Symmetries of KdV and Loop Groups

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Abstract

A simple version of the Segal-Wilson map from the $SL(2,\mathbf{C})$ loop group to a class of solutions of the KdV hierarchy is given, clarifying certain aspects of this map. It is explained how the known symmetries, including Bäcklund transformations, of KdV arise from simple, field independent, actions on the loop group. A variety of issues in understanding the algebraic structure of Bäcklund transformations are thus resolved.

1 Prelude: Symmetries of KdV

This section contains a list, which I believe is exhaustive, of known notions of "symmetry" of the KdV hierarchy. One of the aims of this paper is to obtain a unified, simple understanding of them.

1.1 Translation Invariance

The KdV hierarchy is a system of autonomous differential equations on an infinite dimensional affine space, and hence an infinite dimensional abelian group, associated with translation invariance, acts on the space of solutions.

1.2 Scaling and Galilean Invariance

Two further one parameter groups are known to act on the space of solutions, the $\mathbf{C}^*$ action associated with scaling invariance (see, for example, [4], for the case of the KdV equation), and the $\mathbf{C}$ action associated with Galilean invariance (see, again, [4], for the case of the KdV equation). The full group generated by transformations of these types and translations is easy to identify.

1.3 Wahlquist-Estabrook Bäcklund Transformations

Another well known, but less well understood, symmetry is the one parameter family of Backlund transformation (BTs) of Wahlquist and Estabrook [21] (see [4] sections 5.4.2 and 5.4.3 for a typical exposition). To implement a BT is in two ways harder than the simple transformations listed above: first, implementing a BT involves solving a differential equation, and second, the form of this differential equation depends on the solution being transformed, i.e. the transformation is *field dependent*. Note that together, these two problems mean it is not clear when BTs can be implemented: it is necessary to check that the solution being transformed has the properties that guarantee global solvability of the required differential equation.

When BTs can be applied, several remarkable algebraic properties emerge. Because of an integration constant that appears in each application of a BT, BTs do not map single solutions to single solutions, but rather the image of a single solution is a *set* of solutions. Thus a BT is an operator mapping the space of sets of solutions of KdV to itself; if the parameter in the BT is $\theta \in \mathbf{C}$, let us denote this operator as $\mathcal{O}_\theta$. The known algebraic properties (assuming applicability of the BTs) are (1) commutativity [21], (2) that the image of any set of solutions under the square of a BT (of a given parameter value) includes the original set (this is evident from the equations defining the BT), and (3) the product of two BTs is *not* itself another BT. That is,

$$\begin{aligned} \forall\ \theta_1, \theta_2 \qquad \mathcal{O}_{\theta_1}\mathcal{O}_{\theta_2} &= \mathcal{O}_{\theta_2}\mathcal{O}_{\theta_1} \\ \forall\ \theta \qquad I &\subset \mathcal{O}_\theta^2 \\ \nexists\ \theta_1, \theta_2, \theta_3 \ \text{ s.t. } \ \mathcal{O}_{\theta_1}\mathcal{O}_{\theta_2} &\neq \mathcal{O}_{\theta_3}. \end{aligned} \tag{1}$$

(In the second of these relations, I use the notation that for operators A, B on sets, $A \subset B$ implies the image of any set under A is a subset of the image of that set under B, and I denotes the identity operator on sets.)

In addition to the problem of determining to which sets of solutions BTs can be applied, there is a need to understand the origins of this strange notion of symmetry, and how it interacts with the other symmetries listed in this section. The results of this paper go some way towards resolving these last two questions; in particular I will give in Sec. 5.3 what I consider to be the correct algebraic framework within which to consider BTs, and the results of Sec. 5 give a field independent realization of all the symmetries of KdV listed in this section, making their interrelationships straightforward. With regard to the question of applicability of BTs, the paper [7] studies when the BT $\mathcal{O}_0$ can be implemented on a solution satisfying "reasonable" analyticity conditions while retaining such conditions. A sufficient condition for this is found: that the spectrum of the Schrödinger operator associated with the starting solution have no negative eigenvalues. The solutions of KdV studied here (which include certain singular solutions) all admit the BT $\mathcal{O}_0$, and also BTs of other sufficiently low parameter values, while staying in the class of solutions under study. It seems, however, that the class only includes single soliton solutions that are sufficiently broad, shallow and slow (recall that all these properties are determined by a single parameter), and thus the class of solutions is not sufficiently large to make the question of when a BT can or cannot be applied, while staying within the class, of real interest.

1.4 Galas Bäcklund Transformations

In a relatively unknown recent work [6], Galas found a novel Bäcklund transformation for KdV. Since this is not well known, and presented only very briefly in [6], I give some details. Combining Eqs. (4) to (7) in [6], it emerges that the KdV equation

$$u_t + u_{xxx} + 6uu_x = 0 \tag{2}$$

is invariant under $u \mapsto u + 2(\ln \tau)_{xx}$, where τ (which Galas calls $4+\epsilon p_3$) is related to u by

$$u = \theta - \frac{1}{2}\left(\frac{\tau_{xxx}}{\tau_x} - \frac{\tau_{xx}^2}{2\tau_x^2}\right) = \theta - \frac{(\sqrt{\tau_x})_{xx}}{\sqrt{\tau_x}} \tag{3}$$

and

$$\frac{\tau_t}{\tau_x} = -6\theta - \left(\frac{\tau_{xxx}}{\tau_x} - \frac{3\tau_{xx}^2}{2\tau_x^2}\right). \tag{4}$$

For contrast, the standard BT is $u \mapsto u + 2(\ln \tau)_{xx}$, where τ is related to u by

$$u = \theta - \frac{\tau_{xx}}{\tau} \tag{5}$$

and

$$\frac{\tau_t}{\tau_x} = -6\theta - \left(\frac{\tau_{xxx}}{\tau_x} - \frac{3\tau_{xx}}{\tau}\right). \tag{6}$$

Galas finds the infinitesimal version of his BT as well, and shows that it generates one soliton solutions from the trivial solution $u = 0$. In fact it can generate other interesting solutions from $u = 0$; it is straightforward to check that

$$\tau = (x - 12k^2t) + \frac{\sinh(2k(x - 4k^2t))}{2k} \tag{7}$$

solves Eqs. (3) and (4) with $u = 0$ and $\theta = k^2$. This gives an interesting singular solution of KdV, which should presumably be regarded as a nonlinear superposition of the rational solution $u = 2(\ln \tau)_{xx}$, $\tau = x$, with the "singular soliton" solution $u = 2(\ln \tau)_{xx}$, $\tau = \sinh(2k(x - 16k^2t))$ (at least up to Galilean transformations). Note, however, that like both these solutions, for any real value of t the new solution is only singular for one real value of x (and has a double pole there); is this sense the notion of "superposition" is inappropriate. Such mixed rational-solitonic solutions have been studied in the literature [1].

Algebraic properties of this BT have not as of yet been given; in Sec. 5.4 I will prove commutativity, at least within a limited class of solutions. The possible limitations on applicability, the fact that one BT creates a family of solutions out of a single one, and the fact that this transformation is field dependent, all apply to this BT as they do to the standard one.

1.5 The Hierarchy of Infinitesimal Symmetries Generalizing Scaling and Galilean Invariance

It is of course possible to write generators of translation, scaling and Galilean transformations. The generators of the translation transformations form a *hierarchy*; they are related to each other by application of an operator, with very special properties, known as the recursion operator (see, for example [16]). The "lowest order" translation is generated by the recursion operator out of the generator of the trivial symmetry, i.e. the symmetry that leaves a solution invariant. The generators commute, reflecting the abelian nature of the translation group.

It turns out that the generator of scaling transformations can be obtained from the generator of Galilean transformations by application of the recursion operator. And, furthermore, a hierarchy of symmetry generators can be produced by repeated application of the recursion operator to the generator of Galilean transformations. Let us denote the generators of the translation symmetries by l_n, $n = 0, 1, 2, \ldots$, (where it is understood that l_{n+1} is obtained by the action of the recursion operator on l_n, and l_0 is the lowest order translation generator), and the generators in the new hierarchy by m_n, $n = -1, 0, 1, \ldots$ (where again it is understood that m_{n+1} is obtained by the action of the recursion operator on m_n, and m_{-1} is the generator of Galilean transformations). The algebra obeyed by these infinitesimal transformations is then found to be

$$\begin{aligned} [l_r, l_s] &= 0 \\ [m_r, m_s] &= (s-r)m_{r+s} \\ [m_r, l_s] &= (s+\tfrac{1}{2})l_{r+s} \end{aligned} \tag{8}$$

(where in the last relation it is understood that $l_{-1} = 0$). It is not currently known how to exponentiate the generators m_n, $n > 0$; of particular interest is the generator m_1, since m_{-1}, m_0, m_1 form a closed $sl(2)$ subalgebra.

The history of the m_n symmetries is a little unclear to me. The earliest reference I am aware of is [11]. They are exploited in [15], and a reference is given to a preprint by the authors of [3]; in [3] these infinitesimal symmetries were discovered for the KP hierarchy (although all solutions of KdV are solutions of KP, it does not follow that symmetries of KP can be restricted to symmetries of KdV; in this case they can, as has also been illustrated in [2]). The symmetries are discussed further in [12]. In addition, there has been some discussion [8] of how the m_n hierarchy can be extended to define symmetry generators $m_{-2}, m_{-3}, \ldots$

by application of the inverse of the recursion operator. These results will be reproduced later in the paper (and also results of [9] showing the existence of further infinitesimal symmetries that leave the KdV solution invariant but act on certain "prepotentials" associated with the solution).

1.6 Zakharov-Shabat Dressing Transformations

Somewhat remote from all the above results, but, in a sense that will emerge, inclusive of all of them, are the "dressing transformations" of Zakharov and Shabat (see [22] for a concise explanation in the case of the modified KdV equation). The KdV hierarchy can be interpreted as the consistency condition for a certain homogeneous linear differential system. A solution of this linear system gives rise to a solution of the hierarchy, but there are many solutions of the linear system corresponding to any solution of the hierarchy. Dressing transformations should be thought of as an action of the $SL(2,\mathbf{C})$ loop group on solutions of the linear system. Unfortunately, taking two solutions of the linear system corresponding to the *same* solution of the hierarchy, and acting on them with the *same* dressing transformation, gives, in general, two solutions of the linear system corresponding to *different* solutions of the hierarchy. Thus there is no guarantee that dressing transformations give rise to a sensible notion of symmetry on the hierarchy itself. It turns out that there is a dense subset in the loop group for which dressing transformations can be interpreted as Bäcklund transformations (i.e. as maps that take a single solution to a family of solutions of finite dimensionality); the Galas BTs arise precisely this way. The standard BT, however, can only be expressed as a field *dependent* dressing transformation.

2 Aims and Methods

This paper has two main aims. One, as stated above, is to obtain a unified and simplified understanding of symmetries of KdV. I will do this by exploiting a cornerstone of KdV theory, the *Segal-Wilson correspondence*. The other aim of this paper is to give a simplified reformulation of this correspondence, clarifying several issues, both conceptual and technical.

In [19], Segal and Wilson gave a construction associating a solution of the KdV hierarchy with each point in a certain infinite dimensional grassmanian. The grassmanian is a homogeneous space of the $SL(2,\mathbf{C})$ loop group, which I denote G, a quotient of G by the subgroup of loops which are boundary values of analytic loops on the unit disc; I denote this subgroup G^+. Throughout this paper I regard G as the set of 2×2 matrices, with unit determinant, and entries Laurent series in a parameter λ, convergent for $|\lambda|=1$, and defining a smooth map from the circle $|\lambda|=1$ to $SL(2,\mathbf{C})$; thus G^+ is the subgroup for which entries are power series, which, since they are convergent for $|\lambda|=1$, define analytic functions in $|\lambda| < 1$. The main paper of Segal and Wilson [19] does not really exploit the description of the grassmanian as the quotient G/G^+; however in other papers, Wilson [22] (actually for the case of the modified KdV hierarchy) gives an explicit map from elements of G to solutions of the hierarchy, exploiting the Birkhoff factorization of G. (Writing G^- for the subgroup of loops that are boundary values of loops analytic in $|\lambda| > 1$ and which reduce to the identity at $\lambda = \infty$, the Birkhoff factorization theorem states that for elements g in a dense, open subset of G, there exists a unique factorization $g = g_-^{-1} \cdot g_+$, where $g_- \in G^-$, $g_+ \in G^+$. The product map $G^- \times G^+ \to G$ defined by $(g_-, g_+) \mapsto g_-^{-1} g_+$ is a diffeomorphism from $G^- \times G^+$ to its image. See [17].) The solution of the hierarchy obtained by Wilson's map from an element $g \in G$ is unchanged by right multiplication $g \mapsto g \cdot h$, $h \in G^+$; thus the map actually defines a map from G/G^+ to solutions of the hierarchy.

The reformulation of this map that I will give is inspired by Mulase's results for the KP hierarchy [14] (which I had the good fortune to hear Mulase lecture on in 1989). Mulase emphasizes that integrable systems are really linear systems in disguise. Of course, for KdV this is well-known: the invertible scattering transform converts the nonlinear flow for the function satisfying the KdV hierarchy to a linear flow for associated scattering data. I will consider a simple linear flow on the loop group G, whose solution is determined uniquely by an initial value, i.e. some element of G. A flow on G induces, via Birkhoff factorization, and assuming the flow does not leave the relevant dense open subset of G, flows on G^+ and G^-, and there is no reason why these flows should be linear. The corresponding flow on G^- turns out to be, more or less, the KdV hierarchy! The orbit on G^- is unaffected by right multiplication of the initial value of the flow on G by an element of G^+ and thus Wilson's map from G/G^+ to solutions of KdV is recovered.

The reader may be itching to know what I meant by the phrase "more or less" in the above paragraph. In fact the G^- flow contains somewhat more than the KdV hierarchy. In particular I will identify a further infinite dimensional abelian subgroup H of G, that, acting in an appropriate way on the initial value of the G flow, leaves the associated KdV solution invariant, even though the G^- flow is not invariant. It thus turns out that the Wilson map is a map from the double coset space $H \backslash G/G^+$ to solutions of KdV; a study of the geometry of this space would be very interesting.

Returning for a moment from technical to conceptual issues, the approach I present "explains" the role of the loop group in KdV theory: the loop group is the space of initial value data for the simple linear flow that KdV conceals.

Another technical issue that emerges is that there are two ways to pass from the G flow to the associated KdV solution, using either the G^+ or the G^- flows as intermediaries. These both have an advantage and a disadvantage: the G^+ flow is not invariant under the G^+ action that leaves the KdV solution invariant, but is under the H action, and vice-versa for the G^- flow. For different computational purposes, different approaches are preferable. Note the G^+ flow is also determined by a set of linear equations; this is the linear system referred to in the discussion of dressing transformations in section 1.6. In Wilson's works [22] only the G^+ flow is used; on the other hand, in the work of Drinfeld and Sokolov [5], the G^- flow is used. The results presented here show the rather simple way these approaches are related.

Armed with a firm grasp of the Segal-Wilson correspondence, the symmetries of KdV can be understood. The natural origin of symmetries of KdV should be what will be loosely called "symmetry actions" on the loop group. These include right and left multiplication by elements of G, other automorphisms of G, and the induced action on (subgroups of) G of reparametrizations of λ. Unfortunately most of these actions do not descend to the double coset space. It turns out that there are some actions that do (these give rise to genuine symmetries of KdV), and others that have the property that they map individual cosets to a finite dimensional set of cosets. These latter actions are precisely Bäcklund transformations! In this way field independent symmetry actions on G give rise to all the symmetries of KdV I have listed in section 1. Field independence is an important point here; all of the symmetry actions on G that will be considered can be rewritten as, say, right multiplications, but not necessarily field independent ones.

The contents of the remainder of this paper are as follows. In section 3, I discuss the zero curvature formulation of the KdV hierarchy, and particularly a nonstandard gauge choice that will be important. In section 4 I present the reformulation of the Segal-Wilson correspondence, and, in section 5 I use it to show how field independent symmetry actions on G induce all the symmetries of KdV presented in section 1. Finally, in section 6, I present a few open problems.

Two more comments are in order before concluding this introductory section. First, the reader will have noticed that I am studying the KdV hierarchy without exploiting its embedding in the KP hierarchy. The reason for this is quite simple — although many of the known properties of KdV are inherited from those of KP, it is not the case that this must be so for all interesting properties of KdV, and in particular, I am not certain whether the Galas BT has an analog for KP. Second, the reader will find another presentation of a more "group-theoretic" approach to the Segal-Wilson correspondence in [10], which has some overlap with the ideas being presented here, and shows that the ideas being presented here can be set in a very general framework. The work of Mulase [14] is actually an example of the constructions of [10].

3 A Nonstandard Gauge for the Zero Curvature Formulation of KdV

The KdV hierarchy is defined as follows. The sequence P_n, $n = 0, 1, 2, \ldots$ of differential polynomials in $u(x)$ is defined by the recursion relation

$$\partial_x P_{n+1} = (\tfrac{1}{4}\partial_x^3 - u\partial_x - \partial_x u)P_n \qquad n = 0, 1, 2, \ldots \tag{9}$$

with the initial condition $P_0 = 1$, supplemented by the condition that P_n is homogeneous of weight $2n$, where the n-th x-derivative of u is assigned weight $n+2$. The KdV hierarchy is then the set of differential equations

$$u_{t_n} = -\partial_x P_{n+1} \qquad n = 1, 2, \ldots. \tag{10}$$

Here u is regarded as a function of x and the infinite number of "times" t_n, $n = 1, 2, \ldots$. I use a non-standard numbering of these times to underscore the fact that I am considering KdV without using its embedding in KP. The choice $n = 1$ is the original KdV equation; $P_2 = -\frac{1}{4}u_{xx} + \frac{3}{2}u^2$ (note the conventions used from here on in this paper differ from those used in Sec. 1.4 where I followed the conventions of [6]). Partial derivatives are denoted by suffices, as usual. The proofs that the recursion relation (9) has a unique solution, when supplemented with the homogeneity condition given, and that the flows (10) commute, are by now standard (see, for example, [16]).

The hierarchy can be presented as a set of zero-curvature conditions

$$A_{t_n} = B_x^{(n)} - [A, B^{(n)}] \qquad n = 1, 2, \ldots \tag{11}$$

where

$$\begin{aligned} A &= \begin{pmatrix} 0 & 1 \\ 2u+\lambda & 0 \end{pmatrix} && (12)\\ B^{(n)} &= \begin{pmatrix} -\frac{1}{2}b_x^{(n)} & b^{(n)} \\ (2u+\lambda)b^{(n)} - \frac{1}{2}b_{xx}^{(n)} & \frac{1}{2}b_x^{(n)} \end{pmatrix} && (13)\\ b^{(n)} &= \sum_{r=0}^{n} P_{n-r}\lambda^r. && (14) \end{aligned}$$

Drinfeld and Sokolov [5] introduced a generalization of this scheme. The zero curvature equations (11) are invariant under gauge transformations

$$\begin{aligned} A &\to hAh^{-1} + h_x h^{-1} && (15)\\ B^{(n)} &\to hB^{(n)}h^{-1} + h_{t_n}h^{-1}, && (16) \end{aligned}$$

where h is an $SL(2)$ matrix (which is allowed to be x, t_n and even λ dependent). Taking

$$h = \begin{pmatrix} 1 & 0 \\ -j & 1 \end{pmatrix}, \tag{17}$$

where j is dependent on x, t_n but not λ, gives a zero curvature system with

$$A = \begin{pmatrix} j & 1 \\ 2u - j_x - j^2 + \lambda & -j \end{pmatrix}. \tag{18}$$

Choosing j to be a solution of $j_x + j^2 = 2u$ gives a zero curvature formulation for the modified KdV hierarchy. However, I will be more interested in a less commonly studied gauge (despite the fact it is actually mentioned in [5]). Take j to be a solution of $u = j_x$ in the above; to implement the gauge transformation on the matrices $B^{(n)}$ it is necessary to know j_{t_n}, and the obvious choice is

$$j_{t_n} = -P_{n+1}, \tag{19}$$

which is clearly consistent with the assignment $u = j_x$. This gives a zero curvature formulation for the potential KdV hierarchy (i.e. the coupled system of Eqs (10) and (19) with the relation $u = j_x$). I call such a gauge choice "PKdV gauge". Amongst the family of zero curvature equations related by gauge transformations in the way described above, it turns out that PKdV gauge has a useful characterization:

Lemma. Up to translation of j by a function of x, independent of t_n, PKdV gauge is the unique gauge choice such that $B^{(n+1)} - \lambda B^{(n)}$ is independent of λ.

Proof. From Eq. (14), $b^{(n+1)} - \lambda b^{(n)}$ is independent of λ. After the gauge transformation given by Eq. (16) with h given by Eq. (17),

$$B^{(n)} = \begin{pmatrix} -\frac{1}{2}b_x^{(n)} + jb^{(n)} & b^{(n)} \\ (2u+\lambda)b^{(n)} - \frac{1}{2}b_{xx}^{(n)} + jb_x^{(n)} - j^2 b^{(n)} - j_{t_n} & \frac{1}{2}b_x^{(n)} - jb^{(n)} \end{pmatrix}. \tag{20}$$

Thus $B^{(n+1)} - \lambda B^{(n)}$ is independent of λ if and only if $(\lambda b^{(n+1)} - j_{t_{n+1}}) - \lambda(\lambda b^{(n)} - j_{t_n})$ is independent of λ, which, since the $O(\lambda^0)$ term in $b^{(n+1)}$ is P_{n+1}, is true if and only if $j_{t_n} = -P_{n+1}$. This is the case in PKdV gauge. It also implies that $u - j_x$ is independent of t_n, giving $u = j_x + f(x)$ where $f(x)$ does not depend on the t_n, thus specifying PKdV gauge, up to translation of j by a function of x. $\square$

It will be useful to have a theorem stating clearly the existence and characterization of the zero curvature formulation in PKdV gauge. To this end, let us define M to be the affine space with coordinates $t_{-1}, t_0, t_1, t_2, \ldots$, and $\mathcal{A}$ to be the Lie algebra of traceless 2×2 matrices whose entries are formal power series in λ. I also need the following notion:

Definition. A change of coordinates $t_i \to t_i'$ on M is *admissible* if it is given by

$$t_i = \sum_{j=0}^{\infty} a_j t'_{i+j} \qquad i = -1, 0, 1, \ldots \tag{21}$$

where $a_0 = 1$, and for $j > 0$, the a_j are arbitrary constants, only a finite number of which are nonzero.

I will need the effect of admissible changes of coordinates on the components of a one-form on M. If $\alpha = \sum_{n=-1}^{\infty} \alpha_n dt_n = \sum_{n=-1}^{\infty} \alpha'_n dt'_n$ is a one-form on M, then by a simple substitution,

$$\alpha'_n = \sum_{m=-1}^{n} a_{n-m}\alpha_m. \tag{22}$$

With these preparations I can now state the following

Theorem. Let Z be an $\mathcal{A}$ valued one form on M, i.e. $Z = \sum_{n=-1}^{\infty} Z_n dt_n$, $Z_n \in \mathcal{A}$. Suppose (1) that Z_0 has the form

$$Z_0 = \begin{pmatrix} j & 1 \\ \lambda + u - j^2 & -j \end{pmatrix},$$

with j, u functions on M, (2) that $Z_n - \lambda Z_{n-1}$ is independent of λ, $n = 0, 1, \ldots$, and (3) that $dZ = Z \wedge Z$. Then, possibly after a sequence of admissible changes of coordinates, (1) $u = j_{t_0}$, (2) j, u solve the potential KdV hierarchy, with x identified as t_0, and (3)

$$\begin{aligned} j_{t_{-1}} &= -1 \\ u_{t_{-1}} &= 0. \end{aligned}$$

Proof. As a preliminary, note that it is straightforward to show that an admissible changes of coordinates does not affect assumption (2), and since assumption (3) is written in terms of differential forms, it evidently is not affected either. Furthermore, assumptions (1) and (2) imply

$$Z_{-1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{23}$$

and under an admissible change of coordinates $Z_0 \to Z_0 + a_1 Z_{-1}$, so assumption (1) is also not affected: the function u is just translated. Thus the assumptions in the theorem allow us to perform admissible changes of coordinates. The KdV flows, however, are not left invariant under such changes of coordinates; rather the flows pick up certain linear combinations of the lower flows. This accounts for the need to allow for an admissible change of coordinates in the conclusion of the theorem. Fortunately, this cumbersome but harmless detail will not play any role in the rest of this paper.

The equation $dZ = Z \wedge Z$ is equivalent to the system

$$\frac{\partial Z_m}{\partial t_n} - \frac{\partial Z_n}{\partial t_m} + [Z_m, Z_n] = 0 \qquad m, n = -1, 0, 1, \ldots \tag{24}$$

Conclusion (3) in the theorem follows immediately from taking $m = 0$, $n = -1$. To obtain conclusion (1), let us take $m = 0$, $n \geq 1$. Then

$$\frac{\partial Z_0}{\partial t_n} = \frac{\partial Z_n}{\partial t_0} - [Z_0, Z_n]$$

$$
\begin{aligned}
&= \frac{\partial(Z_n-\lambda Z_{n-1})}{\partial t_0} - [Z_0,(Z_n-\lambda Z_{n-1})] + \lambda\left(\frac{\partial Z_{n-1}}{\partial t_0} - [Z_0, Z_{n-1}]\right) \\
&= \frac{\partial(Z_n-\lambda Z_{n-1})}{\partial t_0} - [Z_0,(Z_n-\lambda Z_{n-1})] + \lambda\frac{\partial Z_0}{\partial t_{n-1}}. \qquad (25)
\end{aligned}
$$

Write $\Delta_n = Z_n - \lambda Z_{n-1}$, $n = 0, 1, 2, \ldots$, which by hypothesis is independent of λ. The LHS of Eq. (25) is independent of λ, and the RHS is linear. Setting the coefficient of λ on the RHS to zero,

$$
\frac{\partial Z_0}{\partial t_{n-1}} = \left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \Delta_n\right], \qquad (26)
$$

which implies

$$
\begin{aligned}
(\Delta_n)_{11} &= \frac{1}{2}\frac{\partial}{\partial t_{n-1}}(u - j^2) \\
(\Delta_n)_{12} &= -\frac{\partial j}{\partial t_{n-1}}. \qquad (27)
\end{aligned}
$$

The terms independent of λ in Eq. (25) give

$$
\frac{\partial Z_0}{\partial t_n} = \frac{\partial \Delta_n}{\partial t_0} - \left[\begin{pmatrix} j & 1 \\ u-j^2 & -j \end{pmatrix}, \Delta_n\right]. \qquad (28)
$$

The 1,2 entry of this gives

$$
\frac{\partial(\Delta_n)_{12}}{\partial t_0} - 2(j(\Delta_n)_{12} - (\Delta_n)_{11}) = 0. \qquad (29)
$$

Substituting from Eq. (27),

$$
\frac{\partial}{\partial t_{n-1}}(u - j_{t_0}) = 0, \qquad (30)
$$

which implies, since it is true for all $n \geq 1$, that $u = j_{t_0} + c$ where c is a function of t_{-1} alone. From the t_{-1} evolution equations as already established, c must in fact be constant. Having shown this, c can be removed via an admissible change of coordinates (with only a_1 nonzero), which as explained above translates u by a constant, thus proving conclusion(1).

To establish conclusion (2) requires looking at the 1,1 and 2,1 entries of Eq. (28). Using Eq. (27) and the relation $u = j_{t_0}$ these give

$$
\begin{aligned}
\frac{\partial j}{\partial t_n} &= -(\Delta_n)_{21} + \frac{1}{2}\frac{\partial^2}{\partial t_0^2}\frac{\partial j}{\partial t_{n-1}} - j\frac{\partial}{\partial t_0}\frac{\partial j}{\partial t_{n-1}} + (j^2 - 2u)\frac{\partial j}{\partial t_{n-1}} \qquad (31) \\
\frac{\partial u}{\partial t_n} &= \frac{\partial(\Delta_n)_{21}}{\partial t_0} + j\frac{\partial^2}{\partial t_0^2}\frac{\partial j}{\partial t_{n-1}} - (u + j^2)\frac{\partial}{\partial t_0}\frac{\partial j}{\partial t_{n-1}} - 2ju\frac{\partial j}{\partial t_{n-1}}. \qquad (32)
\end{aligned}
$$

Eliminating $(\Delta_n)_{21}$ by differentiating the first of these equations with respect to t_0 and adding to the second, gives, after some algebra:

$$\frac{\partial u}{\partial t_n} = (\tfrac{1}{4}\partial_{t_0}^3 - u\partial_{t_0} - \partial_{t_0}u)\frac{\partial j}{\partial t_{n-1}}. \tag{33}$$

This is almost the required result. It is clear that given the correct t_{n-1} flow for j, viz. $j_{t_{n-1}} = -P_n$, then Eq. (33) implies the correct t_n evolution for u, viz. $u_{t_n} = -\partial_{t_0}P_{n+1}$. Integrating this with respect to t_0 gives $j_{t_n} = -P_{n+1}+c_n(t_{-1},t_1,t_2,\ldots)$. To complete an inductive step requires showing c_n can be set to zero. This involves three stages. First, t_{-1} independence follows of c_n follows immediately from the t_{-1} flow equations for u,j that have already been established. The second stage is showing there is no $t_1,t_2,\ldots$ dependence. This involves use of Eqs. (24) for $m,n\geq 1$, and is unfortunately intricate, and is therefore relegated to an appendix. Once it has been established that c_n is constant, the third stage is to observe that it can be eliminated by an admissible change of coordinates with only a_{n+1} nonzero (from Eq. (31), the addition of a constant to j_{t_n} is equivalent to addition of a constant to $(\Delta_n)_{21}$, i.e. addition of a constant multiple of Z_{-1} to Z_n). In this manner it emerges that Eq. (33) is the crucial ingredient in an inductive step, obtaining the correct t_n flow for j from the correct t_{n-1} flow. The induction is started from the relation $j_{t_0} = u$, which has already been proven. $\square$

4 A Map from the $SL(2,\mathbf{C})$ Loop Group to Solutions of KdV

On M, define the following one-form valued in the Lie algebra of G:

$$\Omega = \sum_{n=-1}^{\infty}\begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}^{2n+1} dt_n = \sum_{n=-1}^{\infty}\begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}\lambda^n dt_n. \tag{34}$$

Consider on M the linear differential system

$$dU(t) = \Omega U(t), \tag{35}$$

where $U(t)$ is a G valued function on M. Since $d\Omega + \Omega\wedge\Omega = 0$ (in fact $d\Omega = \Omega\wedge\Omega = 0$) this system is completely integrable in the sense of Frobenius, and its general solution is

$$U(t) = \exp\left(\sum_{n=-1}^{\infty}\begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}\lambda^n t_n\right)U_0, \tag{36}$$

where $U_0 = U(0)$ is an arbitrary element of G. Since the big cell of G, where the Birkhoff decomposition can be applied, is an open dense subset of G, it comes as no surprise that the flow on G defined by (36) only leaves the big cell for discrete values of t [22]. Let us consider therefore the Birkhoff decomposition of U, in an interval of values of t for which it is defined. Writing

$$U(t) = S^{-1}(t) \cdot Y(t), \tag{37}$$

where $Y(t) \in G^+$, $S(t) \in G^-$, and substituting in (35), gives

$$-dS \cdot S^{-1} + dY \cdot Y^{-1} = S\Omega S^{-1}. \tag{38}$$

The first term on the LHS lies in the Lie algebra of G_-, whereas the second term lies in the Lie algebra of G_+. Therefore, using the obvious notation for the projections of an element in the Lie algebra of G to those of G_+ and G_-,

$$\begin{aligned} dS \cdot S^{-1} &= -(S\Omega S^{-1})_- && (39)\\ dY \cdot Y^{-1} &= (S\Omega S^{-1})_+. && (40) \end{aligned}$$

Write $Z = (S\Omega S^{-1})_+$; Z is an $\mathcal{A}$ valued one form on M. Using Eq. (40), $dZ = Z \wedge Z$. But using the definition of Ω, Eq. (34),

$$Z = \sum_{n=-1}^{\infty} Z_n dt_n, \qquad Z_n = \left(S \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} S^{-1} \lambda^n \right)_+ . \tag{41}$$

Now

$$S = I + \frac{1}{\lambda} \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} + O\left(\frac{1}{\lambda^2}\right) \tag{42}$$

for some functions α, β, γ on M. A simple computation now gives

$$Z_0 = \begin{pmatrix} \beta & 1 \\ \lambda - 2\alpha & -\beta \end{pmatrix} . \tag{43}$$

From the definition of the Z_n, Eq. (41), it is apparent that $Z_n - \lambda Z_{n-1}$ is independent of λ. Thus all three of the conditions for the theorem of Sec. 3 hold, and it follows that, up to an admissible change of coordinates, the functions $j = \beta$, $u = \beta^2 - 2\alpha$ satisfy the potential KdV hierarchy. In other words, given any solution of Eq. (35) (a solution of which is specified by the choice of U_0, an element of the loop group), there is an associated solution of the potential KdV hierarchy. The replacement $U_0 \to U_0 \cdot g_+$, where $g_+ \in G^+$, has the effect $U \to U \cdot g_+$ and $Y \to Y \cdot g_+$, but S and Z and therefore the associated KdV solution are left unchanged. Equally easy to see is that the replacement

$$U_0 \to h \cdot U_0, \qquad h = \exp\left(\sum_{m=2}^{\infty} \frac{t_{-m}}{\lambda^m} \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} \right), \tag{44}$$

where $t_{-2}, t_{-3}, \ldots$ are parameters, causes $U \to h \cdot U$, $S \to S \cdot h^{-1}$, but Y and Z are left invariant. Thus the map is actually from the double coset space $H \backslash G / G^+$

to solutions of KdV, where H is the infinite dimensional abelian subgroup of G generated by the commuting matrices

$$\frac{1}{\lambda^n}\begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} \qquad\qquad n = 2, 3, \ldots \tag{45}$$

in the Lie algebra of G. The solutions of KdV obtained this way can have singularities at points where the Birkhoff decomposition is not possible.

Let us pause to reiterate the important points mentioned in Sec. 2. The loop group element U_0 arises as the initial value data for the simple linear flow Eq. (35), which is the linear system "behind" KdV. There are *two* formulae for Z, $Z = (S\Omega S^{-1})_+$ and $Z = dY \cdot Y^{-1}$, meaning that given a solution of Eq. (35) the associated KdV solution can be constructed either going through the flow on G^+ given by Y, or through the flow on G^- given by S. Finally, in addition to the G^+ action, there is an H action on U_0 that leaves the KdV solution invariant.

One way of thinking about this extra invariance is as follows: I have chosen to work on the affine space with coordinates $t_{-1}, t_0, \ldots$, but I could have decided to work on the affine space with coordinates $\ldots, t_{-2}, t_{-1}, t_0, t_1, t_2, \ldots$, taking the sum in the definition of Ω, Eq. (34), to be from $-\infty$ to ∞ rather than from -1 to ∞. The new flows would all leave Y and Z invariant, but not S. It follows that there must be degrees of freedom in S that do not "contribute" to Z, that are acted upon by the new flows. This is indeed the case; the simplest example of such a degree of freedom is γ in Eq. (42) above. Since these new flows evidently commute, when working on the affine space with coordinates $t_{-1}, t_0, \ldots$, there must be an infinite dimensional abelian group action on U_0 leaving Z, but not S, invariant; this is just the H action introduced above.

For later reference I give a number of formulae. First, I give the relation of the first few terms in Y and the solution functions j, u. Writing

$$Y = \sum_{n=0}^{\infty} Y_n \lambda^n, \tag{46}$$

where the matrices Y_n are independent of λ,

$$\begin{aligned} Z_0 &= \frac{\partial Y}{\partial t_0} Y^{-1} \\ &= \left(\frac{\partial Y_0}{\partial t_0} + \frac{\partial Y_1}{\partial t_0}\lambda\right)\left(Y_0^{-1} - Y_0^{-1} Y_1 Y_0^{-1}\lambda\right) + O(\lambda^2) \\ &= \frac{\partial Y_0}{\partial t_0} Y_0^{-1} + Y_0 \frac{\partial (Y_0^{-1} Y_1)}{\partial t_0} Y_0^{-1}\lambda + O(\lambda^2). \end{aligned} \tag{47}$$

Writing

$$Y_0 = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \qquad\qquad Y_0^{-1} Y_1 = \begin{pmatrix} b_1 & b_2 \\ b_3 & -b_1 \end{pmatrix}, \tag{48}$$

the following relationships emerge (in addition to the determinant constraint $a_1a_4 - a_2a_3 = 1$):

$$
\begin{array}{ll}
a_3 = a_{1t_0} - ja_1 & a_{1t_0t_0} = 2ua_1 \\
a_4 = a_{2t_0} - ja_2 & a_{2t_0t_0} = 2ua_2
\end{array}
$$
$$
\begin{aligned}
b_{1t_0} &= -a_1a_2 \\
b_{2t_0} &= -a_2^2 \\
b_{3t_0} &= a_1^2.
\end{aligned} \tag{49}
$$

Evolutions for components of Y can easily be determined using the higher components of $Z = dY \cdot Y^{-1}$; in particular, from $Z_1 = (\partial Y/\partial t_1) \cdot Y^{-1}$,

$$
\begin{aligned}
a_{1t_1} &= \tfrac{1}{2}u_{t_0}a_1 - ua_{1t_0} \\
a_{2t_1} &= \tfrac{1}{2}u_{t_0}a_2 - ua_{2t_0} \\
b_{1t_1} &= a_{1t_0}a_{2t_0} - ua_1a_2 \\
b_{2t_1} &= a_{2t_0}^2 - ua_2^2 \\
b_{3t_1} &= -a_{1t_0}^2 + ua_1^2
\end{aligned} \tag{50}
$$

Next, consider the problem of to what extent S can be reconstructed from Z. Looking at Eq. (41), and recalling from Sec. 3 that $\Delta_n = Z_n - \lambda Z_{n-1}$, $n = 0, 1, \ldots$, we see at once that Δ_n is the coefficient of λ^{-n} in $S\begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}S^{-1}$. That is,

$$
S\begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}S^{-1} = \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} + \sum_{n=0}^{\infty}\frac{\Delta_n}{\lambda^n} = \sum_{n=-1}^{\infty}\frac{\Delta_n}{\lambda^n} \tag{51}
$$

where $\Delta_{-1} \equiv \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Finally, it will be useful to an abbreviation for the matrix appearing in Eq. (36), so define

$$
M = \exp\left(\sum_{n=-1}^{\infty}\begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}\lambda^n t_n\right) = \cosh(z\sqrt{\lambda})I + \frac{\sinh(z\sqrt{\lambda})}{\sqrt{\lambda}}\begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}, \tag{52}
$$

where $z = \sum_{n=-1}^{\infty} t_n\lambda^n$, so that Eq. (36) now reads $U = MU_0$.

Two more notes are in order before concluding this section. First, I must mention again that the above construction is modeled on Mulase's similar construction for KP [14]; I have followed Mulase's notation throughout. Second, the PKdV gauge of the zero curvature formulation evidently plays a pivotal role in the construction; I expect though that a similar formulation for KdV in its standard gauge could be found using a nonstandard Birkhoff decomposition of G (c.f. the different projections used in [5], Sec.3).

5 Symmetry Actions on the Loop Group and Symmetries of KdV

In this section I give the symmetry actions on G that give rise to the symmetries of KdV listed in Sec. 1. The order of subsections differs from that of Sec. 1 in that I deal with infinitesimal symmetries first.

5.1 Infinitesimal symmetries

Consider infinitesimal left multiplications on U_0, i.e. transformations

$$U_0 \mapsto (I + \epsilon P)\, U_0, \tag{53}$$

where ϵ is an infinitesimal parameter and P is in the Lie algebra of G. If P is independent of U_0 this defines a map from G/G^+ to G/G^+, but it is not clear whether this descends to the double coset space $H\backslash G/G^+$. Under the transformation (53),

$$\begin{aligned} U &\mapsto M\,(I+\epsilon P)\,U_0 \\ &= \left(I + \epsilon M P M^{-1}\right) U, \end{aligned} \tag{54}$$

which gives

$$Y \mapsto \left(I + \epsilon (SMPM^{-1}S^{-1})_+\right) Y \tag{55}$$

$$S \mapsto \left(I - \epsilon (SMPM^{-1}S^{-1})_-\right) S. \tag{56}$$

(These are easily computed: an infinitesimal variation δU in U gives rise to infinitesimal variations δY in Y and δS in S where $\delta U = -S^{-1}\delta S S^{-1} Y + S\delta Y$, i.e. $S\delta U Y^{-1} = -\delta S S^{-1} + \delta Y Y^{-1}$. Thus $\delta S S^{-1} = -(S\delta U Y^{-1})_-$ and $\delta Y Y^{-1} = (S\delta U Y^{-1})_+$.) Using now the definition of Z_0 from Eq. (41), a straightforward calculation gives

$$Z_0 \mapsto Z_0 + \epsilon \left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \mathrm{res}(SMPM^{-1}S^{-1}) \right], \tag{57}$$

where $\mathrm{res}(T)$, for T in the Lie algebra of G, denotes the $O(\lambda^{-1})$ term in T.

Consider two special cases. First, take

$$P = P_n \equiv \lambda^n \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}^{2n+1} \qquad n = \ldots, -1, 0, 1, \ldots \tag{58}$$

Evidently $MP_nM^{-1} = P_n$. For $n < -1$ these transformations have no effect on Z_0; they are just infinitesimal H transformations. For $n \geq -1$, using Eq. (51),

$$\begin{aligned} Z_0 &\mapsto Z_0 + \epsilon \left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \Delta_{n+1} \right] \\ &= Z_0 + \epsilon \frac{\partial Z_0}{\partial t_n}, \end{aligned} \tag{59}$$

where in the last equality I have used Eq. (26). Thus the choice $P = P_n$ generates translations in t_n, for $n \geq -1$. Since all the matrices P_n commute, these transformation clearly descend to the double coset space. I denote the generator of these transformations on U_0 by l_n, in accordance with the notation of Sec. 1.5. The generator of the infinitesimal transformation $U_0 \mapsto (1+\epsilon P)U_0$ is defined as the operator whose action on U_0 gives PU_0; so l_n is just multiplication by P_n. Note that here l_n is defined in a wider sense than in Sec. 1.5; there l_n denoted the generator of translations acting on the solutions of KdV, whereas here it denotes the generator of the transformations acting on the loop group that induce translations on solutions of KdV. In particular l_n here is not zero for negative n.

Now let us take

$$P = Q_n \equiv \lambda^n \left(\lambda \frac{dU_0}{d\lambda} U_0^{-1} + \tfrac{1}{4} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \qquad n = \ldots, -1, 0, 1, \ldots \tag{60}$$

These look like field dependent transformations, since U_0 appears in P! But in fact this is just the choice of P for the field independent infinitesimal transformation which is a linear combination of a reparametrization of U_0 and a multiplication by a fixed algebra element:

$$U_0(\lambda) \mapsto U_0(\lambda(1+\epsilon\lambda^n)) + \tfrac{1}{4}\epsilon\lambda^n \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} U_0(\lambda). \tag{61}$$

The associated generators are

$$m_n = \lambda^n \left(\lambda \frac{d}{d\lambda} + \tfrac{1}{4} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \qquad n = \ldots, -1, 0, 1, \ldots \tag{62}$$

In Sec. 5.2 I will give the finite action which these generate; below I will explain why I am looking at these particular transformations. But first let me give the action on the KdV solution. To use Eq. (57) it is necessary to compute $SMQ_nM^{-1}S^{-1}$ with Q_n given by Eq. (60). Substituting $U_0 = M^{-1}U = M^{-1}S^{-1}Y$ in Q_n gives, after some algebra,

$$\begin{aligned} SMQ_nM^{-1}S^{-1} &= \lambda^{n+1}(Y'Y^{-1} - S'S^{-1}) \\ &+ \lambda^n S \left(\tfrac{1}{4} M \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M^{-1} - \lambda M' M^{-1} \right) S^{-1}, \end{aligned} \tag{63}$$

where a prime denotes differentiation with respect to λ. Using Eq. (52)

$$\tfrac{1}{4} M \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M^{-1} - \lambda M' M^{-1} = \tfrac{1}{4} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \sum_{m=-1}^{\infty} (m+\tfrac{1}{2}) t_m \lambda^m \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}. \tag{64}$$

The last two equations and Eq. (51) give the final result

$$\begin{aligned} SMQ_nM^{-1}S^{-1} &= \lambda^{n+1}(Y'Y^{-1} - S'S^{-1}) + \frac{\lambda^n}{4} S \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} S^{-1} \\ &- \lambda^n \left(\sum_{m=-1}^{\infty} (m+\tfrac{1}{2}) t_m \lambda^m \right) \left(\sum_{p=-1}^{\infty} \Delta_p \lambda^{-p} \right). \end{aligned} \tag{65}$$

The residue of this must be computed. Of the four terms, the simplest to discuss is the last, despite its apparent complexity. For all n, it contributes an infinite series of terms to the residue

$$-\sum_{m=\max(-1,-2-n)}^{\infty}(m+\tfrac{1}{2})t_m\Delta_{m+n+1}. \tag{66}$$

Because this term is given purely in terms of j and u, via the Δ_p's, it descends to the double coset space. For the other terms this is not true in general. The first term in (65) will contribute to the residue if $n\leq -2$, since $Y'Y^{-1}$ is a power series in λ; the terms it contributes descend to $H\backslash G$, but not necessarily to the double coset space. The second term will contribute to the residue if $n\geq 0$ and third term if $n\geq -1$; their contributions descend to G/G^+ but not necessarily to the double coset space. So it is not clear that there is ever a genuine symmetry of KdV here! In practice, however, it turns out that for $n=-1,0$ only H-invariant degrees of freedom from S contribute to the variation of Z_0. A little further calculation gives the results

$$\begin{array}{lrcl}
n=-1 & Z_0 & \mapsto & Z_0+\epsilon\left(\begin{pmatrix}0 & 0\\ \frac{1}{2} & 0\end{pmatrix}-\sum_{m=0}^{\infty}(m+\frac{1}{2})t_m\frac{\partial Z_0}{\partial t_{m-1}}\right)\\
 & j & \mapsto & j-\epsilon\sum_{m=0}^{\infty}(m+\frac{1}{2})t_m\frac{\partial j}{\partial t_{m-1}}\\
 & u & \mapsto & u+\frac{\epsilon}{2}-\epsilon\sum_{m=1}^{\infty}(m+\frac{1}{2})t_m\frac{\partial u}{\partial t_{m-1}}
\end{array} \tag{67}$$

$$\begin{array}{lrcl}
n=0 & Z_0 & \mapsto & Z_0+\epsilon\left(\begin{pmatrix}-\beta/2 & 0\\ 2\alpha & \beta/2\end{pmatrix}-\sum_{m=-1}^{\infty}(m+\frac{1}{2})t_m\frac{\partial Z_0}{\partial t_m}\right)\\
 & j & \mapsto & j-\frac{\epsilon j}{2}-\epsilon\sum_{m=-1}^{\infty}(m+\frac{1}{2})t_m\frac{\partial j}{\partial t_m}\\
 & u & \mapsto & u-\epsilon u-\epsilon\sum_{m=0}^{\infty}(m+\frac{1}{2})t_m\frac{\partial u}{\partial t_m}.
\end{array} \tag{68}$$

The $n=-1$ and $n=0$ cases correspond to Galilean transformations and scaling respectively. For $n=1$, a detailed calculation gives

$$\begin{aligned}
u &\mapsto u+\epsilon\left(\tfrac{1}{2}ju_{t_0}+2u^2-\tfrac{1}{2}u_{t_0t_0}-\sum_{m=-1}^{\infty}(m+\tfrac{1}{2})t_m\frac{\partial u}{\partial t_{m+1}}\right)\\
&= u+\epsilon\left(\tfrac{1}{2}(j+t_{-1})u_{t_0}+(2u^2-\tfrac{1}{2}u_{t_0t_0})-\tfrac{1}{2}t_0u_{t_1}-\tfrac{3}{2}t_1u_{t_2}-\ldots\right).
\end{aligned} \tag{69}$$

In the second line here I have emphasized the cancellation between terms that ensures that even though j appears in this transformation law, u develops no t_{-1} dependence. The $n=1$ transformation law for j involves the non H-invariant quantity γ defined in Eq. (42); and for higher n other non H-invariant terms appear in the transformation laws for both u and j. This is perfectly in accord with existing knowledge about the higher symmetries in the hierarchy of Galilean and scaling symmetries; each time the recursion operator is applied to obtain a higher symmetry, a new integration constant appears. It is straightforward to check that while the symmetries m_n, $n \geq 1$ cannot be considered to descend to the double coset space $H\backslash G/G^+$, they do descend to spaces of the form $H_n\backslash G/G^+$, where, for every n, H_n is a subgroup of H of finite codimension.

Turning now briefly to the $n = -2, -3, \ldots$ cases, none of these symmetries descend to G/G^+. Since it is straightforward to compute using Eq. (65), I give the explicit form of the $n = -2$ symmetry:

$$\begin{aligned} u &\rightarrow u + \epsilon\left(2b_1(a_1a_2)_{t_0} - b_2(a_1^2)_{t_0} + b_3(a_2^2)_{t_0} - \sum_{m=2}^{\infty}(m+\frac{1}{2})t_m\frac{\partial u}{\partial t_{m-2}}\right) \\ j &\rightarrow j + \epsilon\left(2b_1a_1a_2 - b_2a_1^2 + b_3a_2^2 - \sum_{m=1}^{\infty}(m+\frac{1}{2})t_m\frac{\partial j}{\partial t_{m-2}}\right), \end{aligned} \tag{70}$$

where here I am using the notation set up in Eq. (48) (c.f. [8]).

As a final comment on the m_n symmetries, I note that the operators defined on the loop group, viz.

$$l_n = \lambda^n \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} \qquad m_n = \lambda^n \left(\lambda\frac{d}{d\lambda} + \tfrac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \tag{71}$$

satisfy the commutation relations of Eq. (9), where now all indices are allowed to run over all the integers. They are related by recursion formulae $l_{n+1} = \lambda l_n$, and $m_{n+1} = \lambda m_n$; λ is playing the role of the recursion operator.

To conclude this section, I need to answer the question of why I have not looked at other infinitesimal symmetries on G. It is easy to check that a generic choice of field independent infinitesimal transformation of the form of Eq. (53) simply does not descend to the double coset space. These are the messy symmetries discussed in [9]. On the other hand, it is currently not at all clear that I have identified the full set of symmetries that do descend to the double coset space. This is the problem in understanding the (local) geometry of the double coset space $H\backslash G/G^+$; particularly it is of interest to identify a basis for vector fields on this space.

5.2 Translations, Galilean and scaling symmetries

Consider now the finite transformations generated by the infinitesimal transformations of Sec. 5.1. Finite translations correspond to transformations

$$U_0 \mapsto \exp\left(\theta\lambda^n \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}\right) U_0 \tag{72}$$

on G. This is evidently equivalent to a translation of t_n by θ, for $n \geq -1$; for $n < -1$ it does not act on KdV solutions. To exponentiate the symmetries in the hierarchy of Galilean and scaling symmetries, observe that

$$m_n = \lambda^{n+1} \begin{pmatrix} \lambda^{-\frac{1}{4}} & 0 \\ 0 & \lambda^{\frac{1}{4}} \end{pmatrix} \frac{d}{d\lambda} \begin{pmatrix} \lambda^{\frac{1}{4}} & 0 \\ 0 & \lambda^{-\frac{1}{4}} \end{pmatrix}, \tag{73}$$

so a finite transformation generated by this takes the form

$$U_0(\lambda) \mapsto \begin{pmatrix} (1+\theta\lambda^n)^{\frac{1}{4}} & 0 \\ 0 & (1+\theta\lambda^n)^{-\frac{1}{4}} \end{pmatrix} U_0\left(\lambda(1+\theta\lambda^n)\right). \tag{74}$$

The quantity on the RHS here will not be in G for arbitrary U_0 and θ, but I assume the necessary restrictions are imposed so that it is. When do these transformations descend to $H \backslash G/G^+$? To descend to the coset space $H \backslash G$, the effect of an H transformation on U_0 followed by a transformation of the above type must be equivalent to the effect of first applying the transformation of the above type followed by a (possibly different) H transformation. That is, for all choices of $t_{-2}, t_{-3}, \dots$ it must be possible to find $s_{-2}, s_{-3}, \dots$ such that

$$\begin{aligned} &\begin{pmatrix} (1+\theta\lambda^n)^{\frac{1}{4}} & 0 \\ 0 & (1+\theta\lambda^n)^{-\frac{1}{4}} \end{pmatrix} \exp\left(\sum_{m=2}^{\infty} \frac{t_{-m}}{(\lambda(1+\theta\lambda^n))^m} \begin{pmatrix} 0 & 1 \\ \lambda(1+\theta\lambda^n) & 0 \end{pmatrix}\right) \\ &\qquad = \exp\left(\sum_{m=2}^{\infty} \frac{s_{-m}}{\lambda^m} \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}\right) \begin{pmatrix} (1+\theta\lambda^n)^{\frac{1}{4}} & 0 \\ 0 & (1+\theta\lambda^n)^{-\frac{1}{4}} \end{pmatrix} \end{aligned} \tag{75}$$

Multiplying this out, various cancellations take place, and it turns out that this is equivalent to the simple requirement

$$\sum_{m=2}^{\infty} \frac{s_{-m}}{\lambda^m} = \sum_{m=2}^{\infty} \frac{t_{-m}}{\lambda^m} (1+\theta\lambda^n)^{-m+\frac{1}{2}} \tag{76}$$

For $n \leq 0$ this can be satisfied, taking $|\theta| < 1$ and expanding in negative powers of λ for $|\lambda| > |\theta|$. However, it is obvious that symmetries of the form of Eq. (74) only descend to G/G^+ if for all $g_+(\lambda) \in G^+$, $g_+(\lambda(1+\theta\lambda^n)) \in G^+$. This requires $n \geq -1$. Thus, as expected from the analysis in the infinitesimal case, the only symmetries that descend to the double coset space are the cases $n = -1, 0$, the cases of Galilean and scaling symmetries respectively. In particular, these results show that *the so-called hidden symmetries of KdV cannot be exponentiated.*

It remains to give formulae for finite Galilean and scaling transformations. For $n = 0$ (scaling transformations)

$$U_0(\lambda) \mapsto \begin{pmatrix} p^{\frac{1}{2}} & 0 \\ 0 & p^{-\frac{1}{2}} \end{pmatrix} U_0(p^2\lambda), \tag{77}$$

where $p=(1+\theta)^{\frac{1}{2}}$. A simple calculation gives

$$U(\lambda,t)\mapsto\begin{pmatrix}p^{\frac{1}{2}} & 0\\ 0 & p^{-\frac{1}{2}}\end{pmatrix}U(p^2\lambda,s), \tag{78}$$

where

$$s_m=\frac{1}{p^{2m+1}}t_m \qquad m=-1,0,\ldots \tag{79}$$

In turn this gives

$$\begin{aligned}S(\lambda,t) &\mapsto \begin{pmatrix}p^{\frac{1}{2}} & 0\\ 0 & p^{-\frac{1}{2}}\end{pmatrix}S\left(p^2\lambda,s\right)\begin{pmatrix}p^{-\frac{1}{2}} & 0\\ 0 & p^{\frac{1}{2}}\end{pmatrix}\\ Y(\lambda,t) &\mapsto \begin{pmatrix}p^{\frac{1}{2}} & 0\\ 0 & p^{-\frac{1}{2}}\end{pmatrix}Y\left(p^2\lambda,s\right),\end{aligned} \tag{80}$$

where I have assumed that p is such that the matrices on the RHS of these expressions are genuinely in G^-, G^+ respectively. Finally from the $O(\lambda^{-1})$ term of the S transformation law, emerge the standard scaling transformations

$$j(t)\mapsto\frac{1}{p}j(s), \qquad u(t)\mapsto\frac{1}{p^2}u(s). \tag{81}$$

For $n=-1$ (Galilean transformations) things are marginally more difficult. The starting point is

$$U_0(\lambda)\mapsto\begin{pmatrix}(1+\theta/\lambda)^{\frac{1}{4}} & 0\\ 0 & (1+\theta/\lambda)^{-\frac{1}{4}}\end{pmatrix}U_0(\lambda+\theta). \tag{82}$$

This gives, after some effort:

$$\begin{aligned}U(\lambda,t) \mapsto\ & \begin{pmatrix}(1+\theta/\lambda)^{\frac{1}{4}} & 0\\ 0 & (1+\theta/\lambda)^{-\frac{1}{4}}\end{pmatrix}\exp\left(\sum_{m=2}^{\infty}\frac{s_{-m}}{(\lambda+\theta)^m}\begin{pmatrix}0 & 1\\ \lambda+\theta & 0\end{pmatrix}\right)\\ & U\left((\lambda+\theta),s\right)\end{aligned} \tag{83}$$

where the "times" $\ldots,s_{-2},s_{-1},s_0,\ldots$ are related to the "times" $t_{-1},t_0,\ldots$ by

$$\sum_{m=-1}^{\infty}t_m\lambda^{m+\frac{1}{2}}=\sum_{m=-\infty}^{\infty}s_m\left(\lambda+\theta\right)^{m+\frac{1}{2}}, \tag{84}$$

This in turn gives

$$\begin{aligned}S(\lambda,t) &\mapsto S\left(\lambda+\theta,s\right)\exp\left(-\sum_{m=2}^{\infty}\frac{s_{-m}}{(\lambda+\theta)^m}\begin{pmatrix}0 & 1\\ \lambda+\theta & 0\end{pmatrix}\right)\\ &\qquad \begin{pmatrix}(1+\theta/\lambda)^{-\frac{1}{4}} & 0\\ 0 & (1+\theta/\lambda)^{\frac{1}{4}}\end{pmatrix}\\ Y(\lambda,t) &\mapsto Y\left(\lambda+\theta,s\right),\end{aligned} \tag{85}$$

where, again, certain assumptions on θ have been made; finally, the Galilean invariance formulae emerge:

$$\begin{array}{rclcrcl} j(t) & \mapsto & j(s) & \qquad & u(t) & \mapsto & u(s)+\frac{\theta}{2} \\ s_m & = & \displaystyle\sum_{n=m}^{\infty}\binom{n+\frac{1}{2}}{n-m}(-\theta)^{n-m}t_n & & & & m=-1,0,\ldots \end{array} \tag{86}$$

The reader may wish to try deriving these from both the transformation law for Y and that for S; the latter is much simpler to use. The expressions in Eq. (86) for finite Galilean transformations are consistent with the infinitesimal transformation law Eq. (67), and indeed it can be checked directly that Eq. (86) does give a symmetry of the whole KdV hierarchy. For future reference, I call this Galilean transformation $A(\theta)$. The range of values of θ for which the transformation $A(\theta)$ is defined on G is limited, and is different for different elements of G. For the action of $A(\theta)$ on u, j there is also a limitation, of convergence of the series in Eq. (86), but these limitations do not correspond. This is just the first symptom to appear of the fact that the class of KdV solutions being studied in this paper is limited. The restrictions on applicability of $A(\theta)$ will be important in the next section.

5.3 Wahlquist-Estabrook Bäcklund Transformations

Let $B : G \to G$ be the involution on G defined by

$$U_0 \mapsto \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} U_0 \begin{pmatrix} 0 & \lambda^{-1} \\ 1 & 0 \end{pmatrix} \tag{87}$$

This descends to an involution on the coset space $H\backslash G$, since U_0 is multiplied on the left by a matrix that commutes with all elements of H. It does not descend to $H\backslash G/G^+$ though, because

$$U_0 \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & \lambda^{-1} \\ 1 & 0 \end{pmatrix} = U_0 \begin{pmatrix} 0 & \lambda^{-1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} d & c\lambda^{-1} \\ \lambda b & a \end{pmatrix}, \tag{88}$$

so right multiplication of U_0 by an element of G^+ followed by application of B is not equivalent to application of B followed by right multiplication by a (possibly different) element of G^+. The above identity shows, however, that B *does* descend to the double coset space $H\backslash G/J$, where J is the subgroup of G^+ consisting of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G^+$ such that c has no constant term in its power series expansion in λ. J is a codimension 1 subgroup of G^+, so B should give rise to a map that generates a one parameter family of solutions of KdV from a given one.

Defining $G^{+(1)}$ to be the subgroup of elements of G^+ that reduce to the identity at $\lambda = 0$, $G^{+(1)} \subset J \subset G^+$, and $G^{+(1)}$ is normal in G^+. $G^+/G^{+(1)}$ is naturally identified as the subgroup of constant elements in G. It follows that if I define the map $\mathcal{O}_0$, mapping G to the space of subsets of G, by

$$U_0 \mapsto \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} U_0 \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} 0 & \lambda^{-1} \\ 1 & 0 \end{pmatrix}, \tag{89}$$

where $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ is an arbitrary $SL(2,\mathbf{C})$ matrix, then this map *will* descend to the double coset space $H\backslash G/G^+$. It seems at first glance that this map should give a three parameter family of solutions from a single one. But the full information in the matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ is not actually relevant; it can be multiplied on the right by any upper triangular matrix without effect. The map (89) gives

$$\begin{aligned} U &\mapsto \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} U \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} 0 & \lambda^{-1} \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} S^{-1} \cdot Y \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} 0 & \lambda^{-1} \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} S^{-1} \begin{pmatrix} \frac{\beta}{\lambda} & \frac{1}{\lambda} \\ 1+\frac{T\beta}{\lambda} & \frac{T}{\lambda} \end{pmatrix} \cdot \\ &\qquad \begin{pmatrix} -T & 1 \\ \lambda + T\beta & -\beta \end{pmatrix} Y \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} 0 & \lambda^{-1} \\ 1 & 0 \end{pmatrix} \end{aligned} \tag{90}$$

where in the last line I have inserted a certain matrix and its inverse, and

$$T = \frac{pa_3 + ra_4}{pa_1 + ra_2} = \frac{\tau_{t_0}}{\tau} - j, \qquad \tau = pa_1 + ra_2. \tag{91}$$

(Throughout, I use the notation of Sec. 4 for the components of S and Y needed. In particular β is defined in Eq. (42), and $a_1, \ldots, a_4$ in Eq. (48).) The reason for the insertion in the last equation is that — as can be checked by a straightforward calculation — the RHS is now written in Birkhoff factorized form, that is

$$\begin{aligned} S &\mapsto \begin{pmatrix} \frac{T}{\lambda} & -\frac{1}{\lambda} \\ -1-\frac{T\beta}{\lambda} & \frac{\beta}{\lambda} \end{pmatrix} S \begin{pmatrix} 0 & -1 \\ -\lambda & 0 \end{pmatrix} \\ Y &\mapsto \begin{pmatrix} -T & 1 \\ \lambda + T\beta & -\beta \end{pmatrix} Y \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} 0 & \lambda^{-1} \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{92}$$

Computing the $O(\lambda^{-1})$ terms of the new S gives the transformation

$$\begin{aligned} u &\mapsto \left(\frac{\tau_{t_0}}{\tau}\right)^2 - u = u - \left(\frac{\tau_{t_0}}{\tau}\right)_{t_0} \\ j &\mapsto j - \frac{\tau_{t_0}}{\tau}, \end{aligned} \tag{93}$$

where in the first of these equations I have used the result $\tau_{t_0 t_0} = 2u\tau$ which is evident from the definition of τ, Eq. (91), and Eq. (49). Using Eq. (50), it is easy to compute $\tau_{t_1} = \frac{1}{2}u_{t_0}\tau - u\tau_{t_0} = \frac{1}{4}(\tau_{t_0t_0t_0} - 3\tau_{t_0t_0}\tau_{t_0}/\tau)$. (Compare Eqs. (5)-(6).)

Having defined and discussed the involution $B : G \to G$, and the map $\mathcal{O}_0$ from G to the space subsets of G, the extension of B which descends to $H\backslash G/G^+$, I now define one parameter generalizations of these maps, $B(\theta) : G \to G$, and $\mathcal{O}_\theta$, mapping G to the space of subsets of G, via

$$\begin{aligned} B(\theta) &= A(-\theta)\ B\ A(\theta) &(94)\\ B(\theta)U_0 &= \sqrt{\frac{\lambda-\theta}{\lambda}}\begin{pmatrix}0 & 1\\ \lambda & 0\end{pmatrix}U_0\begin{pmatrix}0 & (\lambda-\theta)^{-1}\\ 1 & 0\end{pmatrix}\\ \mathcal{O}_\theta &= A(-\theta)\ \mathcal{O}_0\ A(\theta) &(95)\\ \mathcal{O}_\theta U_0 &= \sqrt{\frac{\lambda-\theta}{\lambda}}\begin{pmatrix}0 & 1\\ \lambda & 0\end{pmatrix}U_0\begin{pmatrix}p & q\\ r & s\end{pmatrix}\begin{pmatrix}0 & (\lambda-\theta)^{-1}\\ 1 & 0\end{pmatrix}. \end{aligned}$$

Almost everything I have said about B and $\mathcal{O}_0$ is true of $B(\theta)$ and $\mathcal{O}_\theta$, in particular $B(\theta)$ is an involution, which does not descend to $H\backslash G/G^+$, but does descend to $H\backslash G/J_\theta$, where J_θ is a codimension 1 subgroup of G^+, consisting of the matrices $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ in G^+ such that c has a zero at $\lambda = \theta$. $\mathcal{O}_\theta$ is an extension of $B(\theta)$ that does descend to $H\backslash G/G^+$. There is one significant difference, however, between $\mathcal{O}_0$ and $\mathcal{O}_\theta$ for $\theta \neq 0$: this is that because $A(\theta)$ is not defined on all of G, nor is $B(\theta)$. I will not study in detail the domains of the maps $\mathcal{O}_\theta$, but for $|\theta| < 1$ they are large enough for the maps to be interesting.

Before I compute the effect of $\mathcal{O}_\theta$ on u, j, let us discuss some algebraic properties of these maps. First, a simple computation shows

$$B(\theta_1)B(\theta_2)U_0 = U_0\begin{pmatrix}\sqrt{\frac{\lambda-\theta_1}{\lambda-\theta_2}} & 0\\ 0 & \sqrt{\frac{\lambda-\theta_2}{\lambda-\theta_1}}\end{pmatrix}. \tag{96}$$

In particular, $B(\theta_1), B(\theta_2)$ *do not* commute, for $\theta_1 \neq \theta_2$. On the other hand,

$$\begin{aligned} \mathcal{O}_{\theta_1}\mathcal{O}_{\theta_2}U_0 &= \sqrt{(\lambda-\theta_1)(\lambda-\theta_2)}U_0 &(97)\\ &\begin{pmatrix}p_2 & q_2\\ r_2 & s_2\end{pmatrix}\begin{pmatrix}0 & (\lambda-\theta_2)^{-1}\\ 1 & 0\end{pmatrix}\begin{pmatrix}p_1 & q_1\\ r_1 & s_1\end{pmatrix}\begin{pmatrix}0 & (\lambda-\theta_1)^{-1}\\ 1 & 0\end{pmatrix}, \end{aligned}$$

where $\begin{pmatrix}p_1 & q_1\\ r_1 & s_1\end{pmatrix}, \begin{pmatrix}p_2 & q_2\\ r_2 & s_2\end{pmatrix}$ are arbitrary $SL(2,\mathbf{C})$ matrices. Using the identity

$$\begin{aligned} &\begin{pmatrix}p_2 & q_2\\ r_2 & s_2\end{pmatrix}\begin{pmatrix}0 & (\lambda-\theta_2)^{-1}\\ 1 & 0\end{pmatrix}\begin{pmatrix}p_1 & q_1\\ r_1 & s_1\end{pmatrix}\begin{pmatrix}0 & (\lambda-\theta_1)^{-1}\\ 1 & 0\end{pmatrix} = &(98)\\ &\begin{pmatrix}p_2 - zq_2 & q_2\\ r_2 - zs_2 & s_2\end{pmatrix}\begin{pmatrix}0 & (\lambda-\theta_1)^{-1}\\ 1 & 0\end{pmatrix}\begin{pmatrix}p_1 & q_1\\ r_1 & s_1\end{pmatrix}\begin{pmatrix}0 & (\lambda-\theta_2)^{-1}\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ z & 1\end{pmatrix}, \end{aligned}$$

where $z = (\theta_2 - \theta_1)s_1/r_1$ it emerges that, when interpreted as maps on the coset space G/G^+, $\mathcal{O}_{\theta_1}, \mathcal{O}_{\theta_2}$ *do* commute [1].

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[1] The identity (98) is only good for $r_1 \neq 0$. For $r_1 = 0$ use

$$\begin{pmatrix}p_2 & q_2\\ r_2 & s_2\end{pmatrix}\begin{pmatrix}0 & (\lambda-\theta_2)^{-1}\\ 1 & 0\end{pmatrix}\begin{pmatrix}p_1 & q_1\\ 0 & s_1\end{pmatrix}\begin{pmatrix}0 & (\lambda-\theta_1)^{-1}\\ 1 & 0\end{pmatrix} =$$

The commutativity property of the $\mathcal{O}_\theta$ operators (on G/G^+) is the only one of the properties of Eq. (1) that there is any difficulty establishing; the second property follows from the involutiveness of $B(\theta)$. It remains to check that the operator $\mathcal{O}_\theta$ does indeed act as a Backlund transformation; this is a simple generalization for the calculation for $\mathcal{O}_0$. Assuming Y is analytic in a neighborhood of $\lambda = \theta$,

$$Y = \sum_{n=1}^{\infty} Y_n^\theta (\lambda - \theta)^n, \tag{99}$$

where the matrices Y_n^θ are independent of λ. Writing

$$Y_0^\theta = \begin{pmatrix} a_1^\theta & a_2^\theta \\ a_3^\theta & a_4^\theta \end{pmatrix} \tag{100}$$

gives, from Eq. (47), the structure relations

$$\begin{aligned} a_3^\theta &= a_{1t_0}^\theta - j a_1^\theta & \qquad a_{1t_0t_0}^\theta &= (2u+\theta)a_1^\theta \\ a_4^\theta &= a_{2t_0}^\theta - j a_2^\theta & \qquad a_{2t_0t_0}^\theta &= (2u+\theta)a_2^\theta. \end{aligned} \tag{101}$$

Starting from the transformation Eq.(95),

$$\begin{aligned} U &\mapsto \sqrt{\frac{\lambda-\theta}{\lambda}} \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} U \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} 0 & (\lambda-\theta)^{-1} \\ 1 & 0 \end{pmatrix} \\ &= \sqrt{\frac{\lambda-\theta}{\lambda}} \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} S^{-1} \begin{pmatrix} \frac{\beta}{\lambda-\theta} & \frac{1}{\lambda-\theta} \\ 1+\frac{T^\theta\beta}{\lambda-\theta} & \frac{T^\theta}{\lambda-\theta} \end{pmatrix} \cdot \\ &\qquad \begin{pmatrix} -T^\theta & 1 \\ \lambda-\theta+T^\theta\beta & -\beta \end{pmatrix} Y \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} 0 & (\lambda-\theta)^{-1} \\ 1 & 0 \end{pmatrix} \end{aligned} \tag{102}$$

where in the last line I have indicated the Birkhoff factorization (assuming $|\theta| < 1$), and

$$T^\theta = \frac{pa_3^\theta + ra_4^\theta}{pa_1^\theta + ra_2^\theta} = \frac{\tau_{t_0}^\theta}{\tau^\theta} - j, \qquad \tau^\theta = pa_1^\theta + ra_2^\theta. \tag{103}$$

Computing the $O(\lambda^{-1})$ terms in the transformed S gives the Bäcklund transformation

$$j \mapsto j - \frac{\tau_{t_0}^\theta}{\tau^\theta} \qquad u \mapsto u - \left(\frac{\tau_{t_0}^\theta}{\tau^\theta}\right)_{t_0}, \tag{104}$$

where τ^θ satisfies $\tau_{t_0t_0}^\theta = (2u+\theta)\tau^\theta$, and (after a little further calculation)

$$\frac{\tau_{t_1}^\theta}{\tau_{t_0}^\theta} = \frac{3\theta}{2} + \frac{1}{4}\left(\frac{\tau_{t_0t_0t_0}^\theta}{\tau_{t_0}^\theta} - \frac{3\tau_{t_0t_0}^\theta}{\tau^\theta}\right). \tag{105}$$

$$i\begin{pmatrix} q_2 & p_2 \\ s_2 & r_2 \end{pmatrix} \begin{pmatrix} 0 & (\lambda-\theta_1)^{-1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & (\lambda-\theta_2)^{-1} \\ 1 & 0 \end{pmatrix} (-i) \begin{pmatrix} q_1(\lambda-\theta_1) & p_1 \\ s_1 & 0 \end{pmatrix}.$$

It is worth reiterating the algebraic framework that has been hereby described for Bäcklund transformations. There is a map between cosets of a certain group and solutions of KdV. The action of certain automorphisms of the group does not descend to a well-defined action on the relevant coset space; rather, a single coset finds itself mapped to a family of cosets. This "explains" the appearance of constants of integration in the implementation of BTs. I expect this underlying algebraic idea to be behind many, if not all, other BTs that involve the solution of differential equations.

These comments are relevant to the discussion of BTs of a single parameter value. Another issue to consider is the following: applying BTs of two different parameter values θ_1, θ_2 to a solution gives two families of solutions. Is there a map between the families? The natural guess for such a map would be the map $B(\theta_1)B(\theta_2)^{-1}$ on G, if it descended to the double coset space, which it does not This map has already been given explicitly in Eq. (96) (since $B(\theta)^{-1} = B(\theta)$); it is the right multiplication of G by a certain element of G, and I will consider such transformations, including this one, in Sec.5.5.

Finally in this section, since I have now identified BTs at the level of the loop group, I can write down the loop group elements corresponding to single soliton solutions. Applying the BT $\mathcal{O}_\theta$ to the identity in the loop group gives the loop group elements

$$U_0 = \begin{pmatrix} s\sqrt{\frac{\lambda-\theta}{\lambda}} & \frac{r}{\sqrt{\lambda(\lambda-\theta)}} \\ q\sqrt{\lambda(\lambda-\theta)} & p\sqrt{\frac{\lambda}{\lambda-\theta}} \end{pmatrix}. \tag{106}$$

These are loop group elements only for $|\theta| < 1$; they give the standard soliton solutions

$$\begin{aligned} j &= -t_{-1} - \sqrt{\theta}\tanh(z\sqrt{\theta} + A) \\ u &= -\theta\,\mathrm{sech}^2(z\sqrt{\theta} + A), \end{aligned} \tag{107}$$

where

$$z = \sum_{n=0}^{\infty} t_n \theta^n \qquad\qquad e^A = \sqrt{\frac{p\sqrt{\theta}+r}{p\sqrt{\theta}-r}}. \tag{108}$$

I have assumed that $p\sqrt{\theta} \neq \pm r$; if $p\sqrt{\theta} = \pm r$ the trivial solutions $j = -t_{-1} \mp \sqrt{\theta}$, $u = 0$ emerge. Note that only soliton solutions with $|\theta| < 1$ arise. This is in accord with the results of [19].

5.4 Galas Bäcklund Transformations

Galas BTs are, in their algebraic structure, very similar to standard BTs. The starting point is the map $C : G \to G$ defined by

$$U_0 \mapsto U_0 \begin{pmatrix} 1 & 0 \\ \mu/\lambda & 1 \end{pmatrix} \tag{109}$$

where μ is a nonzero parameter. This descends to $H\backslash G$ but not to the double coset space, since for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G^+$,

$$U_0 \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \mu/\lambda & 1 \end{pmatrix} = U_0 \begin{pmatrix} 1 & 0 \\ \mu/\lambda & 1 \end{pmatrix} \begin{pmatrix} a + \frac{b\mu}{\lambda} & b \\ c + \frac{(d-a)\mu}{\lambda} - \frac{b\mu^2}{\lambda^2} & d - \frac{b\mu}{\lambda} \end{pmatrix}. \qquad (110)$$

It emerges, though, from this, that C does descend to $H\backslash G/K$, where K is the codimension two subgroup of G^+ consisting of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with b having a zero at $\lambda = 0$, and the constant term in the power series expansion of $d - a - b/\lambda$ also vanishing (a simple calculation is required to check these properties indeed define a group). Furthermore, $G^{+(2)} \subset K \subset G^+$, where $G^{+(2)}$ is the normal subgroup of G^+ consisting of matrices which have constant term the identity, and no linear term. Thus, defining the map $\mathcal{P}_0$ via

$$U_0 \mapsto U_0 \begin{pmatrix} p & q \\ r & s \end{pmatrix} \frac{1}{\sqrt{1 - \lambda^2(P^2 + QR)}} \left(I + \lambda \begin{pmatrix} P & Q \\ R & -P \end{pmatrix} \right) \begin{pmatrix} 1 & 0 \\ \mu/\lambda & 1 \end{pmatrix}, \qquad (111)$$

where here $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ is an arbitrary $SL(2, \mathbf{C})$ matrix and $\begin{pmatrix} P & Q \\ R & -P \end{pmatrix}$ is an arbitrary $sl(2, \mathbf{C})$ matrix, gives a map that will descend to the double coset space: I have inserted after U_0 a group element from each class in $G^+/G^{+(2)}$. It should be noted that there is no natural isomorphism between $G^+/G^{+(2)}$ and a subgroup of G^+.

Note that

$$\begin{pmatrix} 1 & 0 \\ \mu/\lambda & 1 \end{pmatrix} = \begin{pmatrix} \mu^{-\frac{1}{2}} & 0 \\ 0 & \mu^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/\lambda & 1 \end{pmatrix} \begin{pmatrix} \mu^{\frac{1}{2}} & 0 \\ 0 & \mu^{-\frac{1}{2}} \end{pmatrix}. \qquad (112)$$

Since the first and third of the matrices on the RHS are both constant, and right multiplication of G by any constant matrix descends to the identity on G/G^+, it follows that, without any loss of generality I can set $\mu = 1$, which I do from here on.

Before computing the effect of $\mathcal{P}_0$ on a KdV solution, I enlarge to a one parameter family of maps in the same way as in Sec. 5.3, defining

$$\begin{aligned} C(\theta) &= A(-\theta)\ B\ A(\theta) && (113) \\ C(\theta)U_0 &= U_0 \begin{pmatrix} 1 & 0 \\ \frac{1}{\lambda-\theta} & 1 \end{pmatrix}, \\ \mathcal{P}_\theta &= A(-\theta)\ \mathcal{P}_0\ A(\theta) && (114) \\ \mathcal{P}_\theta U_0 &= U_0 \begin{pmatrix} p & q \\ r & s \end{pmatrix} \frac{1}{\sqrt{1 - (\lambda-\theta)^2(P^2 + QR)}} \\ & \quad \left(I + (\lambda - \theta) \begin{pmatrix} P & Q \\ R & -P \end{pmatrix} \right) \begin{pmatrix} 1 & 0 \\ \frac{1}{\lambda-\theta} & 1 \end{pmatrix}. \end{aligned}$$

Note that $C(\theta_1)C(\theta_2) = C(\theta_2)C(\theta_1)$, i.e. here the relevant maps on G are commutative. It is not immediately obvious that this implies $\mathcal{P}_{\theta_1}\mathcal{P}_{\theta_2} = \mathcal{P}_{\theta_2}\mathcal{P}_{\theta_1}$; the simplest way to be satisfied that this is indeed the case is to check that if the matrix $\mathcal{M}$ is defined by

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{\lambda-\theta_1} & 1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{\lambda-\theta_2} & 1 \end{pmatrix} = \mathcal{M} \begin{pmatrix} 1 & 0 \\ \frac{1}{\lambda-\theta_2} & 1 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{\lambda-\theta_1} & 1 \end{pmatrix}, \tag{115}$$

with $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in G^+$, then $\mathcal{M} \in G^+$.

To obtain the effect of $\mathcal{P}_\theta$ on a KdV solution, observe that under $\mathcal{P}_\theta$

$$\begin{aligned} U \;\mapsto\;& U \begin{pmatrix} p & q \\ r & s \end{pmatrix} \\ & \frac{1}{\sqrt{1-(\lambda-\theta)^2(P^2+QR)}} \left(I + (\lambda-\theta)\begin{pmatrix} P & Q \\ R & -P \end{pmatrix}\right) \begin{pmatrix} 1 & 0 \\ \frac{1}{\lambda-\theta} & 1 \end{pmatrix} \\ =\;& S^{-1}\left(I - \frac{N}{\lambda-\theta}\right)\left(I + \frac{N}{\lambda-\theta}\right) Y \begin{pmatrix} p & q \\ r & s \end{pmatrix} \\ & \frac{1}{\sqrt{1-(\lambda-\theta)^2(P^2+QR)}} \left(I + (\lambda-\theta)\begin{pmatrix} P & Q \\ R & -P \end{pmatrix}\right) \begin{pmatrix} 1 & 0 \\ \frac{1}{\lambda-\theta} & 1 \end{pmatrix}, \end{aligned} \tag{116}$$

where in the last line I have inserted a certain factor and its inverse, N being a nonzero matrix independent of λ with $N^2 = 0$. A tedious calculation shows that it is possible to choose such a matrix N so that the above is written in Birkhoff factorized form. To write it requires a little more notation. Referring back to Eqs. (99)-(100), I write

$$(Y_0^\theta)^{-1} Y_1^\theta = \begin{pmatrix} b_1^\theta & b_2^\theta \\ b_3^\theta & -b_1^\theta \end{pmatrix}. \tag{117}$$

Then correct choice of N is given by

$$\begin{aligned} N \;=\;& \frac{1}{1+Q+2sqb_1^\theta + s^2 b_2^\theta - q^2 b_3^\theta} \\ & \begin{pmatrix} -(qa_1^\theta + sa_2^\theta)(qa_3^\theta + sa_4^\theta) & (qa_1^\theta + sa_2^\theta)^2 \\ -(qa_3^\theta + sa_4^\theta)^2 & (qa_1^\theta + sa_2^\theta)(qa_3^\theta + sa_4^\theta) \end{pmatrix}. \end{aligned} \tag{118}$$

Using

$$S \mapsto \left(1 + \frac{N}{\lambda-\theta}\right) S \tag{119}$$

and formulae for the t_0 and t_1 derivatives of the b_i^θ which are easily computed, produces, after some labor, the Galas BT

$$u \mapsto u - \left(\frac{\tau_{t_0}}{\tau}\right)_{t_0} \qquad\qquad j \mapsto j - \frac{\tau_{t_0}}{\tau} \tag{120}$$

where $\tau = 1 + Q + 2sqb_1^\theta + s^2 b_2^\theta - q^2 b_3^\theta$ is related to u by $(2u+\theta) = (\sqrt{\tau_{t_0}})_{t_0 t_0}/\sqrt{\tau_{t_0}}$, and

$$\frac{\tau_{t_1}}{\tau_{t_0}} = \frac{3\theta}{2} + \frac{1}{4}\left(\frac{\tau_{t_0 t_0 t_0}}{\tau_{t_0}} - \frac{3\tau_{t_0 t_0}^2}{2\tau_{t_0}^2}\right) \tag{121}$$

(c.f. Eqs. (3)-(4) above). This is, of course, a BT of the entire hierarchy; further t_i derivatives of τ can be computed as desired.

One difference between the Galas BT and standard BTs is that since

$$\begin{pmatrix} 1 & 0 \\ 1/(\lambda-\theta) & 1 \end{pmatrix} = \exp\begin{pmatrix} 0 & 0 \\ 1/(\lambda-\theta) & 0 \end{pmatrix}, \tag{122}$$

an infinitesimal generator can be written for Galas BTs, as discovered by Galas [6]. Galas BTs do not, on the other hand, have the involutiveness property of standard BTs.

It just remains to make some further comments about soliton solutions. As mentioned in Sec. 1.4, Galas BTs can be used to obtain standard soliton solutions, as well as more general solutions. Thus, as in Sec. 5.3, the results of this section give a way to find matrices U_0 corresponding to soliton solutions. Following through the necessary calculations, it emerges that the matrix

$$U_0 = \begin{pmatrix} p & q \\ p\sqrt{\theta} - \frac{1}{q} & q\sqrt{\theta} \end{pmatrix} \begin{pmatrix} 1 & (\lambda-\theta)Q \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{\lambda-\theta} & 1 \end{pmatrix} \qquad q \neq 0 \tag{123}$$

gives the soliton solution

$$\begin{aligned} j &= -t_{-1} - \sqrt{\theta}\ \left(1 + \tanh(z\sqrt{\theta} + A)\right) \\ u &= -\theta\ \mathrm{sech}^2(z\sqrt{\theta} + A), \end{aligned} \tag{124}$$

(c.f. Eq. (107)), with z defined as in Eq. (108) and A determined by

$$e^{-2A} = 1 + \frac{2(1+Q)}{q^2}. \tag{125}$$

I have not, as of yet, succeeded in determining whether this matrix U_0 is (up to a translation in t_{-1}) a representative of the same double coset as the matrix U_0 given in Eq.(106) (after appropriate adjustment of parameters). This begs the general question of whether the map from $H\backslash G/G^+$ to solutions of KdV is 1-1, which is also not considered here. The matrix U_0 given in Eq. (123) is in G for all θ with $|\theta| \neq 1$, and for $|\theta| > 1$ it is in G^+; in performing the Birkhoff decomposition to obtain the soliton solution I have assumed $|\theta| < 1$.

5.5 Zakharov-Shabat Dressing Transformations

I now consider general right multiplications on G, i.e. transformations of the form

$$U_0 \mapsto U_0 \cdot g \qquad\qquad g \in G. \tag{126}$$

Galas BTs are of this form, and indeed all "symmetry actions" can be rewritten in this form (e.g. a left multiplication $U_0 \mapsto g \cdot U_0$ can be rewritten $U_0 \mapsto U_0 \cdot (U_0^{-1} g U_0)$), but here I only want to consider field independent transformations, i.e. the case where the matrix g is not dependent on U_0. Now, such transformations evidently descend to $H \backslash G$, but in general will not descend to the double coset space. But let us for the moment ignore this. Under such a transformation $U \mapsto U \cdot g = S^{-1} \cdot Y \cdot g = S^{-1} \cdot (YgY^{-1}) \cdot Y$, so writing the Birkhoff decomposition for YgY^{-1} in the form $YgY^{-1} = (YgY^{-1})_-^{-1}(YgY^{-1})_+$ gives the result $Y \mapsto (YgY^{-1})_+ Y = (YgY^{-1})_- Yg$. This is almost exactly the formula for dressing transformations given by Wilson [22]; Wilson's formula actually reads $Y \mapsto (YgY^{-1})_-^{-1} Y$, and differs from the one given here by right multiplication by a factor of g, which does not contribute to $dY \cdot Y^{-1}$. Thus the transformations being considered in this section are Zakharov-Shabat dressing transformations. As explained in Sec. 1.6, these transformations are only well-defined on Y; given a solution of the KdV hierarchy, there are many ways to construct an appropriate Y, and dressing transformations do not preserve equivalence classes of Y's. This is precisely the issue that right multiplications do not in general descend to $H \backslash G / G^+$.

In the case of Galas BTs I have shown how to use a right multiplication that does not descend to $H \backslash G / G^+$ to obtain a BT of the hierarchy, a map that takes single solutions to a finite dimensional family of solutions. It is clear that the procedure followed can be replicated whenever the right multiplication descends to a double coset space $H \backslash G / K$ where now K is any subgroup of finite codimension in G^+; in particular this can be done whenever g is a rational element of G, i.e. has entries which are rational functions of λ, with no singularities on $|\lambda| = 1$. It would be interesting to find a set of generators for the group of rational loops and to determine the associated BTs; though BTs can, of course, also arise from transformations other than right multiplications.

The problem that dressing transformations suffer from, viz. the fact that they cannot be really considered as transformations on the space of solutions of the hierarchy, can be resolved by specifying a choice of Y or U_0 corresponding to a solution of the hierarchy. For the case of the trivial solution of the hierarchy it is natural to choose U_0 to be the identity matrix, giving (at least for $t_{-1} = 0$) $Y = M$. This choice was made by Wilson [22], and it allowed him to define the orbit of the trivial solution of KdV under dressing transformations.

An important context in which dressing transformations arise is understanding the relationship between modified KdV (MKdV) flows and Liouville and sine-Gordon flows. So far I have not constructed MKdV flows in this paper. In fact, associated with any solution of Eq. (35), the fundamental linear equation underlying the KdV system, there are two MKdV fields, given, in the notation of Eq. (48), by

$$v_1 = \frac{a_{1t_0}}{a_1} \qquad \text{and} \qquad v_2 = \frac{a_{2t_0}}{a_2}. \tag{127}$$

From Eq. (49), these are related to the KdV field u by the so-called Miura map

$$2u = v_{1t_0} + v_1^2 = v_{2t_0} + v_2^2, \tag{128}$$

and, from Eq. (50), they satisfy the MKdV equation

$$\begin{aligned} v_{1t_1} &= \tfrac{1}{4} v_{1t_0t_0t_0} - \tfrac{3}{2} v_1^2 v_{1t_0} \\ v_{2t_1} &= \tfrac{1}{4} v_{2t_0t_0t_0} - \tfrac{3}{2} v_2^2 v_{2t_0}. \end{aligned} \tag{129}$$

Indeed v_1, v_2 satisfy the full MKdV hierarchy. Let us now introduce two new flows for U, given by

$$\partial_{s_1} U = U \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{130}$$

$$\partial_{s_2} U = U \begin{pmatrix} 0 & 0 \\ \lambda^{-1} & 0 \end{pmatrix}. \tag{131}$$

The first of these flows corresponds to an infinitesimal dressing transformation by a constant matrix, and hence leaves the KdV field u invariant. It is also straightforward to check that this flow leaves the field a_1 invariant, and hence also v_1, but induces a nontrivial flow on v_2 given by

$$v_{2s_1} = -\frac{1}{a_2^2}. \tag{132}$$

Since $v_2 = a_{2t_0}/a_2$, by introducing $h = \log a_2$, this takes the form of the Liouville equation $h_{t_0 s_1} = -e^{-2h}$. The flow in Eq. (131), is an infinitesimal Galas transformation, and does not leave u invariant. The effect on a_1, a_2 can be computed — with some effort — from the Birkhoff decomposition Eq. (116), and the relevant fact for the current discussion is that

$$v_{2s_2} = a_2^2, \tag{133}$$

also a Liouville equation. A suitable linear combination of the s_1 and s_2 flows gives the sine-Gordon (or, more properly, the sinh-Gordon) equation. Both the s_1 and s_2 evolutions for v_2 commute with all the MKdV flows; this is obvious because the MKdV flows are implemented by left multiplications on U_0, and the s_1 and s_2 flows by right multiplication. However, the s_1 and s_2 flows do not commute with each other.

The MKdV equations, and Liouville and sine-Gordon flows will not be discussed in full detail here, but a few more notes are in order. First, that both s_1 and s_2 flows for v_2 commute with the MKdV flow is a symptom of the symmetry of the MKdV hierarchy under the involution $v_2 \mapsto -v_2$, which can be shown to be an effect of the involution B used in the discussion of Wahlquist-Estabrook BTs. Second, the fact that the MKdV fields v_1, v_2 are really "derived" quantities, and the centrality of the fields a_1, a_2 has been stressed in [23] (see also [13]). Finally, a detailed studied of BTs of the sine-Gordon system has been given in [20], and there is much in common between the results of [20] and the current work.

Finally in this section, I reconsider briefly the dressing transformation introduced in Sec. 5.3 as the product of two Wahlquist-Estabrook BTs, Eq. (96). The remarkable fact about this dressing transformation is that despite the fact that it is not multiplication by a rational element of G, it still has an interpretation as a BT. For completeness I write down here the infinitesimal version of this transformation (taking $\theta_2 = 0$ and $\theta_1 = 2\epsilon$, where ϵ is an infinitesimal parameter). This is then the dressing transformation

$$U_0 \mapsto U_0 \begin{pmatrix} 1+\epsilon/\lambda & 0 \\ 0 & 1-\epsilon/\lambda \end{pmatrix}. \tag{134}$$

Following through the necessary calculations, it an be shown that this induces the infinitesimal BT

$$j \mapsto j - 2\epsilon(pa_1 + ra_2)(qa_1 + sa_2) \tag{135}$$

$$u \mapsto u - 2\epsilon\left((pa_1 + ra_2)(qa_1 + sa_2)\right)_{t_0}, \tag{136}$$

where, as usual, $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ is an arbitrary $SL(2, \mathbf{C})$ matrix, and a_1, a_2 are as above. Since the variation of j and the determinant condition $ps - rq = 1$ are invariant under the rescalings $p \mapsto fp,\ r \mapsto fr,\ q \mapsto f^{-1}q, s \mapsto f^{-1}s,\ f \in \mathbf{C}^*$, this infinitesimal map generates a two parameter family of solutions from a given one, which is consistent with its origins in Sec. 5.3. Looking at this formula alone, there is little reason to suspect that this infinitesimal symmetry can be exponentiated to a finite BT.

6 Open Problems

Some open problems have already been mentioned in the text, and will not be repeated. The central open issue that I perceive is that although in this paper a reasonable unified framework has been built for understanding symmetries of KdV, this has been in the context of a restricted set of solutions. It is quite possible that there is a simple modification of the formalism presented here that will allow consideration of much more general spaces of solutions (c.f. [14]).

Another issue that has not been tackled in this paper is that I have not discussed the Hamiltonian structures or the conserved quantities of the KdV hierarchy. There is, of course, in general, a connection between symmetries and conserved quantities, and it is of considerable interest to understand what the conserved quantities (or appropriate generalizations thereof) associated to BTs are. For that matter, while the conserved quantities associated with the translation symmetries of KdV are of course well known, I am unaware of whether the conserved quantities associated to Galilean or scaling symmetries have been discussed.

It is to be hoped that the work presented here will be continued in several directions. The study of KdV presented here is based on two hypotheses, Mulase's belief that beneath each integrable system lies a genuinely simple linear system, like Eq. (35), and the idea that the most logically satisfactory explanation of BTs is that they arise from simple actions on the space of initial data of the relevant simple linear system. These ideas need to be explored for other integrable systems. Initial studies for both the KP and Principal Chiral Model hierarchies [18] confirm this general picture. Far from being an esoteric study of the structure of integrable systems, this program holds promise of the discovery of new BTs of integrable systems, and, through this, an expansion of the repertoire of known solutions. Indeed, I originally found the Galas BT for $\theta = 0$ via the considerations of Sec. 5.4, before finding Galas' paper [6], and there seems no reason why the Galas BT should not occupy just as significant a place in the textbooks on soliton equations as the Wahlquist-Estabrook BT.

From this paper also emerges the need for further studies and classifications of the solutions of KdV, particularly a deeper exploration of the space of solutions that has been considered here, and reconciliation with the zoo of solutions available in the literature [1].

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Appendix

I complete here the proof of the theorem from Sec. 3. Eqs. (24) clearly imply, for $m \geq 0$, $n \geq -1$, that

$$\frac{\partial (Z_m - \lambda Z_{m-1})}{\partial t_n} - \left(\frac{\partial}{\partial t_m} - \lambda \frac{\partial}{\partial t_{m-1}} \right) Z_n + [Z_m - \lambda Z_{m-1}, Z_n] = 0. \tag{137}$$

Using $Z_m - \lambda Z_{m-1} = \Delta_m$ and $Z_n = \sum_{i=0}^{n+1} \Delta_{n-i} \lambda^i$ (where, following the convention of Sec. 4, $\Delta_{-1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$), and equating powers of λ, Eq. (137) gives the following equations:

$$\frac{\partial \Delta_0}{\partial t_{m-1}} + [\Delta_m, \Delta_{-1}] \quad = \quad 0 \qquad\qquad m \geq 0 \tag{138}$$

$$\frac{\partial\Delta_{n+1}}{\partial t_{m-1}} - \frac{\partial\Delta_n}{\partial t_m} + [\Delta_m, \Delta_n] \quad = \quad 0 \qquad\qquad m, n \geq 0 \tag{139}$$

$$\frac{\partial\Delta_m}{\partial t_n} - \frac{\partial\Delta_n}{\partial t_m} + [\Delta_m, \Delta_n] \quad = \quad 0 \qquad\qquad m \geq 0,\ n \geq -1 \tag{140}$$

The first set of these reproduces Eq. (26). Note the other two sets imply the interesting result

$$\frac{\partial\Delta_{n+1}}{\partial t_{m-1}} = \frac{\partial\Delta_m}{\partial t_n} \qquad\qquad m, n \geq 0. \tag{141}$$

Using Eqs. (27), the 1,2 entry of Eq. (139) or Eq. (140) gives

$$\left(\frac{\partial}{\partial t_m}\frac{\partial}{\partial t_{n-1}} - \frac{\partial}{\partial t_n}\frac{\partial}{\partial t_{m-1}}\right) j = \frac{\partial u}{\partial t_{m-1}}\frac{\partial j}{\partial t_{n-1}} - \frac{\partial u}{\partial t_{n-1}}\frac{\partial j}{\partial t_{m-1}} \tag{142}$$

The situation faced at the end of Sec. 3 was that given the correct t_{n-1} flow for j, u $(n \geq 1)$, the correct t_n-flow for u could be deduced, but this was not sufficient to uniquely determine the correct t_n-flow for j, and instead gave us

$$\frac{\partial j}{\partial t_n} = (\tfrac{1}{4}\partial_{t_0}^2 - \partial_{t_0}^{-1} u \partial_{t_0} - u)\frac{\partial j}{\partial t_{n-1}} + c_n(t_1, t_2, \ldots). \tag{143}$$

(Here I have integrated both sides of Eq. (33). The integration operator $\partial_{t_0}^{-1}$ can be precisely defined, see [16].) Differentiating with respect to t_{m-1}, Eq. (143) gives

$$\frac{\partial^2 j}{\partial t_n \partial t_{m-1}} = -\frac{\partial u}{\partial t_{m-1}}\frac{\partial j}{\partial t_{n-1}} + \frac{\partial c_n}{\partial t_{m-1}} + \begin{pmatrix}\text{terms symmetric}\\ \text{in } \ m, n\end{pmatrix}. \tag{144}$$

This, with Eq. (142), implies

$$\frac{\partial c_n}{\partial t_{m-1}} = \frac{\partial c_m}{\partial t_{n-1}} \qquad\qquad m \geq 0,\ n \geq 1. \tag{145}$$

As an application of this, set $n = 1$ to obtain that c_1 is constant (and therefore, by an admissible change of coordinates, can be set to zero). However, it is clear that Eq. (145) by itself is not enough to eliminate the t_n dependence from all the c_n's (it admits solutions of the form $c_n = \partial P/\partial t_{n-1}$ for appropriate functions P). To do this, it is necessary to look at further entries of Eqs. (139)-(140); in particular looking the 2,1 entry of Eq. (139), and using

$$(\Delta_n)_{21} = -c_n + \left(\tfrac{1}{4}\partial_{t_0}^2 - \partial_{t_0} j + j^2 + \partial_{t_0}^{-1} u \partial_{t_0}\right)\frac{\partial j}{\partial_{t_{n-1}}}, \tag{146}$$

which follows from Eq.(31) and Eq.(143), gives, after some manipulations, the further requirement

$$\frac{\partial c_n}{\partial_{t_m}} = \frac{\partial c_m}{\partial_{t_n}}. \tag{147}$$

This with Eq. (145) is sufficient to imply that all the c_n's are constant.

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