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Self-Dual Yang-Mills and the Hamiltonian Structures of Integrable Systems

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Abstract

In recent years it has been shown that many, and possibly all, integrable systems can be obtained by dimensional reduction of self-dual Yang-Mills. I show how the integrable systems obtained this way naturally inherit bihamiltonian structure. I also present a simple, gauge-invariant formulation of the self-dual Yang-Mills hierarchy proposed by several authors, and I discuss the notion of gauge equivalence of integrable systems that arises from the gauge invariance of the self-duality equations (and their hierarchy); this notion of gauge equivalence may well be large enough to unify the many diverse existing notions.

Contents

1. Introduction

Early on in the study of the self-dual Yang-Mills (SDYM) equations it was observed that dimensional reductions of these equations give rise to so-called "integrable systems" [1]. It was conjectured by Ward [2] that SDYM may be a *universal* integrable system, i.e. that *all* integrable systems might be obtained from it by suitable reductions. A remarkable piece of evidence for this was produced a few years ago by Mason and Sparling [3], who showed how to obtain the Korteweg-de Vries (KdV) and nonlinear Schrödinger (NLS) equations from SDYM, and further wrote down a hierarchy of gauge-invariant equations from which the KdV and NLS hierarchies could be obtained. This stimulated much activity. Mason and Sparling just examined SDYM with gauge group $SL(2)$; Bakas and Depireux showed how to obtain certain flows in the $SL(N)$-KdV hierarchy from $SL(N)$ SDYM, and also obtained a apparently new hierarchy from $SL(3)$ SDYM, with a hamiltonian structure given by the classical limit of the W_3^2 conformal algebra [4]. I showed how the reduction method of Mason and Sparling could be extended to obtain certain three dimensional versions of the KdV and NLS equations from SDYM [5]; this was also observed by Strachan, who studied the resulting equations in some depth [6]. In a recent paper Ablowitz *et.al.* introduced an apparently different version of the SDYM hierarchy, and showed, amongst other things, how the KP hierarchy can be obtained from SDYM with an infinite-dimensional gauge group [7].

Certain well-known properties of integrable systems are very obviously inherited from

SDYM. Amongst these are the Lax pair formulation, given for SDYM by Belavin and Zakharov [8], and the Painlevé property [9], shown for SDYM in [10]. Other properties of integrable systems have at least natural analogs in SDYM. We might expect the inverse scattering formalism to be a special case of the twistor formalism for SDYM, and this is indeed the case [3]; we might expect Bäcklund transformations for integrable systems to originate in those for SDYM [11], or maybe in the loop group action on solutions of SDYM [12]; finally, for certain integrable systems there exist direct solution methods ("the Hirota method"), and these presumably arise in cases where we can actually solve the inverse twistor transform and thereby write down solutions exploiting the action of the loop group on the twistor data [12] (the "Hirota variables", which have the property of being entire [13], are presumably related to the patching matrix G in [12]; I draw the reader's attention to a recent work on the relation of the zero-curvature and Hirota formulations of integrable systems [14], which might be exploited to decide these issues).

None of the above issues will be treated further in this paper. Here I wish to devote myself to one property of integrable systems which has an obvious analog in the SDYM equations, and another that does not. There is a notion of "gauge equivalence" between different integrable systems, two systems being said to be equivalent when there is a map from the solutions of one to the solutions of the other. The classic example of such a map is the Miura map between solutions of MKdV and KdV. These maps arise in many diverse settings. The origin of these maps at the level of SDYM is just gauge freedom in the SDYM equations; performing the same reduction in different gauges gives rise to different equations. I will give a detailed analysis of different gauges in which one might consider reduction of SDYM, and show how the usual notions of gauge equivalence (for instance that of Drinfeld and Sokolov [15]) emerge.

The property of integrable systems I wish to explain that does not have an immediate analog in SDYM is that of bihamiltonian structure. It turns out that this has its origins in the fact that the space of solutions of SDYM has three symplectic structures. These are usually thought of as gauge invariant, but this assumes certain boundary conditions; for the types of solution we will be looking at, we will see the symplectic structures have certain restricted gauge invariances, and they give (on reduction) the bihamiltonian structures of integrable systems. This is the main result of this paper; understanding different gauge choices etc. is a necessary tool for this analysis.

The contents of this paper have been given above. While not strictly necessary for the results on hamiltonian structures, I have included in section 2 what I consider to be the most natural definition of the SDYM hierarchy, and I show in section 3 that it contains the hierarchies of [3], [6] and [7]; in section [11] I give a reduction of the hierarchy to the

KP hierarchy (c.f.[7]).

2. The Self-Dual Yang-Mills Equations and Hierarchy

The SDYM equations on $\mathbf{R}^4$ with coordinates $z, \bar{z}, w, \bar{w}$ and metric $ds^2 = dwd\bar{z} - dzd\bar{w}$ are (in standard notation)

$$F_{zw} = 0 \tag{1a}$$

$$F_{\bar{z}\bar{w}} = 0 \tag{1b}$$

$$F_{z\bar{w}} - F_{w\bar{z}} = 0 \tag{1c}$$

These are consistency conditions for the linear system

$$\begin{aligned}(D_z + \lambda D_{\bar{z}})\Psi = 0\\ (D_w + \lambda D_{\bar{w}})\Psi = 0\end{aligned} \tag{2}$$

There is a natural generalization of the above equations we might write down. Let $z_1, ..., z_n, \bar{z}_1, ..., \bar{z}_n$ be coordinates for $\mathbf{R}^{2n}$, $n \geq 2$; the consistency conditions for the linear system

$$(D_{z_i} + \lambda D_{\bar{z}_i})\Psi = 0, \qquad\qquad i = 1, ..., n \tag{3}$$

are

$$\begin{aligned}F_{z_i z_j} &= 0\\ F_{\bar{z}_i \bar{z}_j} &= 0 \qquad\qquad i, j \in \{1, ..., n\}; i \neq j\\ F_{z_i \bar{z}_j} - F_{z_j \bar{z}_i} &= 0\end{aligned} \tag{4}$$

If d is the dimension of the gauge group, then there are $\frac{3}{2}dn(n-1)$ equations here for $d(2n-1)$ unknowns (components of the gauge field modulo gauge transformations); the equations are therefore overdetermined for $n > 2$. For $n = 4$ equations (4) are equations $(3.5)'$ in [16]. I will take equations (4) with $n = \infty$ as my definition of the SDYM hierarchy. Whereas equations (1) can be modeled on any manifold with a metric, I cannot see how to write equations (4) on anything more general than the cartesian product of $2n$ 1-dimensional manifolds.

3. Gauge Choices

There are two ways to approach solving equations (1) I wish to consider, and these extend naturally to the SDYM hierarchy. Yang [17] proposed first solving equations (1a) and (1b) to write

$$\begin{aligned}A_z = g^{-1}\partial_z g\\ A_w = g^{-1}\partial_w g\end{aligned} \tag{5a}$$

$$
\begin{aligned}
A_{\bar{z}} &= h^{-1}\partial_{\bar{z}}h \\
A_{\bar{w}} &= h^{-1}\partial_{\bar{w}}h
\end{aligned} \tag{5b}
$$

where g, h are some gauge group valued functions. Gauge transformations ($A \to \Lambda^{-1}A\Lambda + \Lambda^{-1}d\Lambda$) act on g, h by $g \to g\Lambda$, $h \to h\Lambda$. Equation (1c) reduces to

$$
\partial_z(J^{-1}\partial_{\bar{w}}J) - \partial_w(J^{-1}\partial_{\bar{z}}J) = 0 \tag{6}
$$

where $J = hg^{-1}$ (which is gauge invariant); equation (6) can also be written

$$
\partial_{\bar{z}}(J^{-1}\partial_w J) - \partial_{\bar{w}}(J^{-1}\partial_z J) = 0 \tag{6$'$}
$$

Equation (6) or $(6)'$ is known as the J formulation of SDYM. The other approach to SDYM (which I believe was first given implicitly in [18]) is to solve (1b) to express $A_{\bar{z}}, A_{\bar{w}}$ in terms of h (equation 5b) and then to solve (1c) to get

$$
\begin{aligned}
A_z &= h^{-1}\partial_z h + h^{-1}\partial_{\bar{z}}Nh \\
A_w &= h^{-1}\partial_w h + h^{-1}\partial_{\bar{w}}Nh
\end{aligned} \tag{7}
$$

where N is some gauge-invariant function valued in the Lie algebra of the gauge group. (1a) then yields

$$
(\partial_w\partial_{\bar{z}} - \partial_z\partial_{\bar{w}})N + [\partial_{\bar{w}}N, \partial_{\bar{z}}N] = 0 \tag{8}
$$

If we write $M = \partial_{\bar{z}}N$ then we can write this, at least formally, as

$$
\partial_w M = (\partial_z + [M, \ \ ])\partial_{\bar{z}}^{-1}\partial_{\bar{w}}M \tag{9}
$$

I call this the M formulation of SDYM. Treating w as "time", this has the form of an evolution equation. There is a clear hint from (9) that if we wish to obtain local evolution equations by reduction of SDYM, what we need to do is fix the $\bar{z}$ dependence so that the $\partial_{\bar{z}}^{-1}$ integration symbol can be integrated out. Note that a lagrangian for (8) was given in [19], and one for (6) was given in [20].

In [18] it was noted that there exists an integrable hierarchy of which (9) is the first non-trivial member,

$$
\partial_{z_i}M = [(\partial_z + [M, \ \ ])\partial_{\bar{z}}^{-1}]^{i-1}\partial_{\bar{w}}M \qquad\qquad i = 2, 3, ... \tag{10}
$$

Let us consider our SDYM hierarchy (4). Solving the second and third sets of equations we have

$$
\begin{aligned}
A_{\bar{z}_i} &= h^{-1}\partial_{\bar{z}_i}h \\
A_{z_i} &= h^{-1}\partial_{z_i}h + h^{-1}\partial_{\bar{z}_i}Nh
\end{aligned} \tag{11}
$$

Write $M = \partial_{\bar{z}_1} N$. The $j = 1$ equations in the first set of equations of (4) give

$$\partial_{z_i} M = (\partial_{z_1} + [M, \ \ ])\partial_{\bar{z}_1}^{-1}\partial_{\bar{z}_i} M \qquad\qquad i = 2, 3, ... \tag{12}$$

Writing $z_1 = z, \bar{z}_1 = \bar{z}, \bar{z}_2 = \bar{w}$ it is clear that if we impose the dimensional reductions $\partial_{\bar{z}_i} = \partial_{z_{i-1}}$, $i = 3, 4, ...$ on (12), then we recover (10). This latter hierarchy is essentially a dimensionally reduced version of our hierarchy in $A_{\bar{z}_i} = 0$ gauge. The hierarchy of [7] (equation (21)) is a slightly generalized version of this, where the $A_{\bar{z}_i}$'s are allowed to be arbitrary commuting constant matrices (though the authors are clearly quite aware that one can generate more general possibilities via gauge transformation). The Bogomolnyi hierarchy of [3], which is written in a gauge invariant way, can be obtained from ours by the dimensional reduction $\partial_{\bar{z}_i} = \partial_{z_{i-1}}$, $i = 1, 2, ...$ (so $\partial_{\bar{z}_1} = 0$). The hierarchy of [6] is a generalization of this, obtained from ours by the dimensional reduction $\partial_{\bar{z}_i} = \partial_{z_{i-1}}$, $i = 1, 2, ...$, $i \neq m$, where m is some integer greater than 1. I have not investigated the relationship of the hierarchy (4) with that of [21]. Especially for the purpose of discussing reductions of SDYM, where, as I mentioned in the introduction, reductions in different gauges give rise to equations related by Miura maps, it is important to have the full gauge-invariant version of the hierarchy.

Finally in this section, we will need later the version of M formulation where we first solve (1a) and (1c). Solving (1a) gives us (5a), and solving (1c) gives

$$\begin{aligned} A_{\bar{z}} &= g^{-1}\partial_{\bar{z}} g + g^{-1}\partial_z P g \\ A_{\bar{w}} &= g^{-1}\partial_{\bar{w}} g + g^{-1}\partial_w P g \end{aligned} \tag{13}$$

Equation (1b) now gives

$$(\partial_{\bar{w}}\partial_z - \partial_{\bar{z}}\partial_w)P + [\partial_w P, \partial_z P] = 0 \tag{14}$$

4. Bihamiltonian Integrable Systems [22]

An evolution equation has (local) bihamiltonian form when it can be written in the form

$$\frac{\partial \mathbf{u}}{\partial w} = \mathcal{D}_1 \frac{\delta \mathcal{H}_3[\mathbf{u}]}{\delta \mathbf{u}} = \mathcal{D}_2 \frac{\delta \mathcal{H}_2[\mathbf{u}]}{\delta \mathbf{u}} \tag{15}$$

where $\mathcal{D}_1, \mathcal{D}_2$ are (local) coordinated hamiltonian operators and $\mathcal{H}_2[\mathbf{u}], \mathcal{H}_3[\mathbf{u}]$ are suitable functionals of $\mathbf{u}(x)$. The recursion operator is defined by $\mathcal{R} = \mathcal{D}_2\mathcal{D}_1^{-1}$. The conserved quantities are given recursively by

$$\mathcal{D}_1 \frac{\delta \mathcal{H}_i}{\delta \mathbf{u}} = \mathcal{D}_2 \frac{\delta \mathcal{H}_{i-1}}{\delta \mathbf{u}} \tag{16}$$

and the associated hierarchy of equations is given by

$$\frac{\partial \mathbf{u}}{\partial z_i} = \mathcal{D}_1 \frac{\delta \mathcal{H}_{i+1}}{\delta \mathbf{u}} = \mathcal{D}_2 \frac{\delta \mathcal{H}_i}{\delta \mathbf{u}} = \mathcal{R}^{i-2} \mathcal{D}_2 \frac{\delta \mathcal{H}_2}{\delta \mathbf{u}}, \qquad i = 2, 3, ... \tag{17}$$

The classic example is the KdV equation for which

$$\begin{aligned} \mathcal{D}_1 &= \partial_x \\ \mathcal{D}_2 &= -\partial_x^3 + u(x)\partial_x + \partial_x u(x) \\ \mathcal{H}_2 &= \int dx\ \tfrac{1}{2} u^2 \\ \mathcal{H}_3 &= \int dx\ \tfrac{1}{2}(u_x^2 + u^3) \end{aligned} \tag{18}$$

Without going into details we note the existence of "elementary dimensional deformations" [23] of bihamiltonian integrable systems. For example, the KdV equation can be written $u_w = \mathcal{R} u_x$, and the equation $u_w = \mathcal{R} u_y$ describing the evolution in w of a function $u(x, y, w)$ is also integrable. This equation is non-local, but by the substitution $u = \gamma_x$ is made local; it has bihamiltonian form (1) with the same $\mathcal{D}_1, \mathcal{D}_2$ as for the KdV equation but with

$$\begin{aligned} \mathcal{H}_2 &= \int dxdy\ \tfrac{1}{2}\gamma_x \gamma_y \\ \mathcal{H}_3 &= \int dxdy\ \tfrac{1}{2}(\gamma_{xx}\gamma_{xy} + \gamma_x^2 \gamma_y) \end{aligned} \tag{19}$$

SDYM in M formulation is a bihamiltonian integrable system; on the space of Lie algebra valued functions $M(z, \bar{z})$, $\mathcal{D}_2 = \partial_z + [M, \ \ ]$ and $\mathcal{D}_1 = \partial_{\bar{z}}$ are two coordinated hamiltonian operators. We clearly have

$$\mathcal{H}_2 = \tfrac{1}{2} \int dz d\bar{z} d\bar{w}\ Tr(M \partial_{\bar{z}}^{-1} \partial_{\bar{w}} M) \tag{20}$$

but it is not so easy to write expressions for the higher conserved quantities. Note though we have $\partial_{z_i} M = (\partial_z + [M, \ \ ]) \delta \mathcal{H}_i / \delta M$. This equation is central in Chern-Simons theory; using $M = -\partial_z J J^{-1}$ we deduce we can write

$$\mathcal{H}_n = \int d\bar{z} d\bar{w}\ S^{(i)}_{WZW}[J] \tag{21}$$

where $S^{(i)}_{WZW}$ denotes the WZW action on the plane defined by coordinates z, z_i. This satisfies (up to an irrelevant overall factor)

$$\delta S^{(i)}_{WZW} = \int dz dz_n\ Tr[J^{-1} \delta J \partial_z (J^{-1} \partial_{z_i} J)] \tag{22}$$

(see [24]), from which it is easy to check that $\mathcal{H}_i$ as given in (21) is independent of all the times z_j, $j \geq 2$ (by construction it is independent of z_i). It is not clear to me at the moment whether there are analogs of $\mathcal{H}_i$ for the more general hierarchy (4).

5. Dimensional Reduction of SDYM

Following [3], let us consider reducing (1) by requiring the potentials to be independent of $\bar{z}$. This is not a gauge invariant statement; to avoid problems we restrict ourselves to $\bar{z}$-independent gauge transformations. Since $A_{\bar{z}}$ transforms homogeneously under such gauge transformations, $A_{\bar{z}} \to \Lambda^{-1} A_{\bar{z}} \Lambda$, we can no longer fix the gauge $A_{\bar{z}} = 0$. Restricting ourselves further to the case where $A_{\bar{z}}$ in some gauge is constant, for $SL(2)$ the equivalence classes of such $A_{\bar{z}}$'s under gauge transformations are represented by

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \kappa \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{23}$$

where κ is an arbitrary constant. These give rise to integrable systems related to KdV and NLS respectively [3]. For $SL(N)$, the class represented by $A_{\bar{z}}$ with $(A_{\bar{z}})_{ij} = \delta_{iN}\delta_{j1}$ gives rise to the SL(N)-KdV equation, and the class represented by $A_{\bar{z}} = \mathrm{diag}(\kappa_1, ..., \kappa_N)$, $\kappa_1 + ... + \kappa_N = 0$, gives rise to the generalized NLS equation of Fordy and Kulish [25]. For $SL(3)$ the class

$$A_{\bar{z}} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{24}$$

gives rise to the W_3^2 hierarchy of Bakas and Depireux [4], and the remaining class

$$A_{\bar{z}} = \begin{pmatrix} \kappa & 0 & 0 \\ 1 & \kappa & 0 \\ 0 & 0 & -2\kappa \end{pmatrix} \tag{25}$$

has yet to be investigated.

Having done the dimensional reduction and chosen the class of $A_{\bar{z}}$, the way I propose looking at equations (1) is to regard (1a) and (1c) as evolution equations for $A_z, A_{\bar{z}}$, and (1b) as a "constraint", restricting some of the entries of $A_{\bar{w}}$; explicitly we have evolution equations

$$\begin{aligned} \partial_w A_z &= \partial_z A_w + [A_z, A_w] \\ \partial_w A_{\bar{z}} &= \partial_z A_{\bar{w}} - \partial_{\bar{w}} A_z + [A_z, A_{\bar{w}}] + [A_{\bar{z}}, A_w] \end{aligned} \tag{26}$$

where $\partial_{\bar{w}} A_{\bar{z}} = [A_{\bar{z}}, A_{\bar{w}}]$ constrains certain entries of $A_{\bar{w}}$ (we assume the class of $A_{\bar{z}}$ has been chosen so this has a solution). Note these equations are invariant under the *full* group of ($\bar{z}$-independent) gauge transformations. We obtain the restricted notion of gauge invariance used in Drinfeld-Sokolov [15] and its generalizations [26] when we partially fix the gauge by going to a gauge where $A_{\bar{z}}$ is constant; there is still a residual gauge invariance generated by the elements of the gauge algebra that commute with the gauge-fixed $A_{\bar{z}}$. While this restricted notion of gauge invariance allows us to understand some Miura maps, there are others that it cannot explain. For example, NLS arises from (26) with

$$\begin{aligned} A_{\bar{z}} &= \begin{pmatrix} \kappa & 0 \\ 0 & -\kappa \end{pmatrix} & A_{\bar{w}} &= 0 \\ A_z &= \begin{pmatrix} 0 & \psi \\ \bar{\psi} & 0 \end{pmatrix} & A_w &= \tfrac{1}{2\kappa} \begin{pmatrix} \partial_z^{-1}\partial_{\bar{w}}(\psi\bar{\psi}) & \partial_{\bar{w}}\psi \\ \partial_{\bar{w}}\bar{\psi} & -\partial_z^{-1}\partial_{\bar{w}}(\psi\bar{\psi}) \end{pmatrix} \end{aligned} \tag{27}$$

where κ is a non-zero constant and $\psi, \bar{\psi}$ are functions, and the Heisenberg ferromagnet equation arises from (26) with

$$\begin{aligned} A_{\bar{z}} &= \begin{pmatrix} \lambda & \mu \\ \nu & -\lambda \end{pmatrix} \qquad A_{\bar{w}} = \begin{pmatrix} \lambda F & \mu F + \mu_{\bar{w}}/2\lambda \\ \nu F - \nu_{\bar{w}}/2\lambda & -\lambda F \end{pmatrix} \\ A_z &= A_w = 0 \end{aligned} \tag{28}$$

where λ, μ, ν are functions satisfying $\lambda^2+\mu\nu = \kappa^2$ and F is defined by $4\kappa^2 F_z = \mu(\lambda^{-1}\nu_{\bar{w}})_z - \nu(\lambda^{-1}\mu_{\bar{w}})_z$. (In fact (27) and (28) give dimensional deformations of the NLS and ferromagnet equations; the usual equations are recovered by the further dimensional reduction $\partial_z = \partial_{\bar{w}}$.) The gauge equivalence between NLS and the Heisenberg ferromagnet, as given in [27], emerges from the full gauge freedom of (26)*.

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* For an analysis of many of the equations in the NLS gauge equivalence class see [28]. There I defined the "Ur-NLS" equation

$$\begin{aligned} S_w &= \frac{2S_zT_{zz}}{T_z} + 3S_z^2 - S_{zz} \\ T_w &= T_{zz} + 2T_zS_z \end{aligned}$$

A solution of the $\kappa = \frac{1}{2}$ ferromagnet equation (equation (52)) is obtained from this by setting

$$\begin{aligned} \lambda &= \tfrac{1}{2} + TS_z/T_z \\ \mu &= -T(1 + TS_z/T_z) \\ \nu &= S_z/T_z \end{aligned}$$

In [29] Hirota showed the gauge equivalence of NLS and the ferromagnet equation to a third equation (the equation in the abstract of [29]); this is obtained from Ur-NLS, modulo some minor rescalings, via $\phi = -Te^S$, $\bar{\phi} = S_z e^{-S}/T_z$.

There is a refinement of equations (26) we can make. We can solve the first of equations (26) via equation (5a), and thus replace (26) with the system

$$\begin{aligned} &\partial_w g = g A_w \\ &\partial_w A_{\bar z} = \partial_z A_{\bar w} - \partial_{\bar w}(g^{-1}\partial_z g) + [g^{-1}\partial_z g, A_{\bar w}] + [A_{\bar z}, A_w] \end{aligned} \tag{29}$$

These equations have standard gauge invariance, i.e. invariance under $A_{\bar w} \to \Lambda^{-1}A_{\bar w}\Lambda + \Lambda^{-1}\partial_{\bar w}\Lambda$, $A_{\bar z} \to \Lambda^{-1}A_{\bar z}\Lambda$, $g \to g\Lambda$, but they also have an *extra* invariance under $g \to tg$ where t is a element of the gauge group**. Solutions of (29) related by $g \to tg$ clearly give the same solution of (26). But instead of passing directly from the variable g to the variable A_z which is invariant under the whole group of t transformations, we could pass first to some set of variables invariant under just a subgroup of t transformations; (29) will imply some evolution for these intermediate variables. Using t symmetry alone, from an integrable system obtained from (29) we will obtain a whole series of integrable systems, by "modding out" by any subgroup of the gauge group. This explanation of the relationship of the the various equations related to KdV, which all stem from an equation called Ur-KdV with a $SL(2)$ invariance was first found by Wilson [30]. We will derive Ur-KdV from (29) shortly; equations (29) can also be used to derive the Ur-NLS equation of [28] (see the footnote to the preceding paragraph).

While the general scheme outlined for reductions in the previous two paragraphs seems to be correct, there is one subtlety; in explicitly relating equations (26) or (29) to known integrable systems by writing everything in components and simplifying, the need arises to fix certain integration constants. It seems that the choice of integration constants can be regarded as the fixing of certain gauge invariant quantity, but I am not aware at the moment of a method of picking out the relevant quantities†. To see that in spite of this the general scheme we have explained is correct, an example is now in order.

6. Example: KdV

To illustrate the relation of equations (26) and (29), and the issue of integration constants

** Throughout this paper I have used the term "gauge group" in the sense of physicists, as the structure group of the theory; mathematicians use the term "gauge group" to refer to what I have called the "group of gauge transformations". I point this out here so that it should be clear that t is a constant, unlike Λ above, which is dependent on the coordinates.

† The problem here seems to arise because SDYM typically contains many copies of any particular integrable system; for example, the parameter κ in (23), is a gauge invariant quantity, and each value of κ gives a reduction of SDYM to NLS.

raised above, let us look at (26) and (29) in the KdV case, with the partial gauge fixing

$$A_{\bar{z}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{30}$$

The second equations of both (26) and (29) become constraints. The most general solution of the constraints for (26) yields

$$A_z = \begin{pmatrix} d & e \\ f & -d \end{pmatrix} \qquad A_w = \begin{pmatrix} dj + \frac{1}{2}(f_{\bar{w}} - j_z) & ej - d_{\bar{w}} \\ c & -dj + \frac{1}{2}(j_z - f_{\bar{w}}) \end{pmatrix} \qquad A_{\bar{w}} = \begin{pmatrix} 0 & 0 \\ j & 0 \end{pmatrix} \tag{31}$$

where c, d, e, f, j are some functions of $z, w, \bar{w}$, with $e_{\bar{w}} = 0$. This last condition is where the integration constant becomes necessary; we will take e to be an arbitrary constant. A gauge invariant way of saying this is that we will fix $Tr[A_{\bar{z}}(A_z - h^{-1}\partial_z h)] = Tr[\partial_{\bar{z}} h h^{-1} \partial_{\bar{z}} N]$, but this is not very illuminating. Residual gauge transformations act via

$$\begin{aligned} d &\to d + eu \\ f &\to f + u_z - 2du - eu^2 \end{aligned} \tag{32}$$

where the function u is the parameter of gauge transformations. Fixing the gauge to $d = 0$, the first equation of (26) now reduces to the dimensional deformation of KdV mentioned in section 4:

$$f_w = \tfrac{1}{2}(-\tfrac{1}{2e}\partial_z^2 + f + \partial_z f \partial_z^{-1}) f_{\bar{w}} \tag{33}$$

Let us look at the corresponding reduction of (29). Writing

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{34}$$

where $\alpha\delta - \beta\gamma = 1$, we have

$$g^{-1}\partial_z g = \begin{pmatrix} \delta\alpha_z - \beta\gamma_z & \delta\beta_z - \beta\delta_z \\ \alpha\gamma_z - \gamma\alpha_z & \alpha\delta_z - \gamma\beta_z \end{pmatrix} \tag{35}$$

and it is no surprise that in the reduction we find we have to fix an integration constant, which we do by setting $\beta\delta_z - \delta\beta_z = e$. Residual gauge transformations act via

$$\begin{aligned} \alpha &\to \alpha + u\beta \\ \gamma &\to \gamma + u\delta \end{aligned} \tag{36}$$

with β, δ invariant. We find that under gauge transformation $\delta\alpha_z - \beta\gamma_z \to \delta\alpha_z - \beta\gamma_z + eu$ so the natural gauge choice is to set $\delta\alpha_z - \beta\gamma_z = 0$. Solving all these constraints on g we eventually find

$$\begin{aligned} g &= \begin{pmatrix} \delta^{-1} + e^{-1} q \delta_z & \delta q \\ e^{-1}\delta_z & \delta \end{pmatrix} \\ g^{-1}\partial_z g &= \begin{pmatrix} 0 & e \\ f & 0 \end{pmatrix} \end{aligned} \tag{37}$$

where $\delta = (eq_z^{-1})^{\frac{1}{2}}$ and $f = -\frac{1}{2e}\{q;z\}$ ($\{q;z\}$ denotes the Schwarzian derivative of q). With this choice of g and

$$A_w = \begin{pmatrix} \frac{1}{4}\partial_{\bar{w}} f & \frac{1}{2}e\partial_z^{-1}\partial_{\bar{w}} f \\ \frac{1}{2}f\partial_z^{-1}\partial_{\bar{w}} f - \frac{1}{4e}\partial_z\partial_{\bar{w}} f & -\frac{1}{4}\partial_{\bar{w}} f \end{pmatrix} \qquad A_{\bar{w}} = \begin{pmatrix} 0 & 0 \\ \frac{1}{2}\partial_z^{-1}\partial_{\bar{w}} f & 0 \end{pmatrix} \tag{38}$$

we obtain from (29) the dimensional deformation of the Ur-KdV equation

$$q_w = -\frac{1}{4e} q_z \partial_z^{-1}\partial_{\bar{w}}\{q;z\} \tag{39}$$

It is easy to check directly that if q solves (39) $f \equiv -\frac{1}{2e}\{q;z\}$ solves (33).

7. Some Formal Manipulations with Formulae of Drinfeld and Sokolov

Having discussed issues of choice of gauge in reductions of SDYM in section 5, we are almost ready to explain the origin of bihamiltonian structure, but one more piece of groundwork is necessary. Even if it is known how to obtain a certain integrable system from a larger one, and the larger one has a known hamiltonian structure, if the hamiltonian structure is presented as in section 4, via a hamiltonian operator $\mathcal{D}$, or equivalently by a set of Poisson brackets, it is an arduous procedure to reduce the hamiltonian structure, via "Dirac reduction". As discussed in [30], an alternative to writing a hamiltonian operator is to write a symplectic form on the space of fields. If the hamiltonian operator is $\mathcal{D}$, and the fields are arranged in a column vector $\mathbf{u}$, then the associated form can be formally written $\int dx \delta\mathbf{u}^T \wedge \mathcal{D}^{-1}\delta\mathbf{u}$. If a form can be written then it is simple to perform a reduction by pulling back the form to the reduced space of fields. Problems arise though due to the need to invert the operator $\mathcal{D}$, which can not always be done (there are systems for which the reverse is true, that the operator $\mathcal{D}^{-1}$ is well-defined, and cannot be sensibly inverted). In the context of bihamiltonian systems, as presented in section 4, there are an infinite number of hamiltonian operators we can formally construct, namely $\mathcal{R}^n\mathcal{D}_1 = (\mathcal{D}_2\mathcal{D}_1^{-1})^n\mathcal{D}_1$, where n can be any integer and we might hope that certain ones of these are "inverse local", so the associated form can be written down.

As mentioned before, to obtain the $SL(N)$ Drinfeld-Sokolov [15] system from SDYM we reduce by imposing $\partial_{\bar{z}} = 0$ and take $(A_{\bar{z}})_{ij} = \delta_{iN}\delta_{j1}$; we find from the second equation of (26) that we need to choose integration constants, which we can do by setting $([A_z]_+)_{ij} = \delta_{i+1,j}$, where $[M]_+$ denotes the strictly upper triangular part of the matrix M. The Drinfeld-Sokolov systems describe evolution of such a matrix A_z, modulo certain gauge transformations. Looking at the formulae for the hamiltonian structures in section 3 of [15], we find that appropriate formal expressions for the symplectic forms associated with the first, second and third hamiltonian structures of the Drinfeld-Sokolov systems are

$$\begin{aligned}
\omega_1 &= \int dz\ Tr(\delta A_z \wedge D_{\bar{z}}^{-1} \delta A_z) \\
\omega_2 &= \int dz\ Tr(\delta A_z \wedge D_z^{-1} \delta A_z) \\
\omega_3 &= \int dz\ Tr(\delta A_z \wedge D_z^{-1} D_{\bar{z}} D_z^{-1} \delta A_z)
\end{aligned} \tag{40}$$

Here $D_z = \partial_z + [A_z,\ \ ],\ D_{\bar{z}} = [A_{\bar{z}},\ \ ];\ D_{\bar{z}}^{-1}$ is at first glance much more problematic to define than D_z^{-1}, but in the above formulae it fortunately only acts on lower triangular matrices. Remarkably, there is a simple way to make two of the above forms well-defined; if we write $A_z = g^{-1}\partial_z g$ then $\delta A_z = D_z(g^{-1}\partial_z g)$, so if we use the variable g as fundamental we have

$$\begin{aligned}
\omega_2 &= -\int dz\ Tr(g^{-1}\delta g \wedge D_z(g^{-1}\delta g)) \\
\omega_3 &= \int dz\ Tr(g^{-1}\delta g \wedge D_{\bar{z}}(g^{-1}\delta g))
\end{aligned} \tag{41}$$

While the manipulations in this section have been questionable, the result, equation (41), is perfectly reasonable; it suggests that in terms of the variables g the Drinfeld-Sokolov systems have a well-defined inverse-local bihamiltonian structure. The above manipulations for ω_2 were discussed by Wilson [30].

8. The Bihamiltonian Structure Induced on Reductions of SDYM

On the space of solutions of SDYM there are three natural closed two forms we can write down

$$\Omega^i = \int\ Tr(\delta A \wedge \delta A) \wedge \alpha^i \qquad\qquad i = 1, 2, 3 \tag{42}$$

where α^i, $i = 1, 2, 3$ are three closed two-forms on $\mathbf{R}^4$

$$\begin{aligned}
\alpha^1 &= dz \wedge dw \\
\alpha^2 &= d\bar{z} \wedge d\bar{w} \\
\alpha^3 &= dz \wedge d\bar{w} - dw \wedge d\bar{z}
\end{aligned} \tag{43}$$

These are natural, because assuming we are dealing with gauge fields that behave well at infinity, the self-duality equations (1), which can be written $F \wedge \alpha^i = 0,\ i = 1, 2, 3$, emerge as moment maps for the gauge invariance of Ω^i [31]. We will consider these forms under the circumstances that the fields are well-behaved in the $z, \bar{z}, \bar{w}$ directions, but possibly not in the w direction (we could choose to compactify in the $z, \bar{z}, \bar{w}$ directions). The criterion for gauge invariance of Ω^i on a region V of $\mathbf{R}^4$ under an infinitesimal gauge transformation $A \to A + D\Phi$ is

$$\int_{\partial V} Tr(\Phi \delta A) \wedge \alpha^i = 0 \tag{44}$$

So for us, with the behavior of the fields as specified, Ω^1 is invariant, and the criteria for Ω^2 and Ω^3 to be invariant are, respectively:

$$\begin{aligned} \int_{w=\infty} dz d\bar{z} d\bar{w} Tr(\Phi \delta A_z) - \int_{w=-\infty} dz d\bar{z} d\bar{w} Tr(\Phi \delta A_z) = 0 \\ \int_{w=\infty} dz d\bar{z} d\bar{w} Tr(\Phi \delta A_{\bar{z}}) - \int_{w=-\infty} dz d\bar{z} d\bar{w} Tr(\Phi \delta A_{\bar{z}}) = 0 \end{aligned} \tag{45}$$

So in particular Ω^2 will be invariant if we restrict to a set of solutions with fixed A_z, and Ω^3 will be invariant if we restrict to a set of solutions with fixed $A_{\bar{z}}$.

Using the representation of self-dual gauge fields given at the end of section 3 (equations (5a) and (13)), we can write

$$\begin{aligned} \delta A_w &= D_w(g^{-1}\delta g) \\ \delta A_z &= D_z(g^{-1}\delta g) \\ \delta A_{\bar{w}} &= D_{\bar{w}}(g^{-1}\delta g) + D_w(g^{-1}\delta P g) \\ \delta A_{\bar{z}} &= D_{\bar{z}}(g^{-1}\delta g) + D_z(g^{-1}\delta P g) \end{aligned} \tag{46}$$

Substituting these expressions into the formulae for Ω^2 and Ω^3, it can be seen that the integrands are total derivatives, and thus Ω^2 and Ω^3 naturally define two symplectic forms on the space of w-independent functions

$$\begin{aligned} \tilde{\Omega}^2 &= \int dz d\bar{z} d\bar{w} Tr(g^{-1}\delta g \wedge D_z(g^{-1}\delta g)) \\ \tilde{\Omega}^3 &= \int dz d\bar{z} d\bar{w} Tr(g^{-1}\delta g \wedge [D_{\bar{z}}(g^{-1}\delta g) + 2D_z(g^{-1}\delta P g)] \end{aligned} \tag{47}$$

If we restrict to a subspace of fields where $A_{\bar{z}}$ is constant, then we can use the equation $\delta A_{\bar{z}} = 0$ to eliminate the δP term in $\tilde{\Omega}^3$, and (up to overall normalizations) we obtain the ω_2, ω_3 of section 7, equation (41), the Drinfeld-Sokolov symplectic forms.

Note the following:

1. The above derivation of the Drinfeld-Sokolov forms from Ω^2, Ω^3 is free of any "formal" manipulations.
2. I have taken care to use the representation of the potentials in terms of g and P, as opposed to the other representations of section 3, because if we wish to consider

reductions setting $\partial_{\bar{z}}$ to zero, the representation in terms of g and P is still good, while the others are not.

3. It is straightforward to check directly the gauge invariance properties of $\tilde{\Omega}^2$,$\tilde{\Omega}^3$; we find their invariance requires $\int dzd\bar{z}d\bar{w}Tr(\Phi\delta A_z) = 0$ and $\int dzd\bar{z}d\bar{w}Tr(\Phi\delta A_{\bar{z}}) = 0$ respectively, as we would expect. Clearly the restricted gauge transformations of the Drinfeld-Sokolov systems [15] satisfy these conditions.

We would also like to obtain hamiltonian structures for equations (26) in an $A_z = 0$ gauge. For this we use the standard M formulation of section 3, i.e. equations (5b) and (7). In light of note 2 above we should be cautious. We have

$$\begin{aligned} \delta A_{\bar{w}} &= D_{\bar{w}}(h^{-1}\delta h) \\ \delta A_{\bar{z}} &= D_{\bar{z}}(h^{-1}\delta h) \\ \delta A_w &= D_w(h^{-1}\delta h) + D_{\bar{w}}(h^{-1}\delta N h) \\ \delta A_z &= D_z(h^{-1}\delta h) + D_{\bar{z}}(h^{-1}\delta N h) \end{aligned} \tag{48}$$

These expressions allow us to write the integrands of Ω^1, Ω^3 as total derivatives, so we can write the associated forms $\tilde{\Omega}^1$,$\tilde{\Omega}^3$ (the latter should presumably agree with that above), and then restricting to the subspace $\delta A_z = 0$ we get

$$\begin{aligned} \tilde{\Omega}^1 &= \int dzd\bar{z}d\bar{w}Tr(h^{-1}\delta h \wedge D_{\bar{z}}(h^{-1}\delta h)) \\ \tilde{\Omega}^3 &= \int dzd\bar{z}d\bar{w}Tr(h^{-1}\delta h \wedge D_z(h^{-1}\delta h)) \end{aligned} \tag{49}$$

This is as far as we can go without formal manipulations; but by analog with the successful manipulations of the previous section it is natural to guess that for reductions of SDYM in $A_z = 0$ gauge, hamiltonian structures arise from the following symplectic forms on the space of potentials $A_{\bar{z}}$:

$$\begin{aligned} \tilde{\omega}_1 &= \int dz\ Tr(\delta A_{\bar{z}} \wedge D_z^{-1}\delta A_{\bar{z}}) \\ \tilde{\omega}_2 &= \int dz\ Tr(\delta A_{\bar{z}} \wedge D_{\bar{z}}^{-1}\delta A_{\bar{z}}) \\ \tilde{\omega}_3 &= \int dz\ Tr(\delta A_{\bar{z}} \wedge D_{\bar{z}}^{-1}D_z D_{\bar{z}}^{-1}\delta A_{\bar{z}}) \end{aligned} \tag{50}$$

$\tilde{\omega}_2$ and $\tilde{\omega}_3$ give $\tilde{\Omega}_1$ and $\tilde{\Omega}_3$ above, and $\tilde{\omega}_1$ is motivated by the usual recursion formalism. In equation (50) $D_z = \partial_z$ and $D_{\bar{z}} = [A_{\bar{z}},\ \ ]$. In at least the simplest example, the Heisenberg ferromagnet, it seems these formulae have some meaning.

9. Example: The Heisenberg Ferromagnet

In equation (28) the method of reduction of SDYM to the Heisenberg ferromagnet was given. It is straightforward to see that $\tilde{\omega}_2$ corresponds to the standard Poisson brackets of [27]. Here I wish to look at $\tilde{\omega}_1$. Since $\lambda^2 + \mu\nu = \kappa^2$, we can eliminate $\delta\lambda$ from $\tilde{\omega}_1$ and write it (up to an overall normalization)

$$\int dz\ (\delta\mu \quad \delta\nu) \wedge \begin{pmatrix} \frac{\nu}{\lambda}\partial_z^{-1}\frac{\nu}{\lambda} & 2\partial_z^{-1} + \frac{\nu}{\lambda}\partial_z^{-1}\frac{\mu}{\lambda} \\ 2\partial_z^{-1} + \frac{\mu}{\lambda}\partial_z^{-1}\frac{\nu}{\lambda} & \frac{\mu}{\lambda}\partial_z^{-1}\frac{\mu}{\lambda} \end{pmatrix} \begin{pmatrix} \delta\mu \\ \delta\nu \end{pmatrix} \tag{51}$$

The ferromagnet equations are

$$\begin{aligned} 2\kappa^2\mu_w &= \partial_z(\lambda\mu_z - \mu\lambda_z) \\ -2\kappa^2\nu_w &= \partial_z(\lambda\nu_z - \nu\lambda_z) \end{aligned} \tag{52}$$

The above symplectic form gives a hamiltonian structure since we can write:

$$\begin{aligned} \begin{pmatrix} \frac{\nu}{\lambda}\partial_z^{-1}\frac{\nu}{\lambda} & 2\partial_z^{-1} + \frac{\nu}{\lambda}\partial_z^{-1}\frac{\mu}{\lambda} \\ 2\partial_z^{-1} + \frac{\mu}{\lambda}\partial_z^{-1}\frac{\nu}{\lambda} & \frac{\mu}{\lambda}\partial_z^{-1}\frac{\mu}{\lambda} \end{pmatrix} \begin{pmatrix} \partial_z(\lambda\mu_z - \mu\lambda_z) \\ -\partial_z(\lambda\nu_z + \nu\lambda_z) \end{pmatrix} &= \frac{2\kappa^2}{\lambda} \begin{pmatrix} -\nu_z \\ \mu_z \end{pmatrix} \\ &= 2\kappa^2 \begin{pmatrix} \delta/\delta\mu \\ \delta/\delta\nu \end{pmatrix} \int dz\ \frac{\nu\mu_z - \mu\nu_z}{\lambda + \kappa} \end{aligned} \tag{53}$$

The hamiltonian in the above equation was given in [27]; I am uncertain whether the "group theoretic origin" of this hamiltonian structure for the ferromagnet equation has been realised before.

10. The KP Hierarchy from the SDYM Hierarchy ††

Let us choose a gauge algebra $\mathcal{G}$ such that a) $\mathcal{G}$ can be identified with its universal enveloping algebra, i.e. we can "multiply" two elements of $\mathcal{G}$ to get another element of $\mathcal{G}$, and b) $\mathcal{G}$ "splits", i.e. we can write $\mathcal{G} = \mathcal{G}_+ \oplus \mathcal{G}_-$, where the commutator of two elements of $\mathcal{G}_+$ is in $\mathcal{G}_+$, and the commutator of two elements of $\mathcal{G}_-$ is in $\mathcal{G}_-$. For $M \in \mathcal{G}$ define M_+, M_- as the projections of M onto $\mathcal{G}_+$ and $\mathcal{G}_-$ respectively.

Suppose $L \in \mathcal{G}$ and consider the following evolution equations for L:

$$\frac{\partial L}{\partial z_i} = [(L^i)_+, L] \qquad i = 1, 2, ... \tag{54}$$

†† The results in this section were obtained in collaboration with Didier Depireux. Simliar results have been obtained by I.A.B.Strachan.

These imply (see section 4 of [32])

$$\begin{aligned}\frac{\partial L^j}{\partial z_i} &= [(L^i)_+, L^j] \\ \frac{\partial (L^j)_+}{\partial z_i} - \frac{\partial (L^i)_+}{\partial z_j} &= [(L^i)_+, (L^j)_+] \\ \frac{\partial (L^j)_-}{\partial z_i} - \frac{\partial (L^i)_-}{\partial z_j} &= -[(L^i)_-, (L^j)_-]\end{aligned} \tag{55}$$

It is straightforward to check, using these results, that equations (54) for L are equivalent to the equations of the SDYM hierarchy (4) with a dimensional reduction $\partial_{\bar{z}_i} = \partial_{z_{i-1}}$, $i = 1, 2, ...$ (so $\partial_{\bar{z}_1} = 0$) and an ansatz

$$\begin{aligned}A_{\bar{z}_1} &= L \\ A_{\bar{z}_i} &= (L^{i-1})_-, \qquad i > 1 \\ A_{z_i} &= -(L^i)_+, \qquad i \geq 1\end{aligned} \tag{56}$$

The KP hierarchy is just this with $\mathcal{G}_+$ the algebra of finite order differential operators in one variable x, $\mathcal{G}_-$ the algebra of psuedodifferential operators in x (with no “constant term”), and

$$L = \partial_x + u_2(x)\partial_x^{-1} + u_3(x)\partial_x^{-1} + ... \tag{57}$$

The gauge choice here is one in which $A_{z_1} = \partial_x$; in the Sato theory (see for example [32]) an object W is defined such that $L = W\partial_x W^{-1}$, and this defines a gauge transformation to a gauge where $A_{\bar{z}_1} = \partial_x$.

11. Conclusions and Further Directions

That the bihamiltonian structures of integrable systems arise naturally from structures on the space of solutions of SDYM is a most pleasing result, and lends much support to the idea that the interesting properties of integrable systems all stem from SDYM. But there is a long way to go before the theory of integrable systems can be rewritten from this viewpoint. Many interesting questions remain, of which I will just pose three here:

1) It is interesting that we recover most standard integrable equations and their hierarchies by a reduction of SDYM and its hierarchy by essentially half the number of coordinates. One the other hand, there are interesting approaches in the theory of integrable systems which involve adding extra coordinates; examples are the mysterious Hirota bilinear operators [29], and the unified approach to integrable systems of Fokas and Santini [33]. It would be very interesting if a link could be made between

the auxiliary coordinates in these methods, and the coordinates we have reduced by from SDYM.

2) The fact that we now have a simple reduction of SDYM to KP opens a lot of possibilities. First, using the methods of this paper, we should be able to extract the bihamiltonian structure of KP [34], and understand hierarchies related to KP via gauge transformations, such as the modified KP. These issues are currently under investigation in collaboration with D.Depireux. More importantly, we need to understand how the twistor formalism gives inverse scattering formalism (and hopefully the Hirota method of direct solution) for KP and its relatives (Davey-Stewartson, multi-component KPs etc.).
3) Quantized integrable systems have been extensively studied, and arise as deformations of conformal field theories. It would be interesting too see if there exists a quantization of SDYM that unifies quantized integrable systems in some sense. Quantization of SDYM using the lagrangians of [19] and [20] is problematic, as there seem to be renormalization problems, but it seems the $N = 2$ string can be regarded in some sense as a quantization of SDYM [35], and this might give positive results.

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