IASSNS-HEP-92/66
LAVAL PHY-26-92
hepth@xxx/9210080

On The Hamiltonian Structures and The Reductions of The KP Hierarchy*

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Abstract: Recent work on a free field realization of the Hamiltonian structures of the classical KP hierarchy and of its flows is reviewed. It is shown that it corresponds to a reduction of KP to the NLS system.

Sep. 1992.

Talk given by D.A.D. at the NSERC-CAP Workshop on *Quantum Groups, Integrable Models and Statistical Systems,* Kingston, Canada July 13-17 1992.

0. Introduction

It is well-known that it is possible to reduce the KP hierarchy to any of the SL(N) KdV hierarchies. The KP hierarchy has a bi-Hamiltonian structure [1] and its second Hamiltonian structure has been shown [2] to be related to $\hat{W}_\infty$, a centerless, non-linear deformation of the W_∞ algebra of Pope, Romans and Shen [3]. This naturally leads to the conjecture that the W_N algebras of Zamolodchikov [4], which arise as the second Hamiltonian structure of the SL(N) KdV hierarchies, can be obtained via some sort of reduction of $\hat{W}_\infty$ [2]. While the evidence for this conjecture is convincing, an explicit proof is still lacking.

On the other hand, inspired by the fact that the *linear* algebra W_∞ possesses a representation in terms of two bosons [5], Yu and Wu [6] presented a two boson representation of $\hat{W}_\infty$. In [7], it was shown that this representation is related to the reduction of the KP hierarchy to a $1+1$ dimensional integrable hierarchy. In this talk, we recall some of the results of [6] and of [7], and show further that the integrable hierarchy of [7] is related to the well-known nonlinear Schrödinger (NLS) hierarchy. It seems that the fact that KP admits such a reduction has been known to specialists in integrable systems for some time [8][9].

Related results, albeit from a very different approach, were also presented by C.S.Xiong at this workshop [10].

We will be using differential $\partial \equiv \partial_x$ and pseudo-differential ∂^{-1} operators, where ∂^{-1} is an integration symbol satisfying $\partial\partial^{-1}a = \partial^{-1}\partial a = a$. We have $\partial^{-1}a = a\partial^{-1} - a'\partial^{-2} + a''\partial^{-3} - \ldots$ To shorten the expressions of the Poisson brackets, we write $\{a, b\} = c\delta'$ for $\{a(x), b(x')\} = c(x)\partial_x\delta(x, x')$.

1. The two boson representation of $\hat{W}_\infty$

The KP hierarchy

The KP hierarchy is usually introduced following the approach of the Japanese school [11]; specifically, the r^{th} flow of the KP hierarchy is given by the Lax equation

$$\frac{\partial}{\partial t_r} L = [(L^r)_+, L] \qquad , \qquad r \in N \tag{1.1}$$

where L is the pseudo-differential operator

$$L = \partial + u_0\partial^{-1} + u_1\partial^{-2} + \ldots = \partial + \sum_{i=0}^{\infty} u_i\partial^{-i-1} \quad , \quad \partial = \partial/\partial x \ . \tag{1.2}$$

and $(L^r)_+$ is the differential part of the r^{th} power of the KP operator L. The fields u_i are implicitly understood to depend on x and on an infinite number of time coordinates t_i. For instance, for $r=2$ we get $(L^2)_+ = \partial^2 + 2u_0$ and

$$\begin{aligned} r=2: \quad & u_{0,t_2} = 2u_{1,x} + u_{0,xx} \quad , \\ & u_{1,t_2} = 2u_{2,x} + u_{1,xx} + 2u_0 u_{0,x} \quad , \\ & u_{2,t_2} = 2u_{3,x} + u_{2,xx} + 4u_{0,x}u_1 - 2u_0 u_{0,xx} \quad , \\ & \vdots \end{aligned} \tag{1.3}$$

The flows defined by (1.1) commute ($\partial_{t_i}\partial_{t_j} u_k = \partial_{t_j}\partial_{t_i} u_k$) and furthermore they are *bi-Hamiltonian*, i.e. (1.1) can also be written as

$$\partial_{t_r} u_i = \{u_i, \int \mathcal{H}_{r+1}\}_1 = \{u_i, \int \mathcal{H}_r\}_2 \tag{1.4}$$

where $\{.,.\}_1$ and $\{.,.\}_2$ are two Poisson brackets for the fields u_i and the $\int \mathcal{H}_r$'s are some Hamiltonians. The brackets $\{.,.\}_1$ (historically the first set to be discovered [12]) were shown in [13] to correspond to the *linear* conformal algebra $W_{1+\infty}$ with $c=0$. The second Hamiltonian structure [1] corresponds to $\hat{W}_\infty$ [6]. The Hamiltonians are given explicitly by

$$\mathcal{H}_r = \frac{1}{r}\ res\ L^r \quad , \tag{1.5}$$

where $res\ L^r$ denotes the coefficient of ∂^{-1} in L^r.

We give the first commutators of these algebras, as we will need their expressions shortly. The relation between the u and W fields will be given later. For $W_{1+\infty}$ we have

$$\begin{aligned} & \{W_1, W_1\}_1 = 0 \quad , \quad \{W_2, W_1\}_1 = W_1\delta' \quad , \\ & \{W_2, W_2\}_1 = 2W_2\delta' + W_2'\delta \quad , \quad \{W_1, W_3\}_1 = 2W_2\delta' + 2W_2'\delta \quad , \end{aligned} \tag{1.6}$$

and for $\hat{W}_\infty$ we have

$$\begin{aligned} & \{W_2, W_2\}_2 = 2W_2\delta' + W_2\delta \quad , \quad \{W_2, W_3\}_1 = 3W_3\delta' + 2W_3'\delta \quad , \\ & \{W_2, W_4\}_2 = \frac{2}{3}W_2\delta''' + \frac{4}{3}W_2'\delta'' + (4W_4 + \frac{2}{3}W_2'')\delta' + 3W_4'\delta \quad , \\ & \{W_3, W_3\}_2 = \frac{1}{2}W_2\delta''' + \frac{3}{4}W_2'\delta'' + (4W_4 - \frac{1}{12}W_2'' - 2{W_2}^2)\delta' + (2W_4 - \frac{1}{6}W_2'' - {W_2}^2)'\delta \quad , \\ & \{W_4, W_4\}_2 = \ldots + (6W_6 + \frac{1}{3}W_4'' + \frac{1}{45}{W_2}^{(4)} - 6W_4W_2 - 6{W_3}^2 - 2{W_2}^3 - \frac{1}{2}{W_2'}^2)\delta' \\ & \qquad + (3W_6 - W_4'' + \frac{1}{15}{W_2}^{(4)} - 3W_4W_2 - 3{W_3}^2 - {W_2}^3 \\ & \qquad\quad + \frac{1}{2}(W_2^2)'' - \frac{1}{4}{W_2'}^2)'\delta \end{aligned} \tag{1.7}$$

Towards a representation of $\hat{W}_\infty$

The algebra W_∞ was found to possess a free field representation in terms of two real bosons [5], so in [6] Yu and Wu proposed the existence of a similar representation of $\hat{W}_\infty$. To this end, they introduced the currents $\jmath(x)$ and $\bar{\jmath}(x)$ with the natural Poisson brackets

$$\{\jmath(x),\jmath(x')\}_2 = 0 \ , \ \{\jmath(x),\bar{\jmath}(x')\}_2 = \delta'(x,x') \ , \ \{\bar{\jmath}(x),\bar{\jmath}(x')\}_2 = 0 \ . \tag{1.8}$$

Upon imposing

$$L = \partial + \sum_{i=0}^{\infty} u_i \partial^{-i-1} = \partial + \bar{\jmath}\frac{1}{\partial - (\jmath + \bar{\jmath})}\jmath \quad , \tag{1.9}$$

we get an expression for the fields u_i in terms of the currents, as

$$u_i = \bar{\jmath}[(-\partial + \jmath + \bar{\jmath})^i \ \jmath] \quad . \tag{1.10}$$

If (1.10) were to provide a *faithful* representation of $\hat{W}_\infty$, one could just compute the $\{u_i, u_j\}$ commutators using their expression in (1.10) and the Poisson brackets (1.8). However, more careful examination [14] reveals that this is not the case. It is easy to check that (1.10) implies relations or constraints between the fields u_i; the simplest such relation is

$$u_2 u_0 - u_1^2 + u_1' u_0 - u_1 u_0' = 0 \quad . \tag{1.11}$$

Thus it is not possible to unambiguously translate an expression in $\jmath\ \bar{\jmath}$ into an expression in the fields u_i. The upshot of this is that the two boson representation is related to a *reduction*, as opposed to a *realization* of the KP hierarchy.

Let us show this more explicitly. The map between the fields u_i and W_i was derived in [7] and gives, for the $\hat{W}_\infty$ case

$$\begin{aligned} &W_2 = u_0 \quad , \quad W_3 = u_1 + \frac{1}{2}u_0' \quad , \quad W_4 = u_2 + u_1' + \frac{1}{3}u_0'' + u_0^2 \quad , \\ &W_5 = u_3 + \frac{3}{2}u_2' + u_1'' + \frac{1}{4}u_0''' + 3u_0 u_1 + \frac{3}{2}u_0 u_0' \quad . \end{aligned} \tag{1.12}$$

(note the correction of a typographical error in the expression for W_4 in [7]). In terms of $\jmath$ and $\bar{\jmath}$, we find

$$\begin{aligned} W_2 &= \jmath\bar{\jmath} \quad , \quad W_3 = \frac{1}{2}(\jmath\bar{\jmath}' - \jmath'\bar{\jmath}) + \jmath\bar{\jmath}^2 + \jmath^2\bar{\jmath} \quad , \\ W_4 &= \frac{1}{3}(\jmath\bar{\jmath}'' - \jmath'\bar{\jmath}' + \jmath''\bar{\jmath}) - \jmath\jmath'\bar{\jmath} + \jmath\bar{\jmath}\bar{\jmath}' - \jmath'\bar{\jmath}^2 + \jmath^2\bar{\jmath}' + \jmath^3\bar{\jmath} + 3\jmath^2\bar{\jmath}^2 + \jmath\bar{\jmath}^3 \quad , \\ W_5 &= \frac{1}{4}(\jmath\bar{\jmath}''' - \jmath'\bar{\jmath}'' + \jmath''\bar{\jmath}' - \jmath'''\bar{\jmath}) + \frac{1}{2}(2\jmath^2\bar{\jmath}'' + 2\jmath''\bar{\jmath}^2 + 3\jmath\jmath''\bar{\jmath} + 3\jmath\bar{\jmath}\bar{\jmath}'' \\ &\quad + \jmath'^2\bar{\jmath} + \jmath\bar{\jmath}'^2 - \jmath\jmath'\bar{\jmath}' - \jmath'\bar{\jmath}\bar{\jmath}' + 3\jmath^3\bar{\jmath}' - 3\jmath'\bar{\jmath}^3 - 3\jmath^2\jmath'\bar{\jmath} + 3\jmath\bar{\jmath}^2\bar{\jmath}' \\ &\quad - 9\jmath\jmath'\bar{\jmath}^2 + 9\jmath^2\bar{\jmath}\bar{\jmath}') + \jmath^4\bar{\jmath} + 6\jmath^3\bar{\jmath}^2 + 6\jmath^2\bar{\jmath}^3 + \jmath\bar{\jmath}^4 \quad , \end{aligned} \tag{1.13}$$

so that under the interchange $\jmath \to -\bar{\jmath}$, $\bar{\jmath} \to -\jmath$, we see that $W_n \to (-)^n W_n$. Such a symmetry was already present for the bosonic representation of the *linear* W_∞, [5]. We note that the linear part (in terms of u_i fields) of the field redefinitions (1.12) are given by a formula analogous to (1.10), namely

$$\begin{aligned} W_{n+2}^{lin} &= \{u_0, u_1 + \frac{1}{2}u_0', u_2 + u_1' + \frac{1}{3}u_0'', u_3 + \frac{3}{2}u_2' + u_1'' + \frac{1}{4}u_0''', \ldots\} \\ &= \frac{1}{n+1}\sum_{m=0}^{n}[(-\partial + \jmath + \bar{\jmath})^m \jmath][(\partial + \jmath + \bar{\jmath})^{n-m}\bar{\jmath}] \quad . \end{aligned} \tag{1.14}$$

Given the relations (1.12), (1.11) becomes a constraint on the W fields,

$$W_4 W_2 = W_2^3 + W_3^2 - \frac{1}{4}W_2' W_2' + \frac{1}{3}W_2 W_2'' \quad . \tag{1.15}$$

Note that the $\{W_4, W_4\}$ commutator in (1.7) involves a $W_4 W_2$ term, just like (1.15).

2. The $\jmath$, $\bar{\jmath}$ hierarchy

In [7] a reduction similar to the one given in the last section was considered at the level of the KP flows themselves. The second Hamiltonian structure of this $\jmath$, $\bar{\jmath}$ hierarchy is clearly given by (1.8), so we can immediately write down the flows as

$$\vec{\jmath}_{,t_r} = \begin{pmatrix} \jmath \\ \bar{\jmath} \end{pmatrix}_{t_r} = P_2 \nabla_{\vec{\jmath}} \int \mathcal{H}_r \quad , \tag{2.1}$$

where $\nabla_{\vec{\jmath}} = (\delta/\delta\jmath, \delta/\delta\bar{\jmath})$ and P_2 is the Hamiltonian structure corresponding to (1.8),

$$P_2 = \begin{array}{c} \\ \jmath \\ \bar{\jmath} \end{array} \begin{array}{c} \begin{array}{cc} \jmath & \bar{\jmath} \end{array} \\ \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \end{array} \quad , \tag{2.2}$$

and $\mathcal{H}_r$ is obtained by taking the expression for the KP Hamiltonian (1.5) and writing it in terms of the $\jmath$, $\bar{\jmath}$ fields through (1.10).

Let us consider the second flow in more detail. It is written explicitly as

$$\begin{aligned} r = 2 \quad &: \mathcal{H}_2 = -\jmath'\bar{\jmath} + \jmath^2\bar{\jmath} + \jmath\bar{\jmath}^2 \quad , \\ &\jmath_{,t_2} = (-\jmath' + \jmath^2 + 2\jmath\bar{\jmath})' \quad , \\ &\bar{\jmath}_{,t_2} = (\ \bar{\jmath}' + \bar{\jmath}^2 + 2\jmath\bar{\jmath})' \quad . \end{aligned} \tag{2.3}$$

The $\jmath$, $\bar{\jmath}$ hierarchy turns out to be bi-Hamiltonian, with the first structure P_1 (corresponding to $W_{1+\infty}$) being non local, but the third Hamiltonian structure $P_3 = P_2 {P_1}^{-1} P_2$ being local. P_3 and P_1 are given explicitly in [7].

Let us try to manipulate (2.3) to see if it can be made to correspond to a known integrable system. We notice that upon setting $\bar{\jmath}$ to 0, (2.3) becomes the "derivative" of the Burgers equation, $h_t = h'' + 2hh'$, which can be linearized by the Cole–Hopf transformation $h = u'/u$ into the heat equation $u_t = u''$ [15]. Guided by this analogy, let us first "integrate" the flows (2.3) by introducing h and $\bar{h}$ defined by $h' = -\jmath$ and $\bar{h}' = \bar{\jmath}$. We get:

$$\begin{aligned} h_{,t_2} &= -h'' - h'^2 + 2h'\bar{h}' \quad , \\ \bar{h}_{,t_2} &= \quad \bar{h}'' + \bar{h}'^2 - 2h'\bar{h}' \quad . \end{aligned} \tag{2.4}$$

Using these equations, we find that $\psi = h' e^{h-\bar{h}}$ and $\bar{\psi} = \bar{h}' e^{\bar{h}-h}$ satisfy

$$\begin{aligned} \psi_{,t_2} &= -\psi'' + 2\psi^2\bar{\psi} \quad , \\ \bar{\psi}_{,t_2} &= \quad \bar{\psi}'' - 2\bar{\psi}^2\psi \quad , \end{aligned} \tag{2.5}$$

which is the second flow of the NLS system. A more careful treatment of the relation between our $\jmath$, $\bar{\jmath}$ system and the NLS one in standard form can be found in [16]. In fact it can be checked that the entire $\jmath$, $\bar{\jmath}$ hierarchy can be mapped in this manner to the NLS hierarchy. The simplest way to do this is to exploit the powerful concept of Fréchet derivatives (see for instance [15]) to map the different Hamiltonian structures of the $\jmath$, $\bar{\jmath}$ hierarchy to those of the NLS hierarchy. From $h = -\partial^{-1}\jmath$ and $\bar{h} = \partial^{-1}\bar{\jmath}$, we find that the Fréchet derivative of $(h, \bar{h})^T$ with respect to $(\jmath, \bar{\jmath})^T$ is

$$D = \begin{pmatrix} -\partial^{-1} & 0 \\ 0 & \partial^{-1} \end{pmatrix} \tag{2.6}$$

so that the Hamiltonian structure of the flows (2.4) is

$$DP_2D^\dagger = \begin{pmatrix} -\partial^{-1} & 0 \\ 0 & \partial^{-1} \end{pmatrix} \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \begin{pmatrix} \partial^{-1} & 0 \\ 0 & -\partial^{-1} \end{pmatrix} = \begin{pmatrix} 0 & \partial^{-1} \\ \partial^{-1} & 0 \end{pmatrix} \tag{2.7}$$

(Here we are assuming the possibility of defining ∂^{-1} as an anti-self-adjoint operator). The Fréchet derivative of $(\psi, \bar{\psi})^T$ with respect to $(h, \bar{h})^T$ is

$$\tilde{D} = \begin{pmatrix} e^{h-\bar{h}}(\partial + h') & -e^{h-\bar{h}}h' \\ -e^{\bar{h}-h}\bar{h}' & e^{\bar{h}-h}(\partial + \bar{h}') \end{pmatrix} \tag{2.8}$$

and using this we find

$$\tilde{D}DP_2D^\dagger\tilde{D}^\dagger = \begin{pmatrix} -2\psi\partial^{-1}\psi & -\partial + 2\psi\partial^{-1}\bar{\psi} \\ -\partial + 2\bar{\psi}\partial^{-1}\psi & -2\bar{\psi}\partial^{-1}\bar{\psi} \end{pmatrix} \tag{2.9}$$

This is the second Hamiltonian structure of the NLS system. Similarly one can show explcitly that the P_1 induces the first Hamiltonian structure of NLS.

We can formulate the connection between the KP and NLS hierarchies directly, without going through the $j, \bar{j}$ hierarchy. It is straightforward to show that the Lax operator (1.9) can be written

$$L = \partial + \bar{\jmath}(\partial - \jmath - \bar{\jmath})^{-1}\jmath = \partial - \psi\partial^{-1}\bar{\psi} \tag{2.10}$$

by simply using the map from the $\jmath$'s to the ψ's we just presented. So the reduction from KP to NLS is given by the constraint $L_{KP} = \partial - \psi\partial^{-1}\bar{\psi}$, or, equivalently, by constraining the KP fields as $u_i = (-)^{i+1}\psi\bar{\psi}^{(i)}$. A natural question is to ask what kind of hierarchies one gets by considering the reduction $L^n_{KP} = \partial^n + v\partial^{n-2} + \ldots + \psi\partial^{-1}\bar{\psi}$. This has been considered in some details in [8].

Finally the relation between the NLS hierarchy and the $\jmath\,, \bar{\jmath}$ system allows us to answer the question that was left open in [7], of finding a zero-curvature formulation of the $\jmath$, $\bar{\jmath}$ hierarchy, and of how to obtain it by reduction from the self–dual Yang–Mills system [17]. Since we use slightly different notation than in [7], we recall how such a reduction is introduced. The self–dual Yang–Mills equations in four dimensions are usually written as

$$F_{12} = F_{34} \quad , \quad F_{13} = F_{42} \quad , \quad F_{14} = F_{23} \quad , \tag{2.11}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. Introducing $w = x^1 + ix^2$, $z = x^3 - ix^4$ and their complex conjugates $\bar{w}$ and $\bar{z}$, eqs.(2.11) become

$$\begin{aligned} F_{\bar{z}\bar{w}} &= 0 \quad , \\ F_{z\bar{z}} + F_{w\bar{w}} &= 0 \quad , \\ F_{zw} &= 0 \quad , \end{aligned} \tag{2.12}$$

where now $F_{zw} = \partial_z A_w - \partial_w A_z + [A_z, A_w]$. The gauge freedom in (2.12) is expressed through the invariance of (2.12) under the transformations $A_\mu \to A'_\mu = gA_\mu g^{-1} - \partial_\mu g\, g^{-1}$. Now, we know that to get the NLS equations by reduction of the self–dual Yang–Mills system, we just need to choose

$$A_z = \begin{pmatrix} 0 & \psi \\ \bar{\psi} & 0 \end{pmatrix} \quad , \quad A_{\bar{w}} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad . \tag{2.13}$$

and reduce the system (2.12) first with respect to $\partial_{\bar{w}}$ and then $\partial_z - \partial_{\bar{z}}$ [17]. The potentials $A_{\bar{z}}$ and A_w are determined by (2.12). Here, since we know that the $\jmath\,, \bar{\jmath}$ system is related to the $\psi, \bar{\psi}$ system by the map given above, we consider the effect of a $\bar{w}$-independent gauge transformation on A_z which leaves $A_{\bar{w}}$ invariant. Under a transformation by

$$g = \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} \quad , \tag{2.14}$$

we get

$$A'_z = \begin{pmatrix} \alpha^{-1}\alpha_z & \psi\alpha^{-2} \\ \bar{\psi}\alpha^2 & -\alpha^{-1}\alpha_z \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}(\jmath + \bar{\jmath}) & -\jmath \\ \bar{\jmath} & \frac{1}{2}(\jmath + \bar{\jmath}) \end{pmatrix} \quad , \tag{2.15}$$

where we have taken $\alpha^2 = e^{h-\bar{h}}$.

If we take the new form of A_z from (2.15), and the form of $A_{\bar{w}}$ from (2.13) and plug into (2.12), imposing $\partial_{\bar{w}} = 0$ but *not* $\partial_z = \partial_{\bar{z}}$, we find

$$\begin{aligned} A_{\bar{z}} &= \beta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{with } \beta_z = -\frac{1}{2}(\jmath + \bar{\jmath})_{\bar{z}} \\ \begin{pmatrix} \jmath \\ \bar{\jmath} \end{pmatrix}_z &= \begin{pmatrix} -\partial + \jmath_z \partial^{-1} + (\jmath + \bar{\jmath}) & \jmath_z \partial^{-1} + 2\jmath \\ \bar{\jmath}_z \partial^{-1} + 2\bar{\jmath} & \partial + \bar{\jmath}_z \partial^{-1} + (\jmath + \bar{\jmath}) \end{pmatrix} \begin{pmatrix} \jmath \\ \bar{\jmath} \end{pmatrix}_{\bar{z}} \quad . \end{aligned} \tag{2.16}$$

The matrix operator in the second equation above is equal to $P_3 P_2^{-1}$ in [7]. The method of reduction used here, not imposing immediatly $\partial_z = \partial_{\bar{z}}$, gives us not only the equations of the hierarchy we are looking for, but also its recursion operator (of course, the $r = 2$ flow (2.3) is obtained after setting $\partial_z = \partial_{\bar{z}}$). The same applies to the reduction to the KdV equation, where one gets not only the usual KdV equation, but also its recursion operator.

3. Some open questions

The results presented here open up some potentially interesting avenues of research. The $\hat{W}_\infty$ algebra, which was originally introduced as an algebra that "contains" all the W_N algebras, apparently contains even more. It is clearly of some interest to try to find all reductions of the KP system and to understand the physical significance of the associated reductions of $\hat{W}_\infty$. Work on the algebras associated with the class of reductions of [8], generalizing the result here, is currently in progress.

Another question that arises is as follows: we know that the KdV equation arises by reduction (i.e. constraining) the KP hierarchy. On the other hand, we also know how to reduce the self-dual Yang-Mills system to the KdV equation [17]. In the work presented here, we see that the KP hierarchy can also be reduced to the NLS hierarchy, and again we know how to reduce the self–dual Yang–Mills equations to NLS. It is believed that self–dual Yang–Mills is a universal integrable system, and in particular KP can be obtained by reduction from it, for a suitable infinite dimensional gauge group. It would be interesting to understand better why certain integrable systems can appear both as direct reductions of self–dual Yang–Mills with a finite dimensional gauge group, and as indirect reductions via KP. Work on this is also in progress.

Some work related to that presented here can be found in [18].

Acknowledgements

We wish to thank Pierre Mathieu and Walter Oevel for discussions.

D.A.D was supported by NSERC (Canada), FCAR (Québec), and BSR (Université Laval), and J.S. was supported by a grant in aid from the U.S. Department of Energy, # DE-FG02-90ER40542.

References

[1] L.A. Dickey, *Soliton equations and Hamiltonian systems* , World Scientific, 1991.

[2] F. Yu and Y.-S. Wu, Nucl.Phys. **B373**(1992) 713; J.M. Figueroa-O'Farril, J. Mas and E. Ramos, Phys. Lett. **B266** (1991) 298.

[3] C.N. Pope, L.J. Romans and X.Shen, Phys.Lett. **B236** (1990) 173; Phys.Lett. **B242** (1990) 401; Phys.Lett. **B245** (1990) 72.

[4] A.B. Zamolodchikov, Theor. Math. Phys. **65** (1985) 1205; A.B.Zamolodchikov and V.A. Fateev, Nucl. Phys. **B280** [FS18] (1987) 644; V.A. Fateev and S.L. Lykyanov, Int. J. Mod. Phys. **A3** (1988) 507.

[5] I. Bakas and E. Kiritsis, Nucl. Phys. **B343** (1990) 185; Erratum *ibid.* **B350** (1991) 512.

[6] F. Yu and Y.-S. Wu, Phys.Rev.Lett. **68** (1992) 2996.

[7] D.A.Depireux, Mod. Phys. Lett. **A7** (1992) 1825.

[8] W.Oevel and W.Strampp, *Constrained KP hierarchy and Bi-Hamiltonian Structures*, Loughborough University Mathematics Report A168.

[9] G.V.Chudnosky, in Proc. of the Lecce Workshop 1979, Lect. notes in Phys. **120**(1980) 104; see also Y. Cheng and Y.S.Li, Phys. Lett **A157**(1991) 22.

[10] L.Bonora and C.S.Xiong, Phys.Lett.**B285** (1992) 191; *Matrix models without scaling limit*, SISSA–161/92/EP; C.S.Xiong, contribution to this workshop.

[11] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, in *Proc. of RIMS Symposium on non-linear integrable systems*, eds. M.Jimbo and T.Miwa, (World Scientific, 1983); G.Segal and G.Wilson, Publ. IHES **61** (1985) 1.

[12] Y. Watanabe, Lett. Math. Phys. **7** (1983) 99; Ann. Mat. Pura Appl. **137** (1984) 77.

[13] F. Yu and Y.-S. Wu, Phys.Lett. **B263** (1991) 220; K. Yamagishi, Phys.Lett. **B259** (1991) 436.

[14] J.M. Figueroa-O'Farril, J. Mas and E. Ramos, *On the two-boson picture of the KP hierarchy*, hep-th/9207013, Bonn-HE-92-17.

[15] P.J.Olver, *Applications of Lie groups to differential equations*, Springer-Verlag, 1986.

[16] J.Schiff, *The Nonlinear Schrödinger Equation and Conserved Quantities in the Deformed Parafermion and SL(2,R)/U(1) Coset Models*, hep-th/9210029, IASSNS-HEP-92/57.

[17] L.J. Mason and G.A.J. Sparling, Phys. Lett. **A137** (1989) 29; I. Bakas and D.A. Depireux, Mod. Phys. Lett. **A6** (1991) 399; J. Schiff, PhD Thesis, Columbia University (1991); D.A.Depireux, PhD thesis, University of Maryland (1991).

[18] M.D.Freeman and P.West, *On the quantum KP hierarchy and its relation to the non-linear Schrödinger equation*, King's College preprint KCL-TH-92-2; H.Aratyn, L.A.Ferreira, J.F.Gomes and A.H.Zimmerman, *On Two-Current Realization of KP Hierarchy*, hep-th/9206096, IFT-P/020/92-SAO-PAULO.