infty \frac{dt}{r(t)} \int_x^t q(\xi)d\xi dt > \int_b^\infty \frac{dt}{r(t)} \cdot \int_x^b q(\xi)d\xi &= \infty. \end{aligned}$$

□

Proof of Lemma 3.3. Consider the second equation in (3.5) (the first one can be treated similarly). For a given $x \in \mathbb{R}$, let us introduce the function $\Phi(d)$:

$$\Phi(d) = \int_x^{x+d} \frac{1}{r(t)} \cdot \int_x^t q(\xi)d\xi dt, \qquad d \geq 0.$$

By (1.3) and (3.3), the function $\Phi(d)$ is continuous, non-negative, and does not decrease on $[0, \infty)$. In addition, $\Phi(0) = 0$, $\Phi(d) \to \infty$ as $d \to \infty$,

$$\Phi'(d) = \frac{1}{r(x+d)} \int_x^{x+d} q(\xi)d\xi > 0 \quad \text{if} \quad \Phi(d) > 0.$$

These properties immediately imply the statement of the lemma. □

Proof of Theorem 3.4. Let us prove (3.9). As preparation note that since $(r(\xi)v'(\xi))' = q(\xi)v(\xi)$ for all $\xi \in \mathbb{R}$, we have

$$r(x)v'(x) - r(t)v'(t) = \int_t^x q(\xi)v(\xi)d\xi, \quad x \geq t\ . \tag{4.4}$$

Thus:

$$\begin{aligned}
r(x)v'(x)\varphi(x) &= r(x)v'(x)\int_{\Delta^-(x)}^{x}\frac{dt}{r(t)} && \text{(using (3.6))}\\
&= v(x)-v(\Delta^-(x))+\int_{\Delta^-(x)}^{x}\frac{1}{r(t)}\int_t^x q(\xi)v(\xi)d\xi dt && \text{(using (4.4))}\\
&= v(x)+\int_{\Delta^-(x)}^{x}\frac{1}{r(t)}\int_t^x q(\xi)[v(\xi)-v(\Delta^-(x))]d\xi dt && \text{(using (3.5))}\\
&= v(x)+\int_{\Delta^-}^{x}\frac{1}{r(t)}\int_t^x q(\xi)\int_{\Delta^-(x)}^{\xi} v'(\nu)d\nu d\xi dt\ . && (4.5)
\end{aligned}$$

Now using (3.7) and (3.11) we have:

$$\begin{aligned}
\theta_1(x) &= 1+\int_{\Delta^-(x)}^{x}\frac{1}{r(t)}\int_t^x q(\xi)\int_{\Delta^-(x)}^{x}\frac{v'(\nu)}{v(x)}d\nu d\xi dt\\
&= 1+\int_{\Delta^-(x)}^{x}\frac{1}{r(t)}\int_t^x q(\xi)\int_{\Delta^-(x)}^{\xi}\frac{r(\nu)v'(\nu)}{v(\nu)}\varphi(\nu)\cdot\frac{1}{r(\nu)\varphi(\nu)}\cdot\frac{v(\nu)}{v(x)}d\nu d\xi dt\\
&= 1+\int_{\Delta^-(x)}^{x}\frac{1}{r(t)}\int_t^x q(\xi)\int_{\Delta^-(x)}^{\xi}\frac{\theta_1(\nu)}{r(\nu)\varphi(\nu)}\exp\left(-\int_\nu^x\frac{\theta_1(s)}{r(s)\varphi(s)}ds\right)d\nu d\xi dt\\
&= 1+\int_{\Delta^-(x)}^{x}\frac{1}{r(t)}\int_t^x q(\xi)\mathcal{K}_1(x,\xi,\theta_1)d\xi dt, \quad x\in\mathbb{R}.
\end{aligned}$$

The proof of (3.10) is similar. □

Proof of Corollary 3.5. Inequalities (3.13) for θ_1 and θ_2 are checked in the same way. For example, since

$$0\le\mathcal{K}_1(x,\xi,\theta_1)\le 1,\quad \xi\in[\Delta^-(x),x],\quad \theta_1>0,$$

we obtain from (3.9):

$$\theta_1(x)=1+\int_{\Delta^-(x)}^{x}\frac{1}{r(t)}\int_t^x q(\xi)\mathcal{K}_1(x,\xi,\theta_1)d\xi dt\ge 1+0=1,$$

$$\theta_1(x)=1+\int_{\Delta^-(x)}^{x}\frac{1}{r(t)}\int_t^x q(\xi)\mathcal{K}_1(x),\xi,\theta_1)d\xi dt\le 1+\int_{\Delta^-(x)}\frac{1}{r(t)}\int_t^x q(\xi)d\xi dt=1+1=2.$$

Furthermore, inequalities (3.13) and (3.14) are equivalent. To prove (3.15), we use the definitions of h, ρ and (2.3):

$$\frac{h(x)}{\rho(x)}=\frac{\varphi(x)\psi(x)}{\varphi(x)+\psi(x)}\left[\frac{r(x)v'(x)}{v(x)}+\frac{r(x)|u'(x)|}{u(x)}\right]=\frac{\frac{r(x)v'(x)}{v(x)}+\frac{r(x)(u'(x)}{u(x)}}{\frac{1}{\varphi(x)}+\frac{1}{\psi(x)}},\ x\in\mathbb{R}$$

to obtain

$$\begin{aligned}
\frac{h(x)}{\rho(x)} &\ \ge\ \min\left\{\frac{r(x)v'(x)}{v(x)}\varphi(x),\frac{r(x)|u'(x)|}{u(x)}\psi(x)\right\}\ \ge\ 1,\\
\frac{h(x)}{\rho(x)} &\ \le\ \max\left\{\frac{r(x)v'(x)}{v(x)}\varphi(x),\frac{r(x)|u'(x)|}{u(x)}\psi(x)\right\}\ \le\ 2.
\end{aligned}$$

□

Proof of Corollary 3.6. Since the solution (2.10) is defined for all $x \in \mathbb{R}$, we have $c_1 \cdot c_2 > 0$ (see the proof of Theorem 3.1). Below we use (2.10), (2.9) and (3.15):

$$|y(x)| \le \frac{|c_1|\ r(x)\ |u'(x)| + |c_2|\ r(x)\ v'(x)}{|c_1|u(x) + |c_2|v(x)} \le \max\left\{\frac{r(x)|u'(x)|}{u(x)}, \frac{r(x)v'(x)}{v(x)}\right\}$$
$$= \max\left\{\frac{1 - r(x)\rho'(x)}{2\rho(x)}, \frac{1 + r(x)\rho'(x)}{2\rho(x)}\right\} \le \frac{1}{\rho(x)} \le \frac{2}{h(x)}.$$

□

Proof of Theorem 3.7. Let $(x_0, y_0) \in P$. Below we use (3.1), (3.14) and (3.17):

$$y_1(x_0) = \frac{r(x_0)u'(x_0)}{u(x_0)} \le -\frac{1}{\psi(x_0)} \le y_0 \le \frac{1}{\varphi(x_0)} \le \frac{r(x_0)v'(x_0)}{v(x_0)} = y_2(x_0).$$

The statement of the theorem follows now from Theorem 3.1. The case $(x_0, y_0) \in Q$ is treated similarly. □

Proof of Theorem 3.8. Below we use the following assertion:

Lemma 4.1. *Under conditions* (1.3)–(1.4)*, the equality*

$$\lim_{x\to+\infty} r(x)v'(x)v(x) = \infty \quad \left(\lim_{x\to-\infty} r(x)|u'(x)|u(x) = \infty\right) \tag{4.6}$$

holds if and only if $A \cdot B = \infty \quad (A_1 \cdot B_1 = \infty)$.

Proof of Lemma 4.1. Necessity.

The two equalities in (4.6) are checked in a similar way. Let us check the first one. Suppose (4.6) holds but $A \cdot B < \infty$. Denote $\tau_1 = r(0)v'(0)$, $\tau_2 = v(0)$. For $x \ge 0$ by Theorem 2.1, we have

$$r(x)v'(x) = \tau_1 + \int_0^x q(t)v(t)dt \le \tau_1 + v(x)\int_0^x q(t)dt \le \tau_1 + B\cdot v(x) \quad \Rightarrow$$
$$\frac{v'(x)}{v(x)} \le \frac{\tau_1}{r(x)v(x)} + \frac{B}{r(x)} \le \left(\frac{\tau_1}{\tau_2} + B\right)\frac{1}{r(x)}, \quad x \ge 0 \quad \Rightarrow$$
$$v(x) \le \tau_2 \exp\left(A\left(\frac{\tau_1}{\tau_2} + B\right)\right) := \tau_3 < \infty, \quad x \ge 0 \quad \Rightarrow$$
$$r(x)v'(x) \le \tau_1 + Bv(x) \le \tau_1 + \tau_3 B := \tau_4 < \infty, \quad x \ge 0 \quad \Rightarrow$$
$$r(x)v'(x)v(x) \le \tau_3 \cdot \tau_4 < \infty, \quad x \ge 0.$$

This provides a contradiction.

Proof of Lemma 4.1. Sufficiency. Denote $\beta(x) = r(x)v'(x)v(x)$, $x \in \mathbb{R}$, $\tau = \min\{\tau_1^2, \tau_2^2\}$ ($\tau > 0$, see (2.2), (2.6)). Then by Theorem 2.1 we have for

$$\beta'(x) = (r(x)v'(x))'v(x) + r(x)v'^2(x) = q(x)v^2(x) + \frac{(r(x)v'(x))^2}{r(x)}$$
$$\ge q(x)v(0)^2 + \frac{(r(0)v'(0))^2}{r(x)} \ge \tau\left\{q(x) + \frac{1}{r(x)}\right\} \quad \Rightarrow$$
$$\beta(x) = \beta(0) + \int_0^x \beta'(t)dt \ge \tau\int_0^x \left(q(t) + \frac{1}{r(t)}\right)dt \to \infty \quad \text{as} \quad x \to \infty.$$

□

Returning now to the proof of Theorem 3.8, let us check assertion A). (Assertion B) is checked in a similar way.) Since $y \neq y_1$, we have $c_2 \neq 0$ in (2.10). Below we use (2.10), (2.4), (2.3) and Lemma 4.1:

$$\lim_{x\to\infty}\frac{y(x)}{y_2(x)} = \lim_{x\to\infty}\frac{1+\frac{\theta_1}{\theta_2}\frac{u'(x)}{v'(x)}}{1+\frac{\theta_1}{\theta_2}\frac{u(x)}{v(x)}} = \lim_{x\to\infty}\left[1+\frac{\theta_1}{\theta_2}\frac{r(x)u'(x)v(x)}{r(x)v'(x)v(x)}\right] = 1.$$

Assertion C) follows from the following relations:

$$\infty = \int_{-\infty}^{x}\frac{1}{r(t)}\int_t^x q(\xi)d\xi dt \leq \int_{-\infty}^{x}\frac{dt}{r(t)}\cdot\int_{-\infty}^{x} q(\xi)d\xi \leq \infty, \quad x\in\mathbb{R},$$
$$\infty = \int_{x}^{\infty}\frac{1}{r(t)}\int_t^x q(\xi)d\xi dt \leq \int_{x}^{\infty}\frac{dt}{r(t)}\cdot\int_{x}^{\infty} q(\xi)d\xi \leq \infty, \quad x\in\mathbb{R}.$$

□

5. Numerical Studies

We briefly consider a numerical approach to approximating $y_1(x_0), y_2(x_0)$, $x_0 \in \mathbb{R}$. Standard methods for numerical integration of ODEs experience problems dealing with singularities, but specifically for the case of the Riccati equation there exists a class of methods, the *Möbius schemes* [7], that permit accurate integration near and through poles of the solution. Introduce a grid with spacing $h > 0$ on $\mathbb{R}$ and grid points $x_n = x_0 + nh$, $n \in \mathbb{Z}$. We seek approximations y_n to the exact values $y(x_n)$ of the solution to the Cauchy problem (1) at the grid points. In the simplest Möbius scheme these are determined via the recursion

$$y_{n+1} = \frac{y_n + hq\left(x_n+\frac{h}{2}\right)}{1+hy_n/r\left(x_n+\frac{h}{2}\right)}\,, \qquad n\in\mathbb{Z}\,.$$

Under suitable smoothness assumptions on the functions $q(x), r(x)$ it can be shown that this method is *second order*, i.e., that errors in the method, defined in an appropriate sense, scale (approximately) as h^2 as h decreases to 0.

Figure 1 shows results of the Möbius scheme above for $q(x) = \cos^2 x$ and $r(x) = (1+x^2)^{-1}$. Numerical solutions, obtained using $h = 0.01$, are shown for the Cauchy problem with $x_0 = 0$ and y_0 taking a range of values between -1.25 and 1.25. The regions P and Q are also displayed on the plot. We see, as expected, that solutions that pass through P do not develop singularities (at least on the range shown) and all solutions that pass through Q do.

Using the Möbius scheme for fixed h we can find approximations to $y_2(0)$, the unique positive number such that if $y_0 > y_2(0)$, then the solution develops a singularity and if $0 \leq y_0 \leq y_2(0)$, the solution is defined on the whole axis. For the above choice of $q(x), r(x)$, the analytic bounds in this paper give $1/\varphi(0) \leq y_2(0) < 2/\varphi(0)$ where $\varphi(0) \approx 1.83$. In the table below we give the approximate values of $y_2(0)$ found using the Möbius scheme for various values of h, and their errors (based on an "exact" value obtained with a very small value of h). The h^2 scaling is evident.

h	0.04	0.02	0.01	0.005	0.0025	0.00125
$y_2(0)$	0.6972904	0.6972039	0.6971823	0.6971769	0.6971756	0.6971752
abs. error	1153×10^{-7}	288×10^{-7}	72×10^{-7}	18×10^{-7}	5×10^{-7}	1×10^{-7}

We hope in a future publication to return to the subject of using such numerical schemes to obtain rigorous bounds for $y_1(x_0), y_2(x_0)$, $x_0 \in \mathbb{R}$.

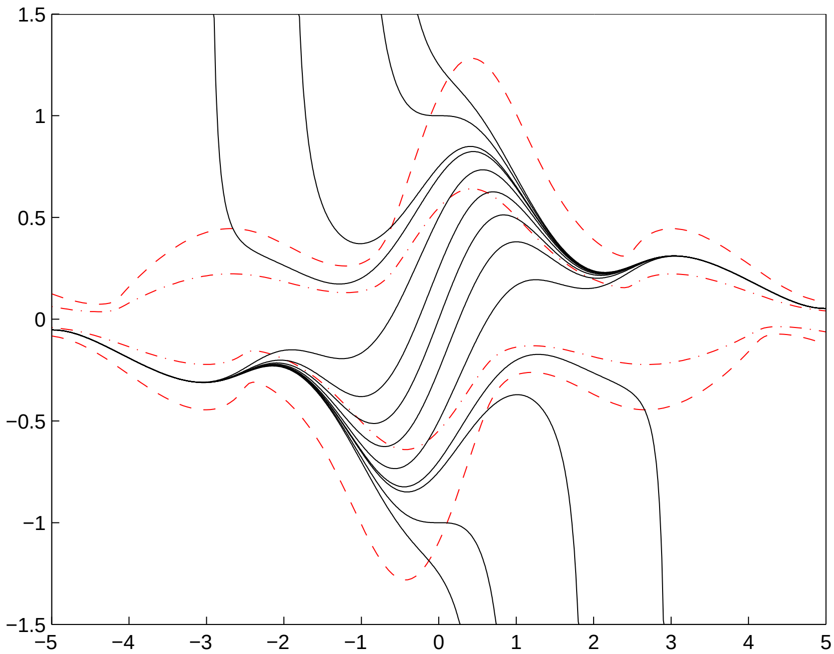

FIGURE 1. Numerical solutions of the Cauchy problem (1) with $q(x) = \cos^2 x$, $r(x) = (1+x^2)^{-1}$, $x_0 = 0$ and values of y_0 between -1.25 and 1.25. The dashed lines bound the region Q (initial conditions for which the solution does not extend to the whole real axis). The dot-and-dashed lines bound the region P (initial conditions for which the solution does extend to the whole real axis).

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