hep-th/9205105
IASSNS-HEP-92/28
May 1992, revised Aug 1992

The KdV Action and Deformed Minimal Models

Jeremy Schiff
Institute For Advanced Study
Olden Lane, Princeton, NJ 08540

Abstract

An action is constructed that gives an arbitrary equation in the KdV or MKdV hierarchies as equation of motion; the second Hamiltonian structure of the KdV equation and the Hamiltonian structure of the MKdV equation appear as Poisson bracket structures derived from this action. Quantization of this theory can be carried out in two different schemes, to obtain either the quantum KdV theory of Kupershmidt and Mathieu or the quantum MKdV theory of Sasaki and Yamanaka. The latter is, for specific values of the coupling constant, related to a generalized deformation of the minimal models, and clarifies the relationship of integrable systems of KdV type and conformal field theories. As a generalization it is shown how to construct an action for the $SL(3)$-KdV (Boussinesq) hierarchy.

An action for the KdV equation should have two basic properties:

(a) The associated equation of motion should be the KdV equation (or some equation in the KdV hierarchy).

(b) The associated Poisson bracket structure (we will reiterate below how to derive Poisson brackets from an action) should be the second hamiltonian structure of the KdV equation.

We should be able to define a quantum theory using our action; given (b) we might expect this to coincide in some sense with the quantum KdV theory as described in [1]. This is clearly desirable, given the correspondence of the conserved quantities of the quantum KdV equation of [1] and the conserved quantities in deformed minimal conformal theories. So we add one further non-essential but desirable property to our list above:

(c) The Heisenberg equation of motion associated with our action should be the quantum KdV equation of [1].

In this note I construct an action that has properties (a),(b),(c). The action can also be regarded as an action for the MKdV equation. In this form the action has a kinetic term that describes a theory which is "nearly" free, and an infinite number of potential terms. In quantizing the theory defined by just the kinetic term, we find many of the features of the Feigin-Fuchs construction for the minimal models; in particular it becomes clear that the quantum analogs of the terms in the potential (the quantum MKdV hamiltonians) describe an infinite number of possible integrable deformations of minimal conformal models. These deformations are more general than those considered by Zamolodchikov [2]. Zamolodchikov's deformations of a conformal field theory are ones that preserve both the integrability and Lorentz invariance of the theory, and there are an infinite number of other perturbations that preserve just the integrability. In the simplest case, the one that we shall consider, the Zamolodchikov deformation gives rise to the integrable, Lorentz-invariant Sine-Gordon theory (as recognized in [3]), whereas the deformations we will consider give rise to theories with equations of motion in the MKdV hierarchy (as is well-known, all the MKdV flows commute with the Sine-Gordon flow). The correspondence of the conserved quantities of the quantum KdV equation and the conserved quantities of deformed minimal models becomes very clear.

In the last part of this note I also show how to construct an action for the SL(3)-KdV equation.

Doubtless many physicists, on meeting the KdV equation for the first time, investigate whether it can be derived from an action. It certainly seems that the action has to be non-local in the KdV field u, and the simplest actions one might guess appear unenlightening (see for example [4]). Indeed, to satsify condition (b) above we need our action to be non-local in u. An explanation of this is as follows: in a classical mechanical system with phase space coordinates X^i, $i = 1, ..., 2n$, a hamiltonian structure is specified either by giving a non-degenerate symplectic form on the phase space

$$\Omega = \tfrac{1}{2}\omega_{ij} dX^i \wedge dX^j \tag{1}$$

or the corresponding set of Poisson brackets

$$\{X^i, X^j\} \;=\; (\omega^{-1})^{ij} \tag{2}$$

The second hamiltonian structure of the KdV equation is given by the Poisson brackets

$$\{u(x), u(y)\} \;=\; -\tfrac{24\pi}{c}\big(\partial_x^3 + u(x)\partial_x + \partial_x u(x)\big)\delta(x-y) \tag{3}$$

and we see at once that the corresponding symplectic form must contain a highly non-local operator. But if the action is local, then the symplectic form must also be local.

The above observations also suggest a way to look for a good action. Wilson [5] has noted that while the symplectic form associated with the second hamiltonian structure of the KdV equation is non-local, the symplectic form associated with the corresponding hamiltonian structure of the "Ur-KdV equation" (the name is Wilson's) *is* local. I review his result. Any solution q of the Ur-KdV equation

$$q_t \;=\; q_{xxx} - \tfrac{3}{2} q_{xx}^2 q_x^{-1} \tag{4}$$

gives a solution of the KdV equation

$$u_t \;=\; u_{xxx} + 3uu_x \tag{5}$$

via the "Miura map"

$$u \;=\; \{q;x\} \;=\; q_{xxx}q_x^{-1} - \tfrac{3}{2}q_{xx}^2 q_x^{-2} \tag{6}$$

($\{q;x\}$ denotes the Schwarzian derivative of q with respect to x). If we take the brackets

$$\{q(x), q(y)\} \;=\; \tfrac{24\pi}{c}\partial_x^{-1}q_x\partial_x^{-1}q_x\partial_x^{-1}\delta(x-y) \tag{7}$$

then u defined by (6) will satisfy (3). Now the *inverse* of the operator preceding the delta function on the right hand side of (7) is clearly local, i.e. the associated symplectic form is local. So we look for an action that is local in the function q.

One action that gives the correct symplectic form/Poisson brackets is the geometric Virasoro action of Polyakov, Bershadsky, Ooguri and others [6]:

$$S_0 = -\frac{c}{48\pi}\int dxdt \;\; q_{xt}q_{xx}q_x^{-2} \tag{8}$$

At this point I briefly digress to give an account (which I learned from V.P.Nair) of how to find the symplectic form determined by an action. Suppose, for definiteness, that we have an action S for a single field ϕ in $1+1$ dimensions. Integrating by parts if necessary, we can find an expression for the variation of the action in the form

$$\delta S = \int dxdt \;\; \Big(p[\phi]\delta\phi + \partial_t\alpha + \partial_x\beta\Big) \tag{9}$$

Here $p[\phi]$ is some density depending on ϕ and its derivatives, and α and β are densities which depend on ϕ and its derivatives, and also are linear in the variation of ϕ and its derivatives. The first term in (9) yields the equation of motion $p=0$, but there is clearly further information. From the term in (9) which is a total derivative with respect to the time t, we obtain a one-form on the space of functions ϕ which are independent of t:

$$\tilde{\alpha} \;=\; \int dx \;\; \alpha \tag{10}$$

Adding a total derivative term to the action S would change $\tilde{\alpha}$ (it does not, of course, change the equations of motion), but it is easy to see that the term that would be added would be exact, so the two-form

$$\Omega \;=\; \delta\tilde{\alpha} \tag{11}$$

is unaffected. This is the symplectic form determined by S (or, more precisely, determined by S and a choice of "time" direction). Using this method it is easy to check that the action (8) gives the symplectic form associated with the brackets (7). Now, classically the action S_0 just describes a free theory, since writing $h = \ln q_x$ we have

$$S_0 = -\frac{c}{48\pi} \int dxdt \;\; h_x h_t \tag{12}$$

So S_0 is clearly not a candidate KdV action. But we can add to S_0 terms dependent only on q and its x-derivatives *without* changing the Poisson brackets. Before we do this, though, we note the other crucial property of S_0, that (ignoring the boundary terms crucial for the derivation of the symplectic form)

$$\delta S_0 = -\frac{c}{24\pi} \int dxdt \;\; u_t q_x^{-1} \delta q \tag{13}$$

Thus the equation of motion derived from an action with "kinetic" term S_0 will give evolution equations for u, just as we desire.

To complete the construction is now easy. Let $p[u]$ be some density in u and its derivatives; write

$$H \;=\; \int dx \; p[u] \tag{14}$$

and define $\delta p/\delta u$ by

$$\delta H \;=\; \int dx \; \frac{\delta p}{\delta u} \delta u \tag{15}$$

(Throughout this work we take x to be defined on some finite range, and assume all functions to satisfy periodic boundary conditions, so that we can integrate by parts with respect to x without boundary terms appearing). Then we find

$$\delta H \;=\; -\int dx \;\; \frac{\delta q}{q_x} \big(\partial_x^3 + u(x)\partial_x + \partial_x u(x) \big) \frac{\delta p}{\delta u} \tag{16}$$

So we consider the action

$$S \;=\; S_0 \;+\; \sum_{n=1}^{\infty} \lambda_n \int dxdt \; p_n[u] \tag{17}$$

where the λ_n, $n = 1, 2, ...$, are constants and the $p_n[u]$, $n = 1, 2, ...$, are the densities of the conserved quantities of the KdV equation (see, for example, [4]) e.g.

$$
\begin{aligned}
p_1[u] &= u \\
p_2[u] &= \tfrac{1}{2}u^2 \\
p_3[u] &= \tfrac{1}{2}(u^3 - u_x^2)
\end{aligned}
\tag{18}
$$

The $p_n[u]$ are related by the Lenard recursion relation

$$
\partial_x \frac{\delta p_n}{\delta u} = \left(\partial_x^3 + u(x)\partial_x + \partial_x u(x)\right)\frac{\delta p_{n-1}}{\delta u} \tag{19}
$$

S is the classical KdV action, which has properties (a) and (b) we listed at the start of this paper; indeed any equation in the KdV hierarchy (or any "linear combination" of the equations, in the obvious sense) can be obtained by suitable choice of the constants λ_n.

To quantize the theory defined by S requires a little care, but is essentially straightforward thanks to existing results in the literature. We start by discussing the action S_0. To quantize any theory, we select a set of local Poisson brackets and elevate it to the level of an operator commutation relation. As we will see below (3) is not the only set of local Poisson brackets we can derive from S_0. But to obtain the quantum KdV theory of [1] we indeed choose (3) as our "fundamental bracket", which now becomes an operator commutation relation. Comparing (3) with the formulae of Gervais [7], we see our quantum theory is characterized by a Virasoro algebra of central charge c. To obtain S from S_0 in the classical theory we added a "potential" consisting of an infinite sum of terms proportional to the conserved quantities of the classical KdV equation. Already in [7] Gervais conjectured that there are quantum analogs of these, i.e. given that the *operator* field $u(x)$ satisfies the commutation relations associated with the bracket (3), for an arbitrary central charge c, there exist an infinite number of operators $\mathcal{P}_n$, $n = 1, 2, ...$, all integrals of normal-ordered densities in u and its derivatives, that mutually commute. Gervais' conjecture received substantial support from the work of Sasaki and Yamanaka [8], who computed $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5$, and it was finally proved in [9] (see also [10]). Thus the obvious way to define quantum KdV theory is, as in [1], via the Heisenberg equation of motion

$$
u_t = \left[u, \sum_{n=1}^{\infty} \lambda_n \mathcal{P}_n\right] \tag{20}
$$

The equation of [1] is just this with $\lambda_2 = 1$, and $\lambda_n = 0$ for $n \neq 2$.

At this juncture it is probably appropriate to point out that the title "the conserved quantities of the quantum KdV equation" for the operators $\mathcal{P}_n$ is somewhat misleading. The operators $\mathcal{P}_n$ exist because the modes of $u(x)$ satisfy a Virasoro algebra. Quantum KdV theory can only be defined *because* the mutually commuting operators $\mathcal{P}_n$ exist, and is defined in a manner that makes it obvious that the $\mathcal{P}_n$'s remain conserved quantities. Calling them "the conserved quantities of the quantum KdV equation" is therefore putting the cart before the horse.

Returning to the classical theory, our next observation is that the action S can also be considered as an action for the field $h = \ln q_x$. S_0 is given in (12), and for the other terms we just use $u = h_{xx} - \frac{1}{2}h_x^2$. Introducing $j = h_x$, the equation of motion is

$$\frac{c}{24\pi} j_t = -\partial_x(\partial_x + j) \sum_{n=1}^{\infty} \lambda_n \frac{\delta p_n}{\delta u} \tag{21}$$

This is the general equation in the MKdV hierarchy. The Poisson bracket for j is simply

$$\{j(x), j(y)\} = \frac{24\pi}{c} \partial_x \delta(x-y) \tag{22}$$

which is the usual Poisson bracket structure for the MKdV hierarchy. The equation of motion (21) for j implies the equation of motion for $u = j_x - \frac{1}{2}j^2$, but of course the equation for u does not imply that for j, since varying h is more general than varying q. We will from now on mostly be interested in S as an action for h, but before we proceed we note that we can also regard S as a *non-local* action for either j or u. We find

$$\delta S_0 = -\frac{c}{24\pi} \int dxdt\ h_t \delta j = \frac{c}{24\pi} \int dxdt\ q_t q_x^{-1} \delta u \tag{23}$$

so regarding S as an action for j yields the equation of motion

$$\frac{c}{24\pi} h_t = -(\partial_x + h_x) \sum_{n=1}^{\infty} \lambda_n \frac{\delta p_n}{\delta u} \tag{24}$$

and regarding it as an action for u yields the equation of motion

$$\frac{c}{24\pi} q_t = -q_x \sum_{n=1}^{\infty} \lambda_n \frac{\delta p_n}{\delta u} \tag{25}$$

This last equation is the general equation in the Ur-KdV hierarchy. The hierarchy defined in (23) is known as the potential KdV hierarchy.

Now, when we regard S as an action for the field h it turns out to be natural to add two further terms to the action. There are two more local functionals of h which commute with all the classical KdV hamiltonians $\int dxdt\ p_n$, viz. $V_+ = \int dxdt\ e^h$ and $V_- = \int dxdt\ e^{-h}$ (to prove that V_- commutes with all the KdV hamiltonians requires use of the result that the KdV hamiltonians are symmetric under $j \to -j$). So we consider the modified action

$$S_M = S + \lambda_+ V_+ + \lambda_- V_- \tag{26}$$

S is invariant under a constant shift of the field h, so if λ_+ and λ_- are both non-zero we can without loss of generality take them to be equal. If one of λ_+,λ_- is zero, then since S is invariant under $h \to -h$ we can without loss of generality take it to be λ_- that is zero, and then by shifting h we can set $\lambda_+ = 1$. Thus we see the effect of adding these terms to the equation of motion (21); they add either a term proportional to $\sinh h$ or a term e^h on the right hand side of (21). Thus the action S_M is an action for the general MKdV/Sinh-Gordon flow or the general MKdV/Liouville flow. Note that the quantities V_+ and V_- *do not* commute with each other. Note also that we might have considered adding terms proportional to V_+ and V_- to S considered as an action for q; but V_+ vanishes when we set $h = \ln q_x$ and assume periodic boundary conditions for q, and V_- gives an extra evolution equation for u which can not be expressed simply in terms of u, but requires use of the function q.

The first step in quantization of the theory defined by S_M is to quantize S_0 by using the Poisson bracket for the field j as our "fundamental bracket". At first glance this is just quantization of a free theory. But let us try to quantize the theory "remembering" that $h = \ln q_x$. If we take to q to satisfy periodic boundary conditions (in x), and we also want h to satisfy periodic boundary conditions, this will have some important consequences. First, assuming we want S_0 to be real, we need h to be either real or pure imaginary; the first possibility is not consistent with periodic boundary conditions on q, so we must take h pure imaginary (this implies a very nasty constraint on q, but this need not concern us).

Let us write $h = -i\beta\phi$ and assume c is positive, so that by correct choice of β we can take

$$S_0 \;=\; \frac{1}{8\pi}\int dxdt\ \phi_x\phi_t \tag{27}$$

Next, having decided that h is imaginary, we recall that the imaginary part of a logarithm is only defined mod 2π, so ϕ can only be defined mod $2\pi/\beta$ (i.e. it is a compactified field). This is in fact good. Supposing the range of x to be $2\pi L$, it is clear that we could take $q = (L/in)e^{inx/L}$, where L is an integer. This would give $\phi = -nx/L\beta$, which is not periodic unless ϕ is a compactified field and L is restricted*. The final deduction we can make from the $h = \ln q_x$ relation is that $\int dx\ e^{-i\beta\phi} = 0$ must vanish. Thus we should investigate the quantum theory of a compactified field satisfying free field commutation relations (determined from (27)) subject to the constraint $\int dx\ e^{-i\beta\phi} = 0$, where β^{-1} is the radius of compactification.

I do not intend here to pursue this quantization to the end. The main point we need is that states in the theory will be defined as states in a free field theory that are in some sense annihilated by the (normal ordered) constraint**, and permitted operators which are polynomial in ϕ and its x derivatives will have to commute with the constraint. In particular, if we seek a permitted operator of form

$$T = -\tfrac{1}{4}\phi_x^2 + i\alpha\phi_{xx} \tag{28}$$

we find we need $\alpha = \frac{1}{2}(\beta - \beta^{-1})$. The modes of T satisfy a Virasoro algebra of central charge $\tilde{c} = 13 - 6(\beta^2 + 1/\beta^2)$, (reproducing (4.18) in the second paper of [8], with $\hbar = 1/2$). For $\beta = \sqrt{m/(m+1)}$, $m = 3, 4, \ldots$, this reproduces the central charges of the unitary minimal models. We see thus that the quantum theory based on S_0 constructed thus is very much related to the Feigin-Fuchs construction for the minimal models.

---

* This seems very strange; but it must be remembered that in writing (27) we have fixed the coupling constant of the theory, and we can in fact formulate the theory with an arbitrary range for x but with a specific coupling constant.

** Looking at the work of Felder [11] we see that that to make this precise it is necessary to define a somewhat extended action of the constraint operator. I do not intend to enter into the details of this here.

It is straightforward now to exploit the results of [8] to understand the full quantum theory associated with S_M. First we note that since we have identified a Virasoro field in the theory defined by S_0, we can add to it the operators $\mathcal{P}_n$ defined above appropriate for the central charge $\tilde{c}$. This gives us the quantum version of S. For the quantum analogs of $V_\pm$, which we will call $\mathcal{V}_\pm$, we take the normal ordered versions of their classical counterparts, $\mathcal{V}_\pm = \int dxdt\ :e^{\mp i\beta\phi}:$. Clearly the $\mathcal{P}_n$'s, which are constructed out of T, all commute with $\mathcal{V}_+$ and since the $\mathcal{P}_n$ are symmetric under $\phi \to -\phi$ (as in the classical case), they must therefore also commute with $\mathcal{V}_-$***. The significance of the commutation of $\mathcal{V}_-$ with the $\mathcal{P}_n$'s in conformal field theory is exactly the statement that the quantum MKdV hamiltonians are conserved quantities in the $\Phi_{(1,3)}$ perturbation of the minimal models [2]; this is because (as noted by Eguchi and Yang [3]) in the Feigin-Fuchs formulation of the minimal models with energy momentum tensor T, the vertex operator $:e^{i\beta\phi}:$ is just the $(1,3)$ primary field (of conformal dimension $(m-1)/(m+1)$) ($:e^{-i\beta\phi}:$ is one of the screening currents, of dimension 1). We see in fact that the most general integrable perturbation of the minimal models including the $\Phi_{(1,3)}$ perturbation consists of adding terms proportional to $\mathcal{P}_n$ and $\mathcal{V}_+$. While it is a somewhat obvious statement that you can add to the hamiltonian of an integrable field theory an arbitrary sum of the conserved quantities, the fact that quantum MKdV theory is an integrable perturbation of the minimal models seems not to be widely appreciated.

Generalizations

We would naturally like to extend this work to the $SL(N)$-KdV equations for $N > 2$, and ultimately to other integrable systems, both for the sake of having actions for integrable systems *per se* and for the sake of understanding all integrable deformations of conformal field theories. Here I only intend to briefly present the construction of the action for the $SL(3)$-KdV case, i.e. I discuss actions for the Boussinesq hierarchy; it seems likely that this will extend smoothly to the $SL(N)$ case.

The crucial ingredients are the analog of S_0 (which we will call $S_0^{(3)}$) and the identi-

*** Again, some deeper analysis is required to understand in what sense $\mathcal{V}_-$ is defined in the theory.

fication of appropriate fields, the analogs of q, h, j, u from above. We expect $S_0^{(3)}$ to be a constrained WZW action of the kind discussed in [6], which in appropriate variables (the analogs of h from above) is a free field theory. Explicitly, we take

$$S_0^{(3)} = \nu \int dxdt \left(\sum_{i=1}^{3} h_{ix} h_{it} \right) \tag{28}$$

where ν is a constant and h_1, h_2, h_3 are three fields satisfying $h_1 + h_2 + h_3 = 0$. If we eliminate h_2 from $S_0^{(3)}$ it essentially reproduces equation (166) in [6]. We define $j_i = \partial_x h_i$, $i = 1, 2, 3$. The analogs of the field u from above will presumably be related to the fields j_i by a standard Miura map. The only question that needs to be resolved is the identification of the analogs of q.

The analogs of q should provide us with a parametrization of $SL(3)$ matrices g such that

$$[g^{-1} g_x]_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{29}$$

Here $[M]_+$ denotes the strictly upper triangular part of M. This subset of $SL(3)$ is invariant under $g \to gU$, where

$$U = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & \gamma & 1 \end{pmatrix} \tag{30}$$

Taking inspiration from the further work of Wilson [12], we observe that if (29) is satisfied we can write

$$g = \begin{pmatrix} a_{xx} + Aa_x + Ba & a_x + Ca & a \\ b_{xx} + Ab_x + Bb & b_x + Cb & b \\ c_{xx} + Ac_x + Bc & c_x + Cc & c \end{pmatrix} \tag{31}$$

where a, b, c, A, B, C are functions, with the Wronskian of a, b, c equal to 1. The group action on g essentially corresponds to translations of A, B, C. To complete the parametrization we just need a parametrization of the three functions a, b, c whose Wronskian is 1. One way to obtain this is to write

$$\begin{aligned} a &= c\phi \\ b &= c\psi \end{aligned} \tag{32}$$

Then the Wronskian condition is solved to give $c = (\phi_{xx}\psi_x - \psi_{xx}\phi_x)^{-1/3}$ and thus a, b, c are written in terms of ϕ, ψ.

We call different choices of A, B, C in g different "gauges". Two gauges are of particular interest; if A, B, C are set to zero we find

$$g^{-1}g_x = \begin{pmatrix} 0 & 1 & 0 \\ u_2 & 0 & 1 \\ u_3 & 0 & 0 \end{pmatrix} \tag{33}$$

and if A, B, C are chosen so that g is upper triangular (which is clearly possible if $bc_x - cb_x \neq 0$)

$$g^{-1}g_x = \begin{pmatrix} j_1 & 1 & 0 \\ 0 & j_2 & 1 \\ 0 & 0 & j_3 \end{pmatrix} \tag{34}$$

with $j_1 + j_2 + j_3 = 0$. These formulae complete the relationships between the different variables we need, which I now summarize:

$$\begin{aligned} h_1 &= -\ln(c^2\psi_x) \\ h_3 &= \ln c \end{aligned} \tag{35}$$

$$c = (\phi_{xx}\psi_x - \psi_{xx}\phi_x)^{-\frac{1}{3}} \tag{36}$$

$$j_i = \partial_x h_i, \quad i = 1, 3 \tag{37}$$

$$\begin{aligned} u_2 &= (j_3 - j_1)_x + j_1^2 + j_1 j_3 + j_3^2 \\ u_3 &= j_{3xx} + j_3(2j_3 + j_1)_x - j_1 j_3(j_1 + j_3) \end{aligned} \tag{38}$$

All that remains is to do some hard calculations. We find (ignoring boundary contributions)

$$\begin{aligned} \delta S_0^{(3)} &= -2\nu \int dxdt \;\; (u_{3t} - u_{2xt} + j_1 u_{2t})\psi_x c^3 \delta\phi - (u_{3t} - u_{2xt} + L u_{2t})\phi_x c^3 \delta\psi \\ &= -2\nu \int dxdt \;\; (2j_1 + j_3)_t \delta h_1 + (2j_3 + j_1)_t \delta h_3 \\ &= \;\; 2\nu \int dxdt \;\; (2h_1 + h_3)_t \delta j_1 + (2h_3 + h_1)_t \delta j_3 \\ &= \;\; 2\nu \int dxdt \;\; (\phi_t\psi_x - \psi_t\phi_x)c^3 \delta u_3 + (\psi_x(c\phi_t)_x - \phi_x(c\psi_t)_x)c^2 \delta u_2 \end{aligned} \tag{39}$$

where $L = -\partial_x \ln(c^2\phi_x)$. The local Poisson brackets derived from $S_0^{(3)}$ are

$$\begin{pmatrix} \{j_1(x), j_1(y)\} & \{j_1(x), j_3(y)\} \\ \{j_3(x), j_1(y)\} & \{j_3(x), j_3(y)\} \end{pmatrix} = -\frac{1}{6\nu} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \partial_x \delta(x - y) \tag{40}$$

$$\begin{pmatrix} \{u_2(x),u_2(y)\} & \{u_2(x),v_3(y)\} \\ \{v_3(x),u_2(y)\} & \{v_3(x),v_3(y)\} \end{pmatrix} =$$

$$-\frac{1}{2\nu}\begin{pmatrix} -2\partial_x^3+u_2\partial_x+\partial_x u_2 & v_3\partial_x+2\partial_x v_3 \\ 2v_3\partial_x+\partial_x v_3 & \frac{2}{3}\partial_x^5-\frac{5}{3}(u_2\partial_x^3+\partial_x^3 u_2)+(u_{2xx}\partial_x+\partial_x u_{2xx})+\frac{8}{3}u_2\partial_x u_2 \end{pmatrix}\delta(x-y) \tag{41}$$

In (41) I have made a standard field redefinition, setting $v_3 = 2u_3 - u_{2x}$. The matrix in (41) agrees, up to a trivial rescaling, with the second hamiltonian structure of the Boussinesq equation as given in [13], example 7.28. From (40) we see that the combinations $J = j_1 + j_3$ and $K = j_1 - j_3$ commute with respect to the Poisson bracket, and these are useful for certain calculations. From the 1,1 entry of (41) and the formulae of [7] we identify the central charge that will charcterize quantum Boussinesq theory to be $-96\pi\nu$ (this calculation actually verifies a conjecture made in [6b] just before equation (173); our action is related to the action of [6b] by $S_0^{(3)} = 4\pi\nu S_{W_3}$).

The full Boussinesq action is taken to be

$$S^{(3)} \;=\; S_0^{(3)} \;+\; \sum_{\substack{n=1 \\ n\neq 0 \ mod\ 3}}^{\infty} \lambda_n \int dxdt\; p_n[u_2,v_3] \tag{42}$$

where $p_n[u_2,v_3]$, $n = 1,2,4,5,...$, are the densities of the conserved quantities of the Boussinesq equation. These satisfy

$$\begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}\begin{pmatrix} \frac{\delta p_{n+3}}{\delta u_2} \\ \frac{\delta p_{n+3}}{\delta v_3} \end{pmatrix} =$$

$$\begin{pmatrix} -2\partial_x^3+u_2\partial_x+\partial_x u_2 & v_3\partial_x+2\partial_x v_3 \\ 2v_3\partial_x+\partial_x v_3 & \frac{2}{3}\partial_x^5-\frac{5}{3}(u_2\partial_x^3+\partial_x^3 u_2)+(u_{2xx}\partial_x+\partial_x u_{2xx})+\frac{8}{3}u_2\partial_x u_2 \end{pmatrix}\begin{pmatrix} \frac{\delta p_n}{\delta u_2} \\ \frac{\delta p_n}{\delta v_3} \end{pmatrix} \tag{43}$$

The first few p_n's are

$$\begin{aligned} p_1 &= u_2 \\ p_2 &= v_3 \\ p_4 &= u_2 v_3 \\ p_5 &= v_3^2+\frac{4}{9}u_2^3-\frac{1}{3}u_{2x}^2 \end{aligned} \tag{44}$$

All the equations of the Boussinesq, modified Boussinesq and Ur-Boussinesq hierarchies are found as the equations of motion, treating the action as a functional of appropriate variables. I just illustrate for the case where $\lambda_2 = \nu$ and all the other λ_n are zero; we obtain the equations

$$\begin{aligned} u_{2t} &= -v_{3x} \\ v_{3t} &= \tfrac{1}{3}u_{2xxx} - \tfrac{2}{3}(u_2^2)_x \end{aligned} \tag{45}$$

$$\begin{aligned} j_{1t} &= \tfrac{1}{3}(2j_{3x} + j_{1x} + 2j_3^2 - j_1^2 + 2j_1j_3)_x \\ j_{3t} &= \tfrac{1}{3}(-2j_{1x} - j_{3x} + 2j_1^2 - j_3^2 + 2j_1j_3)_x \end{aligned} \tag{46}$$

$$\begin{aligned} \phi_t &= -\phi_{xx} - 2c^{-1}c_x\phi_x \\ \psi_t &= -\psi_{xx} - 2c^{-1}c_x\psi_x \end{aligned} \tag{47}$$

where in (47) c is given by (36). Equation (46) is transformed into equation (2.7) of [14] via the substitution

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 3^{-1/2} & -3^{-1/2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} j_1 \\ j_3 \end{pmatrix} \tag{48}$$

and by rescaling the coordinates. It remains to remark that we can add to the action $S^{(3)}$ a sum of terms proportional to $\int dxdt\ \exp(2h_1 + h_3)$, $\int dxdt\ \exp(-h_1 - 2h_3)$ and $\int dxdt\ \exp(h_3 - h_1)$, without ruining the integrability. These terms give rise to the $SL(3)$-Toda flow that commutes with the MKdV flows; note the first two terms vanish upon writing h_1 and h_3 in terms of ϕ and ψ. This completes construction of the classical $SL(3)$-KdV action.

Conclusions

The work presented here goes some way towards clarifying the magical properties of both classical and quantum integrable systems. We started our constructions with the choice of a "free" action (S_0 or $S_0^{(3)}$) which, being first order in time derivatives, determined a Poisson bracket algebra but gave a vanishing Hamiltonian. This Poisson bracket structure (and the corresponding set of operator commutation relations) has the property that it admits an infinite number of mutually commuting quantities of the type now familiar. Of course, this is not a simple result (particularly in the quantum case), and I have not here said anything about the proof of this result. But it seems important to stress that once this property has been established, as a property of the Poisson brackets/commutation relations of the "free" theory, the existence and integrability of classical and quantum

KdV hierarchies becomes essentially obvious; it is just necessary to add a Hamiltonian consisting of a sum of the mutually commuting quantities.

As I have already pointed out, it is of interest to generalize this work to find actions for other integrable systems. One essential part of this is to examine if and how Wilson's antiplectic formalism can be extended to systems such as the non-linear Schrödinger hierarchy, and this will be tackled in a forthcoming paper. Another question outstanding is to ask, since we have obtained quantum MKdV theory as a deformation of the minimal models, whether there is a statistical mechanical meaning to such deformations. If so, it may be interesting to see if via conformal field theoretical techniques one can do computations in quantum MKdV theory.

One aspect of KdV theory that has been completely absent from this paper is the τ-function formalism. The relationship of the KdV field u and the τ-function is $u = 2\partial_x^2 \ln \tau$, but there is no simple relation between τ and any of the fields q, h, j, so it is not clear how to write an action to yield Hirota's form of the KdV equation (note though that if we write $q_x = p^2$ we find $u = 2p\partial_x p^{-1}\partial_x \ln p$). The relationship between integrability of the KdV hierarchy from the point of view of the existence of an infinite number of conserved quantities, and integrability from the point of view of an infinite-dimensional group action on the space of solutions remains somewhat mysterious.

Acknowledgements

I thank D.Depireux for bringing my attention to references [1],[5],[10] and [12], and for some comments. Useful discussions with R.Dijkgraaf, A.Giveon, O.Lechtenfeld, V.P.Nair, E.Verlinde and E.Witten are acknowledged. This work was supported by the U.S.Department of Energy under grant DE-FG02-90ER40542.

References

[1] B.A.Kupershmidt and P.Mathieu, *Phys.Lett.B* **227** (1989) 245.

[2] A.B.Zamolodchikov, *Adv.Stud.in Pure Math.* **19** (1989) 641.

[3] T.Eguchi and S.Yang *Phys.Lett.B* **224** (1989) 373; T.Hollowood and P.Mansfield, *Phys.Lett.B* **226** (1989) 73.

[4] A.Das, *Integrable Models*, World Scientific (1989).

[5] G.Wilson, *Phys.Lett.A* **132** (1988) 45.

[6] a) A.M.Polyakov, *Mod.Phys.Lett.A* **2** (1987) 893; b) M.Bershadsky and H.Ooguri, *Comm.Math.Phys.* **126** (1989) 49 and references therein.

[7] J.-L. Gervais, *Phys.Lett.B* **160** (1985) 277.

[8] R.Sasaki and I.Yamanaka, *Comm.Math.Phys.* **108** (1987) 691, *Adv.Stud.in Pure Math.* **16** (1988) 271.

[9] B.Feigin and E.Frenkel, *Phys.Lett.B* **276** (1992) 79.

[10] P.Di Francesco, P.Mathieu and D.Sénéchal, *Mod.Phys.Lett.A* **7** (1992) 701.

[11] G.Felder, *Nucl.Phys.B* **317** (1989) 215.

[12] G.Wilson, *Quart.J.Math.Oxford* **42** (1991) 227, *Nonlinearity* **5** (1992) 109, and in *Hamiltonian Systems, Transformation Groups and Spectral Transform Methods*, ed. J.Harnad and J.E.Marsden, CRM (1990).

[13] P.J.Olver *Applications of Lie Groups to Differential Equations*, Springer-Verlag (1986).

[14] P.Mathieu and W.Oevel, *Mod.Phys.Lett.A* **6** (1991) 2397.