Self-Duality in Gauge Theory, and Integrable Systems

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ABSTRACT

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Three papers are presented. In "Hyperbolic Vortices and Some Non-Self-Dual Classical Solutions of $SU(3)$ gauge theory", a proposal of Burzlaff [Phys.Rev.D **24** (1981) 546] is followed to obtain a series of non-self-dual classical solutions of four-dimensional $SU(3)$ gauge theory; this is done by finding solutions of the classical equations of motion of an abelian Higgs model on hyperbolic space. The lowest value of the Yang-Mills action for these solutions is roughly 3.3 times the standard instanton action.

In "Kähler-Chern-Simons Theory and Symmetries of Anti-Self-Dual Gauge Fields", Kähler-Chern-Simons theory, which was proposed as a generalization of ordinary Chern-Simons theory, is explored in more detail. The theory describes anti-self-dual instantons on a four-dimensional Kähler manifold. The phase space is the space of gauge potentials, whose symplectic reduction by the constraints of anti-self-duality leads to the moduli space of instantons. Infinitesimal Bäcklund transformations, previously related to "hidden symmetries" of instantons, are canonical transformations generated by the anti-self-duality constraints. The quantum wave functions naturally lead to a generalized Wess-Zumino-Witten action, which in turn has associated chiral current algebras. The dimensional reduction of the anti-self-duality equations leading to integrable two-dimensional

theories is briefly discussed in this framework.

In "The Self-Dual Yang-Mills Equations as a Master Integrable System" a systematic method of dimensional reduction of the self-dual Yang-Mills equations to obtain two-dimensional integrable systems, and simple three dimensional extensions thereof, is examined. This unifies existing knowledge about such reductions. The method produces the recursion operators of various two-dimensional integrable systems; for gauge group $SL(2, \mathbf{C})$ the recursion operators of the KdV, MKdV, Gardner KdV and NLS hierarchies appear, and for $SL(3, \mathbf{C})$ the recursion operators of the Boussinesq and fractional KdV hierarchies. We also obtain the Sine-Gordon and Liouville equations. The different possible reductions for $SL(N, \mathbf{C})$) are classified, giving a conjecture on the existence of large numbers of new integrable systems, and possibly even a scheme for classification of integrable systems.

Table of Contents

Table of Contents i

Acknowledgements iii

Preface v

I. Hyperbolic Vortices and Some Non-Self-Dual Classical Solutions of SU(3) Gauge Theory 1

1. Introduction 1

2. Hyperbolic Vortices 3

3. Non-Self-Dual SU(3) Yang-Mills Solutions 15

References 19

Figures 21

- Figure 1 21
- Figure 2 22
- Figure 3 23
- Figure 4 27
- Figure 5 28
- Figure 6 30

II. Kähler-Chern-Simons Theory and Symmetries of Anti-Self-Dual Gauge Fields 31

1. Introduction 31

2. Action, Equations of Motion and Phase Space 34

3. Propeties of the Symplectic Form and the Algebra of Constraints 40

4. Symplectic Reductions, Classical and Quantum 47

5. Analysis on R^4 56

6. Concluding Remarks 61

References 63

III. The Self-Dual Yang-Mills Equations as a Master Integrable System 66

1. Introduction 66

2. Background Material 70

3. The Reduction Scheme and the Case of SL(2,C) 80

4. SL(N,C) 90

5. SL(3,C) 95

6. Concluding Remarks 108

Note Added in Proof 110

References 111

Acknowledgements

The Talmud (Chulin 24a) notes a contradiction between Numbers 4,3 and Numbers 8,14; the latter verse states that the Levites entered the Temple service at age 25, whereas the former verse states that the age was 30. The Talmud resolves the contradiction by stating that at age 25 the Levites started to learn the Temple service, but only at age 30 did they actually start to perform it. The Talmud continues

> "From here we deduce that a student who has not seen a positive
> sign in his learning for five years will never see such a sign."

It therefore gives me great satisfaction and much cause to give thanks that at the end of five years of graduate work (one at the Hebrew University, Jerusalem and four at Columbia University, New York) I am able to submit this dissertation to complete my PhD requirements.

I owe thanks to many, many people, far too numerous to list all by name. I have learnt much from the Columbia Faculty members with whom I have had contact, through my course work, research and diverse teaching activities; in particular I must mention Roman Jackiw, Erick Weinberg and my thesis advisor V.P.Nair. It is impossible to imagine a better thesis advisor than V.P.Nair; he combines an encyclopedic knowledge of physics with rare depth of understanding, and his door has been perpetually open for the last three years to share this with me, and to give me the best education possible. Chapter 2 of this thesis was coauthored by V.P.Nair, and I also thank him for permission to use this work.

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Finally, I must thank my family for their support throughout the long endeavors that have produced this dissertation. My wife Susie, and my sons Akiva and Yair are a constant source of joy to me, and this work is dedicated to them.

Preface

This thesis consists of three of the research papers that I have written while a graduate student at Columbia University. The first and third are predominantly my own work; the second was written in collaboration with V.P.Nair. While there is a thread of connection between the three, as I shall shortly explain, they are essentially independent, and therefore I have presented them as three "chapters", each of which can be read without reference to the others. Equations and references are numbered separately for each paper, and individual lists of references are given at the end of each paper.

The connection between the papers is that they are all in some way relevant to the study of the self-dual Yang-Mills (SDYM) equations. The central paper is the second of the three, in which a five-dimensional Kähler-Chern-Simons theory is presented as an analogue of Witten's three-dimensional Chern-Simons theory. The equations of motion of this theory are exactly the SDYM equations, on certain types of 4-manifold. The theory affords a rich new viewpoint on the SDYM equations, as well as possibly a door for the generalization of two-dimensional conformal-field-theoretic techniques to four dimensions. A number of questions arise out of this work. It has been known for some time now that the SDYM equations are of some relevance in four-dimensional physical gauge theories, as they provide solutions of the equations of motion of Yang-Mills theories in Euclidean space. For many years it was suspected that (for $SU(2)$ gauge theory and possibly $SU(3)$) they possibly give all solutions with finite action. In 1989 this was proven incorrect by Sibner, Sibner and Uhlenbeck. Thus from working on the self-dual equations, I was naturally led to consider the question of non-self-dual solutions, and in the first of the three papers here I show how to construct some non-self-dual solutions in $SU(3)$ gauge theory. The third paper in this collection is of more direct relevance to the SDYM equations. One of the facts that makes a theory of self-dual gauge fields look very attractive as a possible generalization of conformal field theory is that the SDYM equations are “integrable”. It is known that two-dimensional integrable field theories generalize conformal field theories. It has been proposed that the SDYM equations, being a four-dimensional integrable system might contain *all* two-dimensional integrable systems, the latter being obtained by some suitable reduction. Some recent progress has been made in showing how known integrable systems are embedded in SDYM. In the third paper of this collection, I therefore consider reductions of SDYM.

There is a slight overlap between the second and third papers; pages 58-61 in the second paper are essentially an announcement of the most basic results of the third paper. For the sake of independence of the papers, I have left this slight duplication in. Note that the second paper is also referred to in the third, as reference [10].

The first of these papers has been accepted for publication in *Physical Review D*, and is due to appear in July 1991. The second, which was coauthored with V.P.Nair, has been submitted for publication to *Nuclear Physics B*. The third was written specifically for this thesis, and is not planned for publication in the current form, though the results will be used for publication at some later date.

I. Hyperbolic Vortices and Some Non-Self-Dual Classical Solutions of SU(3) Gauge Theory

1.Introduction

There has been some recent interest in finding finite action, non-self-dual classical solutions in (Euclidean) four dimensional non-abelian gauge theory (on flat space), in the wake of the proof of Sibner, Sibner and Uhlenbeck [1] that such objects do indeed exist for gauge group $SU(2)$. For many years after the discovery [2] and subsequent development [3] of the instanton solutions in gauge theories, it was an open question as to whether these were the only finite action solutions (this is often known in the literature as the Atiyah-Jones conjecture, see [4]). Some progress in this direction was made by Bourguignon and Lawson [5], who proved (for certain gauge groups) that the only local minima of the Yang-Mills functional were given by instantons, so other solutions would have to correspond to saddle points. Furthermore, in [6] Taubes proved that in the two dimensional abelian Higgs theory with critical coupling, both in flat and in hyperbolic space, the only finite action solutions of the equations of motion were given by the solutions of the relevant self-duality equations; this result, in hyperbolic space, implied the non-existence of finite action, non-self-dual solutions in four dimensional $SU(2)$ gauge theory with "cylindrical symmetry", as introduced by Witten [7]. We now realise that this result cannot be generalized as we might have hoped. In addition to the proof of existence of finite action, non-self-dual solutions for group $SU(2)$ [1], a set of such solutions has been explicitly constructed by Sadun and Segert [8], following a proposal of Bor and Montgomery [9].

The significance of the non-self-dual solutions, to both physics and mathematics, is currently not clear. In physics, despite the fact that the non-self-dual solutions correspond to saddle points, and not minima, of the Yang-Mills functional, to do a correct semi-classical approximation by a saddle point evaluation of the path integral, it is certainly necessary to include a contribution due to non-self-dual solutions, and if it should be the case that there is a non-self-dual solution with action lower than the instanton action (this question is currently open, and of substantial importance), then such a contribution would even dominate. Unfortunately, it is questionable whether the semi-classical approximation can give a reliable picture of quantized gauge theories; it has been argued that in four dimensional gauge theory small quantum fluctuations around classical solutions can not be responsible for confinement, unlike in certain lower dimensional theories. But it may still be possible to extract some physics from the semi-classical approach. A first step in such a direction would be to obtain a good understanding of the full set of non-self-dual solutions and their properties.

In this paper, we pursue an old idea, due to Burzlaff [10], for obtaining a non-self-dual, "cylindrically symmetric" solution for gauge group $SU(3)$. If we write $\mathbf{R}^4 = \mathbf{R}\times\mathbf{R}^3$, and identify some $SU(2)$ (or $SO(3)$) subgroup of $SU(3)$, with generators that we will denote T^i, then we can look at the set of $SU(3)$ gauge potentials which are invariant under the action of the group generated by the sum of the T^i's and the generators of rotations on the $\mathbf{R}^3$ factor of $\mathbf{R}^4$ (we choose the T^i's and the $\mathbf{R}^3$ rotation generators to satisfy the same commutation relations).

We call such potentials "cylindrically symmetric" (in analogy to the standard notion of cylindrical symmetry in $\mathbf{R}^3$, which involves writing $\mathbf{R}^3 = \mathbf{R} \times \mathbf{R}^2$ and requiring rotational symmetry on the $\mathbf{R}^2$ factor). Such potentials will be specified by a number of functions of two variables, the coordinate on the $\mathbf{R}$ factor of $\mathbf{R}^4$ (which we will denote x), and the radial coordinate of the $\mathbf{R}^3$ factor (which we will denote y). Clearly the equations of motion for such cylindrically symmetric potentials (if they are consistent) will reduce to equations on the space $\{(x,y) : y \geq 0\}$. In [10] Burzlaff gave an ansatz for a cylindrically symmetric $SU(3)$ potential that would give a finite action, non-self-dual solution, with vanishing topological charge density, for every finite action solution of the equations of motion in a particular two dimensional abelian Higgs model in hyperbolic space (which is just the space $\{(x,y) : y > 0\}$, equipped with a certain metric). Most of this paper is, therefore, devoted to the study of the abelian Higgs model in hyperbolic space with arbitrary couplings; using the ball model for hyperbolic space, we argue that there should exist radially symmetric vortex solutions for a range of values of the coupling constants. For the couplings of Burzlaff we find solutions by straightforward numerical techniques. We also perform numerical experiments for other couplings; it seems quite possible that the same model, with different couplings, may emerge when examining other ansätze for non-self-dual solutions. We make some brief comments on the resulting non-self-dual solutions we have found.

2.Hyperbolic Vortices

The standard two dimensional abelian Higgs model on a spacetime with

(Euclidean) metric $g_{\mu\nu}$ is given by the action

$$S = \int d^2x\sqrt{g}\Big(\frac{\kappa}{2}g^{\mu\nu}D_\mu\phi\overline{D_\nu\phi} + \frac{\mu}{4}g^{\mu\mu'}g^{\nu\nu'}F_{\mu\nu}F_{\mu'\nu'} + \frac{\lambda}{8}(|\phi|^2-1)^2\Big) \qquad (1)$$

Here ϕ is a complex scalar field, A_μ is an abelian gauge potential, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength, and D denotes a covariant derivative, $D_\mu\phi = (\partial_\mu - iA_\mu)\phi$. κ, λ and μ are coupling constants; since classically an overall factor in the action is irrelevant, we can without loss of generality set $\kappa = 1$. For the case of flat space ($g_{\mu\nu} = \delta_{\mu\nu}$) we can make a scale transformation $x^\mu \to \xi x^\mu$, $A_\mu \to A_\mu/\xi$ to set μ to 1, to be left with one physical parameter λ.

For the case of flat space, the above action has been thoroughly studied. Since for finite action we need $|\phi| \to 1$ at infinity, we can define, for finite action configurations, an integer-valued topological invariant, the *vorticity*

$$n = \frac{1}{2\pi}\int_{circle\ at\ \infty} d\arg\phi \qquad (2)$$

Furthermore, for finiteness of the scalar field kinetic energy term in the action, it follows that if $\phi \to e^{i\chi}$ at infinity, then A_μ must tend to the pure gauge configuration $\partial_\mu\chi$ towards infinity. From this follows the *flux-vorticity relation*

$$n = \frac{1}{2\pi}\int d^2x F_{12} \qquad (3)$$

For further analysis it is convenient to separate the cases $\lambda = 1$ and $\lambda \neq 1$. For $\lambda = 1$ it is possible to find solutions to the second-order equations of motion by solving a first-order set of equations, the "self-duality" or "Bogomolnyi" equations [11,12]. One can establish the existence of a radially symmetric solution of these equations with arbitrary vorticity n, and then, by use of an index theorem, one can show that there is in fact a $2|n|-$parameter family of solutions with vorticity n [13]. More precisely, Taubes has shown that the parameter space of $n-$vortex solutions is exactly $\mathbf{R}^{2|n|}$ [14]. The action for all $n-$vortex solutions is the same, $S = |n|\pi$, and it is convenient to consider an $n-$vortex solution, for $n > 0$ ($n < 0$) as a superposition of $|n|$ $1-$vortices ($(-1)-$vortices) at $|n|$ arbitrary points on the plane. Finally, as mentioned in the introduction, Taubes [6] has shown that, for $\lambda = 1$, the solutions of the self-duality equations give *all* finite action solutions of the equations of motion.

For $\lambda \neq 1$, one has to attack the equations of motion directly. In [15] it was established that there is a radially symmetric solution to the equations of motion for any vorticity n, for (apparently) arbitrary λ, but [11] that for $\lambda > 1$, $n > 1$ these solutions were unstable (i.e. did not correspond to minima of the action). A detailed numerical study by Jacobs and Rebbi [16] revealed that for $\lambda > (<)1$ the action for the radially symmetric 2-vortex was greater (less) than twice that for the 1-vortex and thus the solutions with $n > 1$ were unstable (stable). Their results show convincingly that for $\lambda \neq 1$ there are no solutions of the equations of motion corresponding to two 1-vortices at some non-zero, finite separation; for $\lambda > (<)1$ the vortices will repel (attract). It seems reasonable to suggest from this that the only solutions of the equations of motion for $\lambda \neq 1$ are the radially symmetric ones, but for our purposes it is only important to note that as we go away from “critical” coupling, the radially symmetric solutions of the equations of motion *do* persist. Another result of [16] that we will see reproduced for hyperbolic vortices is that the action for the 1-vortex is an increasing function of λ.

We now turn to the hyperbolic case. There are several useful representations of hyperbolic space; in [7] and [10], hyperbolic space appears naturally in the upper half plane model, $\{(x,y) : y > 0\}$ with metric $g_{\mu\nu} = \delta_{\mu\nu}/y^2$. But in this model there is no concept of radial symmetry, so it is much easier for our purposes to work with the ball model, $\{(x^1,x^2) : r = \sqrt{(x^1)^2+(x^2)^2} < R\}$ with the metric $g_{\mu\nu} = \delta_{\mu\nu}/h$ where

$$h = \frac{(R^2-r^2)^2}{4R^2} \tag{4}$$

Here R is an arbitrary parameter. The two models of hyperbolic space are related by the conformal transformation

$$x^1 + ix^2 = R\left(\frac{iR-(x+iy)}{iR+(x+iy)}\right) \tag{5}$$

We note that the point $(0,R)$ in the upper half plane model maps to the origin in the ball model. Using the ball model, our action is simply

$$S \;=\; \int_{r<R} d^2x\left(\frac{1}{2}D_\mu\phi\overline{D_\mu\phi} + \frac{h\mu}{4}F_{\mu\nu}F_{\mu\nu} + \frac{\lambda}{8h}(|\phi|^2-1)^2\right) \tag{6}$$

A scaling transformation here, $x^\mu \to \xi x^\mu$, $A_\mu \to A_\mu/\xi$, $R \to \xi R$ *cannot* be used to remove one of the parameters λ, μ (though it does show us that the choice of R is arbitrary). So in the hyperbolic abelian Higgs model we have two coupling constants.

Another difference between the hyperbolic and flat space cases is that we cannot, in the hyperbolic case, write down an immediate flux-vorticity relation, simply by finiteness of the action arguments. We can still define vorticity, as since $h \to 0$ as $r \to R$, we need $|\phi| \to 1$ as $r \to R$; we therefore define

$$n = \frac{1}{2\pi}\int_{circle\;\; r=R} d\arg\phi \tag{7}$$

Unlike the flat case though, we have no *finiteness* reason to insist that $|D\phi| \to 0$ as we approach the spacetime boundary. However, to make our theory well-defined we need to specify some specific behavior for the fields at the boundary, and, specifically, we would like to choose behavior such that the surface term, that appears when we vary the action to obtain equations of motion, vanishes. For this, the obvious condition to impose is $|D\phi| \to 0$ as $r \to R$; the solutions we obtain are consistent with this. We then have the flux-vorticity relation (3).

We approach the action (6) as we do in the flat case. It is first useful to establish when we can write a set of self-duality equations. Using the identity

$$D_\mu \phi \overline{D_\mu \phi} = |(D_1 \pm iD_2)\phi|^2 \pm |\phi|^2 F_{12} \pm i\big(\partial_1(\phi \overline{D_2 \phi}) - \partial_2(\phi \overline{D_1 \phi})\big) \tag{8}$$

we can integrate by parts to write the action

$$S = \frac{1}{2}\int_{r<R} d^2x \left(|(D_1 \pm iD_2)\phi|^2 + \left(\sqrt{h\mu}F_{12} \pm \frac{1}{2\sqrt{h\mu}}(|\phi|^2 - 1)\right)^2 \pm F_{12}\right) \tag{9}$$

provided $\lambda\mu = 1$, which is the condition for self-duality. In this case we can at once write down the self-dual equations

$$\begin{aligned} (D_1 \pm iD_2)\phi &= 0 \\ F_{12} \pm \frac{\lambda}{2h}(|\phi|^2 - 1) &= 0 \end{aligned} \tag{10}$$

Here, and in all that follows, the upper sign is appropriate for positive n, and the lower sign for negative n.) If we write $\phi = fe^{i\omega}$, we can solve the first of these to obtain

$$A_\mu = \pm \epsilon_{\mu\nu}\partial_\nu \ln f + \partial_\mu \omega \tag{11}$$

and the other equation yields a single equation for f (ω is just the gauge degree of freedom), which we can write in the form

$$\nabla^2 \ln\left(\frac{\lambda f^2}{h}\right) = \left(\frac{\lambda f^2}{h}\right) + \left(\frac{2-\lambda}{h}\right) \tag{12}$$

In writing this we have exploited the fact that $\nabla^2 \ln h = -2/h$. We see straight away that the case $\lambda = 2$, $\mu = 1/2$ is very special; in this case we obtain the Liouville equation, an integrable equation. This is the case that Witten considered in [7], where he found explicitly $2|n|$ solutions of vorticity n. For general λ, however, Painlevé analysis suggests that (12) is not integrable [17]. For later reference let us write down the equations for a radially symmetric solution to the self-duality equations; the appropriate ansatz for a radially symmetric n-vortex is

$$\phi = f(r)e^{in\theta} \tag{13}$$

where θ is the usual polar coordinate. Equation (11) gives

$$\begin{aligned} A_\mu &= -n\epsilon_{\mu\nu}x_\nu \frac{a(r)}{r^2} \\ a(r) &= 1 - \frac{rf'}{|n|f} \end{aligned} \tag{14}$$

and the second of equations (10) tells us

$$\frac{|n|a'}{r} = \frac{\lambda}{2h}(1-f^2) \tag{15}$$

or, equivalently, f must satisfy equation (12), which reduces to

$$(\ln f)'' + \frac{(\ln f)'}{r} = \frac{\lambda}{2h}(f^2-1) \tag{16}$$

For the integrable case, $\lambda = 2$, we can write down the solutions to this equation satisfying the necessary boundary conditions

$$f = p\left(\frac{(r/R)-(R/r)}{(r/R)^p-(R/r)^p}\right) \tag{17}$$

where $p = |n| + 1$. It is straightforward to check that these solutions have the following asymptotic behaviors; near $r = 0$

$$
\begin{aligned}
f(r) &\sim p\left(\frac{r}{R}\right)^{|n|} \\
a(r) &\sim \frac{2}{p-1}\left(\frac{r}{R}\right)^{2}
\end{aligned}
\tag{18}
$$

and near $r = R$

$$
\begin{aligned}
1 - f(r) &\sim \frac{p^2-1}{6}\left(1 - \frac{r}{R}\right)^{2} \\
1 - a(r) &\sim \frac{(p-1)^2(p+1)}{3}\left(1 - \frac{r}{R}\right)
\end{aligned}
\tag{19}
$$

We will later be able to use these as a check for the asymptotic behaviors for general λ, μ.

Let us now look at the action (6) for arbitary λ, μ. The equations of motion are

$$
\begin{aligned}
D_\mu D_\mu \phi + \frac{\lambda \phi}{2h}(1 - |\phi|^2) &= 0 \\
\mu \partial_\nu (h F_{\mu\nu}) + \frac{i}{2}(\overline{\phi} D_\mu \phi - \phi D_\mu \overline{\phi}) &= 0
\end{aligned}
\tag{20}
$$

We look for a radially symmetric $n-$vortex solution in the form

$$
\begin{aligned}
\phi &= f(r) e^{in\theta} \\
A_\mu &= -n \epsilon_{\mu\nu} x_\nu \frac{a(r)}{r^2}
\end{aligned}
\tag{21}
$$

The equations of motion reduce to

$$
\begin{aligned}
f'' + \frac{f'}{r} - \frac{n^2 f (1-a)^2}{r^2} + \frac{\lambda f}{2h}(1 - f^2) &= 0 \\
a'' - \left(\frac{1}{r} - \frac{h'}{h}\right) a' + \frac{f^2}{\mu h}(1 - a) &= 0
\end{aligned}
\tag{22}
$$

(Note that these reduce to equations (2.18) in ref [16] if we set $h = 1$, and suitably redefine coupling constants.) At this point it is useful to introduce the variable

$t = r/R$ to eliminate the constant R from the problem. Using a dot to denote differentiation with respect to t, we obtain

$$\begin{aligned} \ddot{f} + \frac{\dot{f}}{t} - \frac{n^2 f(1-a)^2}{t^2} + \frac{2\lambda}{(1-t^2)^2}(1-f^2) = 0 \\ \ddot{a} - \left(\frac{1}{t} + \frac{4t}{1-t^2}\right)\dot{a} + \frac{4f^2}{\mu(1-t^2)^2}(1-a) = 0 \end{aligned} \tag{23}$$

It is straightforward to compute the action density for the ansatz (21), and we obtain

$$\begin{aligned} S = 2\pi \int_0^1 dt E(t) \\ E(t) = \frac{t\dot{f}^2}{2} + \frac{n^2 f^2(1-a)^2}{2t} + \frac{\mu n^2(1-t^2)^2\dot{a}^2}{8t} + \frac{\lambda t(1-f^2)^2}{2(1-t^2)^2} \end{aligned} \tag{24}$$

We need to analyze the system (23) with the requisite boundary conditions. The first step is to write Frobenius-type expansions for the solutions of (23) near the points $t = 0$ and $t = 1$, both of which are singular points of (23). We obtain the following results: near $t = 0$ the nonsingular solutions of (23) have form

$$\begin{aligned} f = At^{|n|}\left(1 + \sum_{q=1}^{\infty} f_q t^{2q}\right) \\ a = Bt^2\left(1 + \sum_{q=1}^{\infty} a_q t^{2q}\right) \end{aligned} \tag{25}$$

Here A, B are some unspecified constants, and the f_q's and a_q's are constants determined by A, B. It is possible to write a recursion relation for f_q, a_q in terms of $A, B, f_1, a_1, ..., f_{q-1}, a_{q-1}$, but here we just give the first few coefficients explicitly

$$\begin{aligned}
f_1 &= -\left(\frac{Bn^2+\lambda}{2(|n|+1)}\right) \\
f_2 &= \frac{1}{8(|n|+2)}\left(\frac{(Bn^2+\lambda)^2}{|n|+1} - 4\lambda + n^2B(B-2) + \left(2\lambda + \frac{n^2}{\mu}\right)A^2\delta_{|n|1}\right) \\
a_1 &= 1 - \frac{A^2}{2\mu}\delta_{|n|1} \\
a_2 &= 1 - \frac{A^2}{6\mu B}\delta_{|n|2} - \frac{A^2}{6\mu B}\left(4 - B - \frac{Bn^2+\lambda}{|n|+1}\right)\delta_{|n|1}
\end{aligned} \tag{26}$$

Near $t = 1$ we find that solutions of (23) with $f(1), a(1)$ finite are given by series

$$\begin{aligned}
1 - f &= \alpha(1-t)^\zeta\left(1 + \sum_{q=1}^{\infty} g_q(1-t)^q\right) \\
1 - a &= \beta(1-t)^d\left(1 + \sum_{q=1}^{\infty} b_q(1-t)^q\right)
\end{aligned} \tag{27}$$

Here α, β are arbitrary constants, the g_q's and b_q's are defined by a recursion relation, and ζ, d are given as the positive roots of

$$\begin{aligned}
\lambda &= \zeta(\zeta - 1) \\
\frac{1}{\mu} &= d(d+1)
\end{aligned} \tag{28}$$

Equations (28) are very pleasing. For Witten's case [7], $\lambda = 2$ and $\mu = 1/2$, so we have $\zeta = 2$ and $d = 1$. For Burzlaff's case [10], $\lambda = 2$ and $\mu = 1/6$, so we have $\zeta = 2$ and $d = 2$. Uhlenbeck [18] has shown that any solution of the Yang-Mills equations on $\mathbf{R}^4$ with finite action can be obtained (in a suitable gauge) from a smooth gauge field on S^4; thus if we are to obtain finite-action solutions of the Yang-Mills equations from either the Witten or Burzlaff ansätze, we need the coefficients ζ and d to be integers, and we see they are.

To summarize our problem, we see that we need to find solutions of (23), with f, a given by (25) (for some A, B) near $t = 0$, and by (27) (for some α, β)

near $t = 1$. Intuitively this problem is solvable; essentially we just need to choose A, B, α, β in such a way that $f, a, \dot{f}, \dot{a}$ are continuous. We use a straightforward numerical method to actually solve the problem. For a specific A, B we use the power series (25) with the coefficients (26) to obtain f, a up to $t = 0.002$. We then use the Runge-Kutta method (introducing extra dependent functions $g = \dot{f}$, $b = \dot{a}$ to obtain a first order system), with a step length of 10^{-5}, to integrate up to $t = 1$. All work was performed with double precision arithmetic. There is an inherent instability as we approach $t = 1$, corresponding, roughly speaking, to the negative roots d, ζ of equations (28). For generic A, B the functions f, a will be unbounded as we approach $t = 1$. We label the functions f, a arising from the numerical integration with a "+" if they are monotonically nondecreasing, and with a "−" otherwise. Thus we can plot two curves in the $A, B-$plane corresponding to the values of A, B where f and a change from "+" to "−" behavior. The critical values of A, B required for the vortex solution, A_{crit}, B_{crit}, will be at the intersection of these two curves. Plots of the curves in the $A, B-$plane are shown in figure 1* for $n = 1$ for the Witten case $\lambda = 2$, $\mu = 1/2$, and similar plots are found for $n = 2, 3, 4$. We obtain the results

$$\begin{aligned} A_{crit} &= |n| + 1 \\ B_{crit} &= \frac{2}{|n|} \end{aligned} \tag{29}$$

as expected from (18).

In general we find we require very accurate values of A_{crit}, B_{crit} (accuracy of about one part in 10^9) to obtain "reasonable" vortex solutions. There is a

* In this and later figures, the label "+−" denotes that the function f has "+" behavior and the function a has "−" behavior in the marked region; the other labels are defined similarly.

useful method of checking the "reasonableness" of a vortex solution; the action density $E(t)$ defined in (24) is linear in t for $t \approx 0$, and for $t \approx 1$ goes as $(1-t)^{2z}$, where

$$z = \min(d, \zeta - 1) \tag{30}$$

In general, because of the instability at $t = 1$, we will find the numerical $E(t)$ has a slight "tail", that is, instead of tending to zero in the expected way at $t = 1$, it will, after a point, display a slight increase. We have aimed to obtain A_{crit}, B_{crit} to an accuracy such that this "tail" affects the numerical approximation to the action by less than one part in 10^3. The algorithm performs correctly, to well within the required accuracy, for the Witten case.

It remains to give some results. First we check a self-dual case, $\lambda = 6$, $\mu = 1/6$. For any self-dual case, it is easy to check that for the $n-$vortex

$$\begin{aligned} B_{crit} &= \frac{\lambda}{|n|} \\ S &= \pi|n| \end{aligned} \tag{31}$$

We reproduce these results accurately, and we find the values for A_{crit} given in table 1.

n	A_{crit}
1	3.13728895
2	6.80933129
3	12.40261138
4	20.30221717

Table 1: A_{crit} values for $\lambda = \mu^{-1} = 6$

(Note that while we quote A_{crit} values to the accuracy necessary to make our numerical algorithm produce reasonable vortex solutions, it is possible that the A_{crit} of our numerical procedure is only the same as the real A_{crit} to a lower degree of accuracy.) In figure 2 we display the curves in the $A, B-$plane for this case, for $n = 1$, and in figure 3 we display the functions f, a for $n = 1, 2, 3, 4$ for both the Witten case and this case: note the difference in the behaviors at $t = 1$.

Now we move to the Burzlaff case, $\lambda = 2$, $\mu = 1/6$. We obtain the results in table 2.

n	A_{crit}	B_{crit}	$S/2\pi$
1	2.32258782	4.55248618	0.412
2	4.18191496	2.18876301	0.783
3	6.57417323	1.43867781	1.145
4	9.51117487	1.07160765	1.504

Table 2: Vortex solution data for $\lambda = 2,\ \mu = 1/6$

The $A, B-$plane plot, for $n = 1$, is shown in figure 4 and f, a plots, for $n = 1, 2, 3, 4$, are in figure 5. The $A, B-$plane plot shows an interesting feature: the curve marking the change of behavior of f apparently has a cusp at (A_{crit}, B_{crit}). This feature is reproduced for higher n, and we have found this feature in general for $\lambda\mu < 1$ (but it seems that the curve straightens as $\lambda\mu \nearrow 1$, and we have not noticed a cusp in plots for $\lambda\mu > 1$). One proviso is in order here: our numerical algorithm is not necessarily reliable anywhere but exactly at the vortex solution.

Finally we present results for $1-$ and $2-$vortices for $\lambda = 2$ and a range of values of μ. The results are summarized in table 3, where the superfix on A_{crit} etc. denotes the value of n.

μ	A^1_{crit}	B^1_{crit}	A^2_{crit}	B^2_{crit}	$S^1/2\pi$	$S^2/2\pi$
1/6	2.3225878	4.5524862	4.1819150	2.1887630	0.412	0.783
0.4	2.0546757	2.3706522	3.1859077	1.1771755	0.481	0.954
0.5	2.0000000	2.0000000	3.0000000	1.0000000	0.500	1.000
0.75	1.9131831	1.4605004	2.7169828	0.7383847	0.534	1.085
1.0	1.8607006	1.1626815	2.5531907	0.5917803	0.557	1.145
1.5	1.7984541	0.8362130	2.3661313	0.4290016	0.589	1.227
5.0	1.6807265	0.2949309	2.0341395	0.1535178	0.665	1.421

Table 3: 1− and 2−vortex solution data

for $\lambda = 2$ and various values of μ

The main result here is that for $\lambda\mu < 1$ ($\lambda\mu > 1$) it seems that the 2−vortex is stable (unstable) against decay into two 1−vortices. This, and the fact that the action is an increasing function of $\lambda\mu$ are in accordance with the flat space results.

3.Non-Self-Dual $SU(3)$ Yang-Mills Solutions

We feel no need to reproduce verbatim the analysis of Burzlaff [10], save for one point that requires a little clarification. Witten's ansatz [7] for cylindrically symmetric $SU(2)$ gauge fields can be embedded into $SU(3)$ in two distinct ways. One uses the the $SU(2)$ subalgebra of $SU(3)$ with the generators $\sigma^i = (\lambda_1/2, \lambda_2/2, \lambda_3/2)$ and the other uses the $SU(2)$ subalgebra with generators $E^i = (\lambda_7, -\lambda_5, \lambda_2)$ (Here the λ_a's are the usual Gell-Mann matrices). Both σ^i and E^i satisfy the $SU(2)$ commutation relation

$$[T^i, T^j] = i\epsilon^{ijk}T^k \tag{32}$$

but we have different trace formulae

$$\begin{aligned} Tr(\sigma^i\sigma^j) &= \tfrac{1}{2}\delta^{ij} \\ Tr(E^iE^j) &= 2\delta^{ij} \end{aligned} \tag{33}$$

Because of this difference, when we use the E^i's for the embedding we obtain the hyperbolic space action (6) with $\lambda = 2$, $\mu = 1/2$, with a prefactor of 32π, as compared to a prefactor of 8π which we obtain when using the σ^i's. Burzlaff's construction for non-self-dual solutions gives the action (6) with $\lambda = 2$, $\mu = 1/6$, with a prefactor of 32π.

Having stated this we can at once give the main result of this paper: we have non-self-dual $SU(3)$ solutions with action given by $64\pi^2$ times the figures in the last column of table 2. This is in units where the standard instanton action is $8\pi^2$, so the lowest action of our solutions is roughly 3.3 times the instanton action. We would speculate that there exists a solution of the type we have looked at for any n, and we see that the action for the "$n-$solution" is less than n times the action of the basic solution. The asymptotic behavior for large n is clearly of interest. We note that the solution for $n = 4$ is almost as low as only three times the $n = 1$ action. The lowest action for a non-self-dual solution in $SU(2)$ found by Sadun and Segert [8] is roughly 5.4 times the instanton action. In $SU(4)$ gauge theory, it is clear that we can find a non-self-dual solution with action twice that of the instanton: we pick two commuting $SU(2)$ subalgebras and consider the potential which is composed of an instanton in one $SU(2)$ and an anti-instanton in the other $SU(2)$.

Let us investigate just briefly the geometry of our solutions. To do this we must revert to the upper half plane model of hyperbolic space. In section 2 we introduced Cartesian coordinates (x, y) on hyperbolic space in the upper half plane model, and (x^1, x^2) on hyperbolic space in the ball model, and we also used polar coordinates (r, θ) on the ball model. Let us now introduce polar coordinates (ρ, ϕ) on the upper half plane model $(0 < \rho < \infty,\ 0 < \phi < \pi)$ via

$$\begin{aligned} \rho &= \sqrt{x^2 + y^2} \\ \phi &= \tan^{-1}(y/x) \end{aligned} \tag{34}$$

ρ is the standard radial coordinate of $\mathbf{R}^4$ (that is, if the $\mathbf{R}^4$ coordinates are (y^1, y^2, y^3, y^4), then $\rho = \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2 + (y^4)^2}$). The action density of our solutions can be expressed in the ball model as a function of r alone, but in the upper half plane model it is a function of the two variables ρ, ϕ. Explicitly we have, for the Yang-Mills action,

$$\begin{aligned} YM &= 32\pi \int_0^R \int_0^{2\pi} d\left(\frac{r}{R}\right) d\theta \; E(r/R) \\ &= 32\pi \int_0^\infty \int_0^\pi d\left(\frac{\rho}{R}\right) d\phi \; JE(r/R) \end{aligned} \tag{35}$$

where J is the necessary Jacobian

$$J = \frac{s}{t}\left(\frac{1}{1 + s^2 + 2s\sin\phi}\right)^2 \tag{36}$$

where we have written $t = r/R$, as before, and we have introduced $s = \rho/R$. In terms of ρ, ϕ, we have

$$t = \frac{r}{R} = \sqrt{1 - \frac{4s\sin\phi}{1 + s^2 + 2s\sin\phi}} \tag{37}$$

The functions $E(t)$ for our solutions, with $n = 1, 2, 3, 4$ are displayed in figure 6; as mentioned above, for $t \approx 1$, $E(t)$ behaves as a multiple of $(1-t)^2$. This completes all the necessary information to work out the large ρ behavior of the action density, that is the integrand of (the second part of) (35). We find that the action density drops off (for fixed ϕ) as $(\rho/R)^{-5}$. This is identical to the behavior of the standard instanton, except of course we must remember that our solutions do not have full "spherical" symmetry in $\mathbf{R}^4$. The reproduction of the instanton result here is essentially due to the fact that the parameter z of equation (30) is the same for the Witten case and the Burzlaff case. We note that our solutions, like standard instantons, have a scale parameter R associated with them. It seems, in fact, that there is an eight parameter family of our solutions (for each n), as opposed to a five parameter family for the standard instanton: we have in addition to the usual "center" and "scale" parameters, three extra parameters associated with the choice of the time axis, which we use to define the cylindrical symmetry. Possible subtleties could arise in this naive counting, however, due to gauge transformations. We can also consider the effect of the full conformal group on our solutions: applying special conformal transformations to our eight parameter family could generate up to a twelve parameter family of solutions (compare [19]); the form of the potentials for the solutions thus generated would, it seems, be messy, and the task of checking that these solutions are not gauge equivalent might be very tricky.

In conclusion, we just mention a few more points worthy of study, in addition to the various points that have been mentioned in passing above. It is important to examine the stability of our solutions as solutions of the Yang-Mills equations (as solutions of the hyperbolic abelian Higgs model it seems they correspond to genuine minima of the action). By virtue of the results of [5] they do not correspond to minima of the action functional, but to saddle points, and it is of interest to count (if it is finite) the number of small variations away from the solutions that reduce the action functional. Intuitively, the objects that we have found are potentially of more relevance to physics if this number is low. Another question that could produce interesting results is to generalize Burzlaff's ansatz to larger groups, using other $SU(2)$ embeddings, to see if we can obtain hyperbolic abelian Higgs models with other couplings. We expect the couplings to be such that the numbers ζ,d defined by equation (28) are integers.

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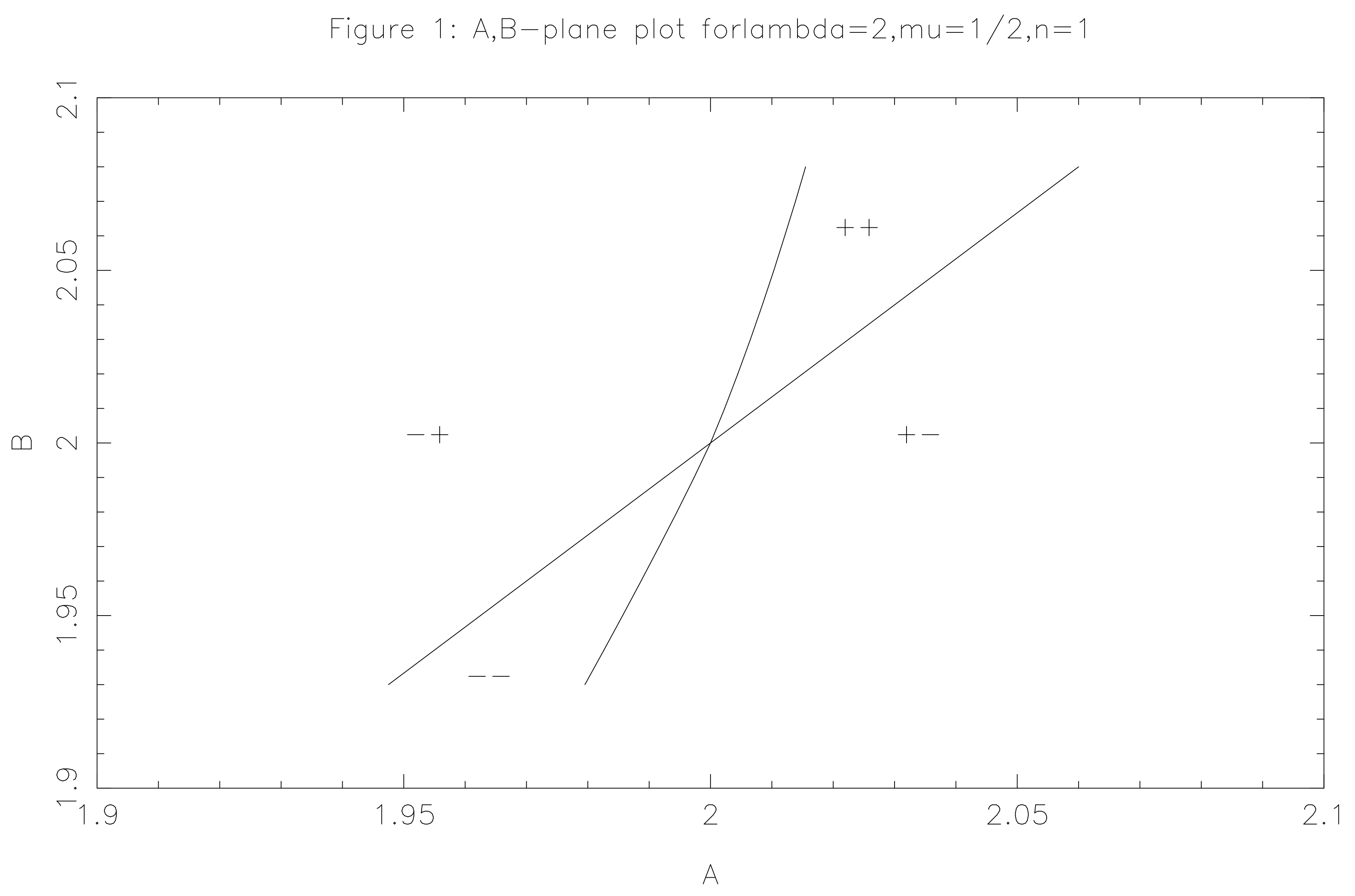


Figure 1: A,B−plane plot forlambda=2,mu=1/2,n=1
B
2.1
2.05
2
1.95
1.9
1.9
1.95
2
2.05
2.1
A
−+
++
+−
−−

Figure 2: A,B−plane plot forlambda=6,mu=1/6,n=1

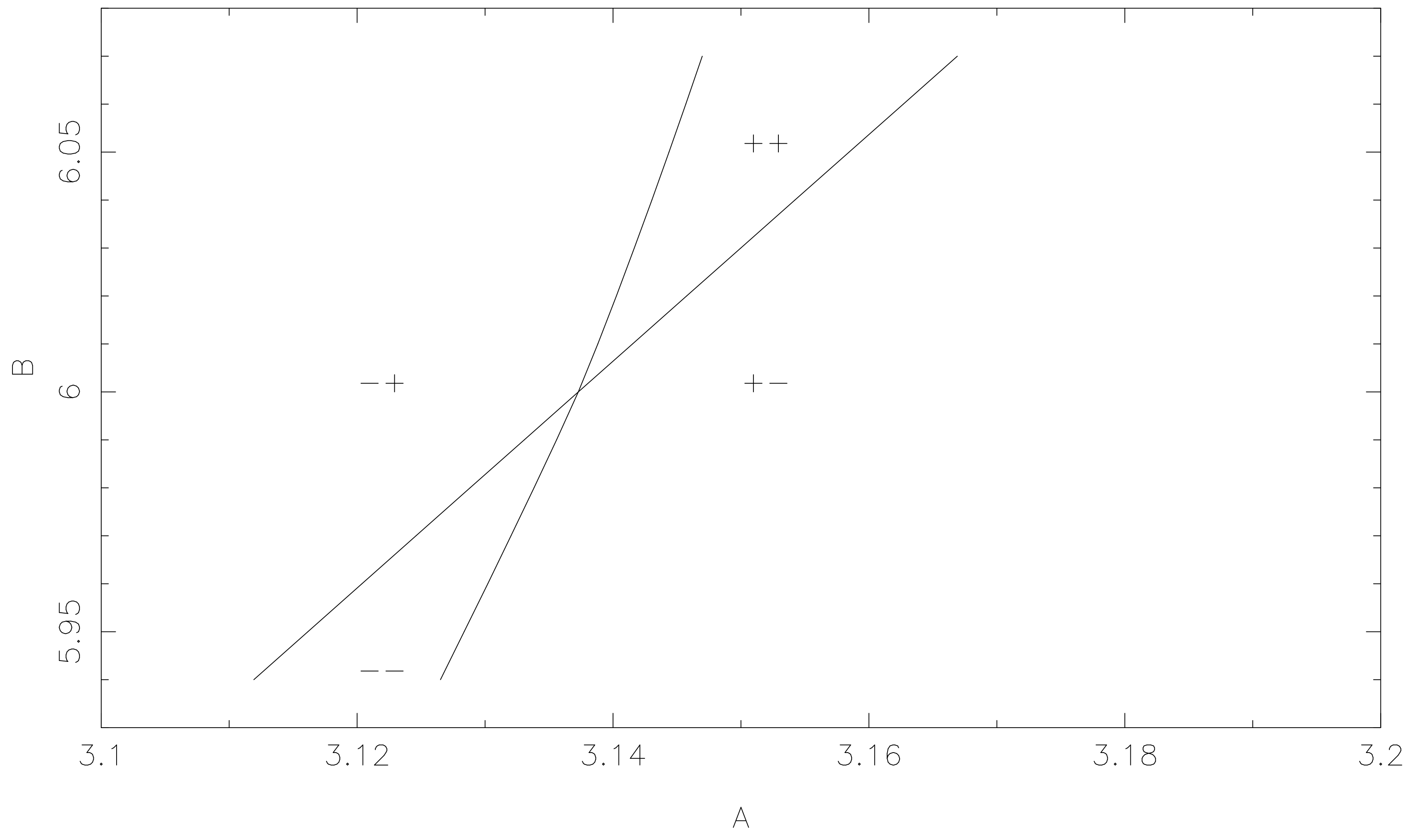


Figure 3a: f(t) for lambda=2,mu=1/2,n=1,2,3,4

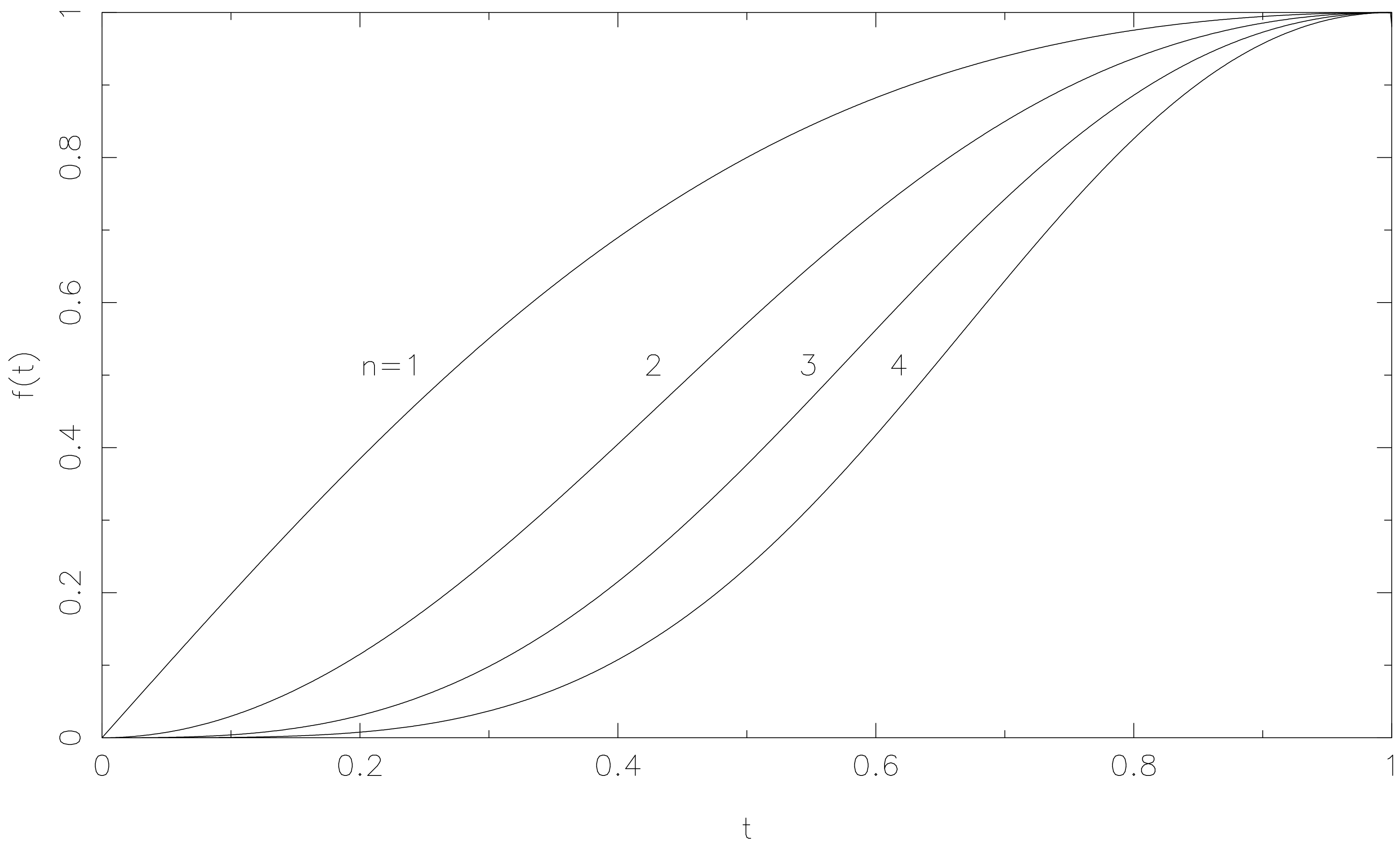


Figure 3b: a(t) for lambda=2,mu=1/2,n=1,2,3,4

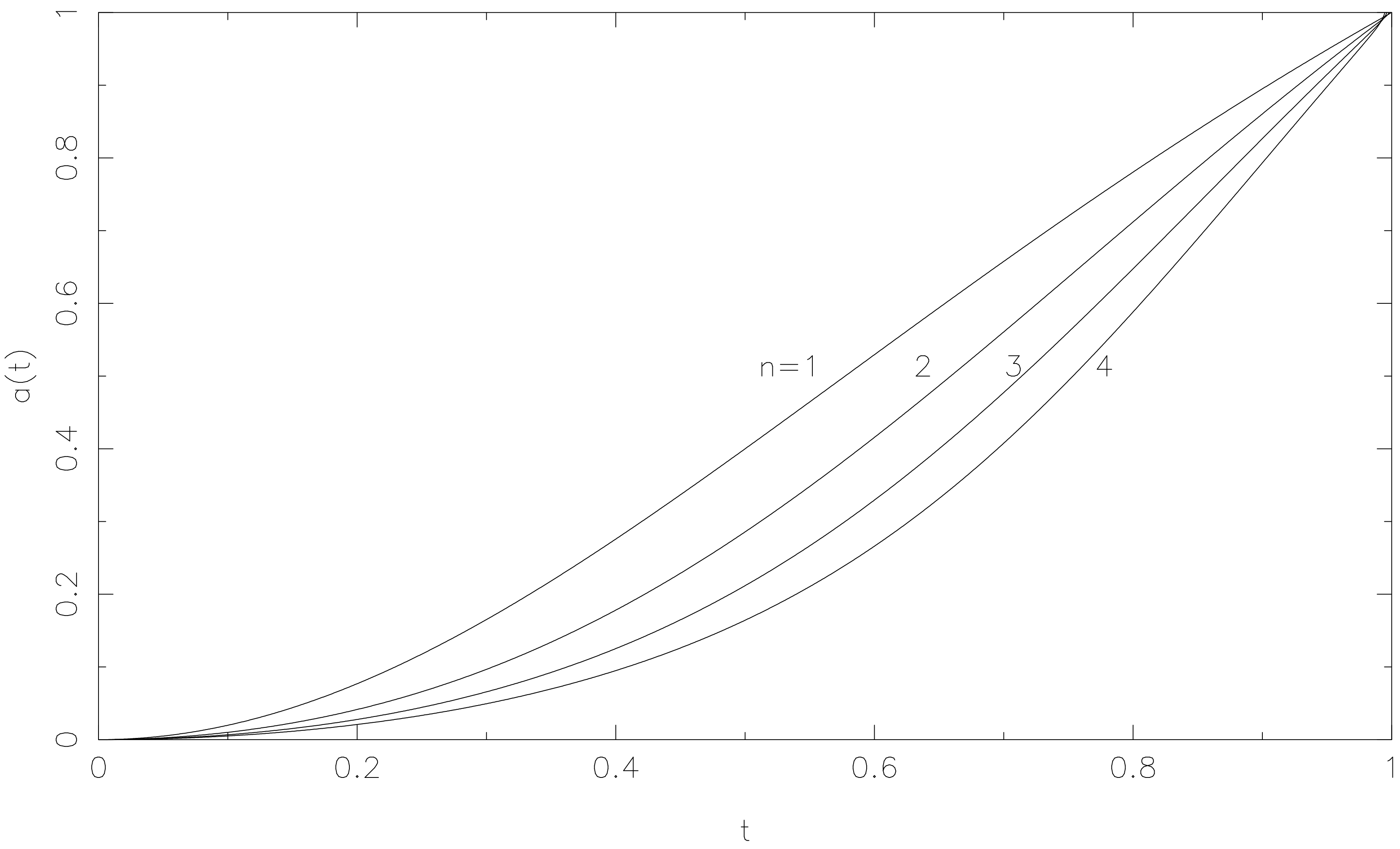


Figure 3c: f(t) for lambda=6,mu=1/6,n=1,2,3,4

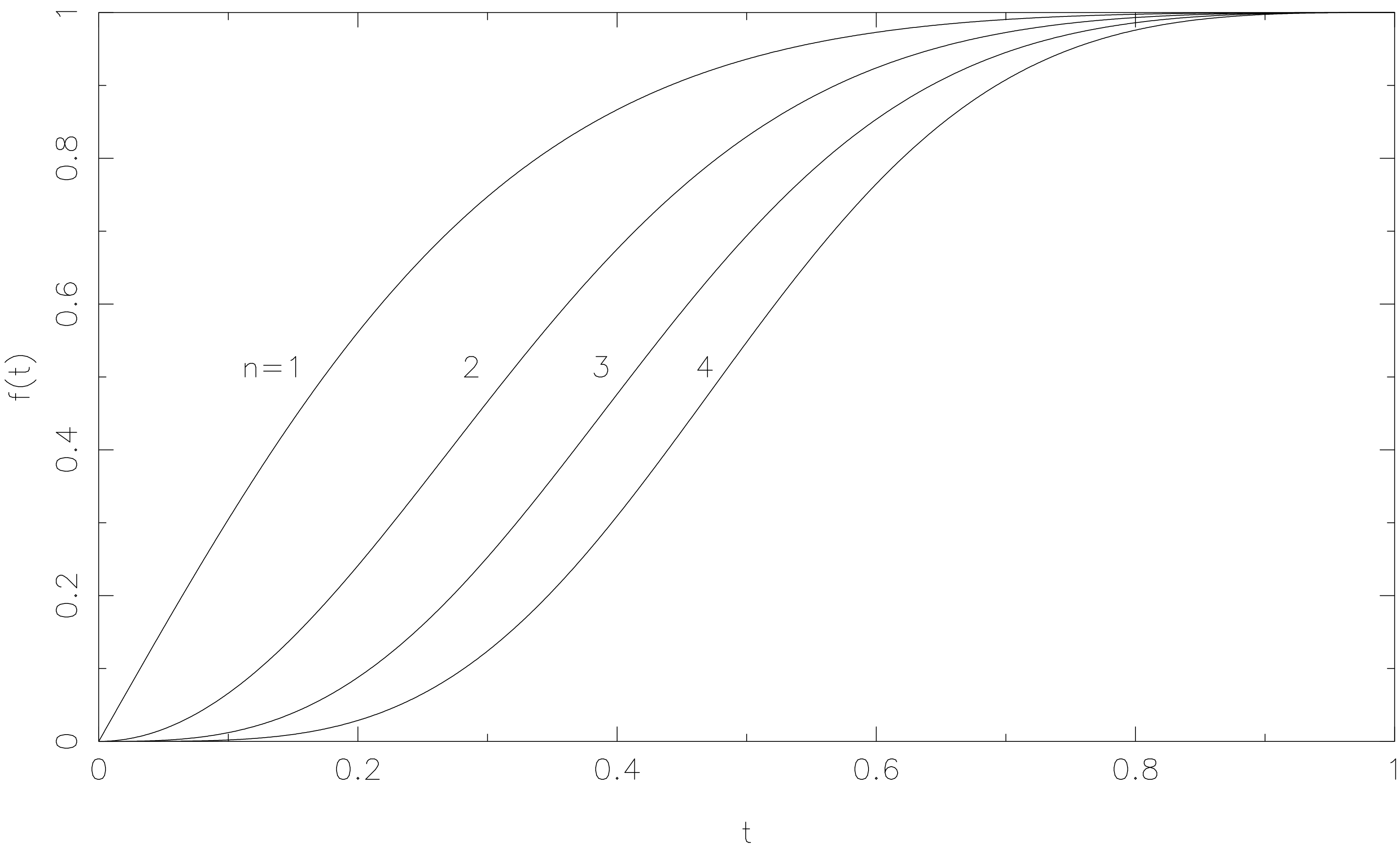


Figure 3d: a(t) for lambda=6,mu=1/6,n=1,2,3,4

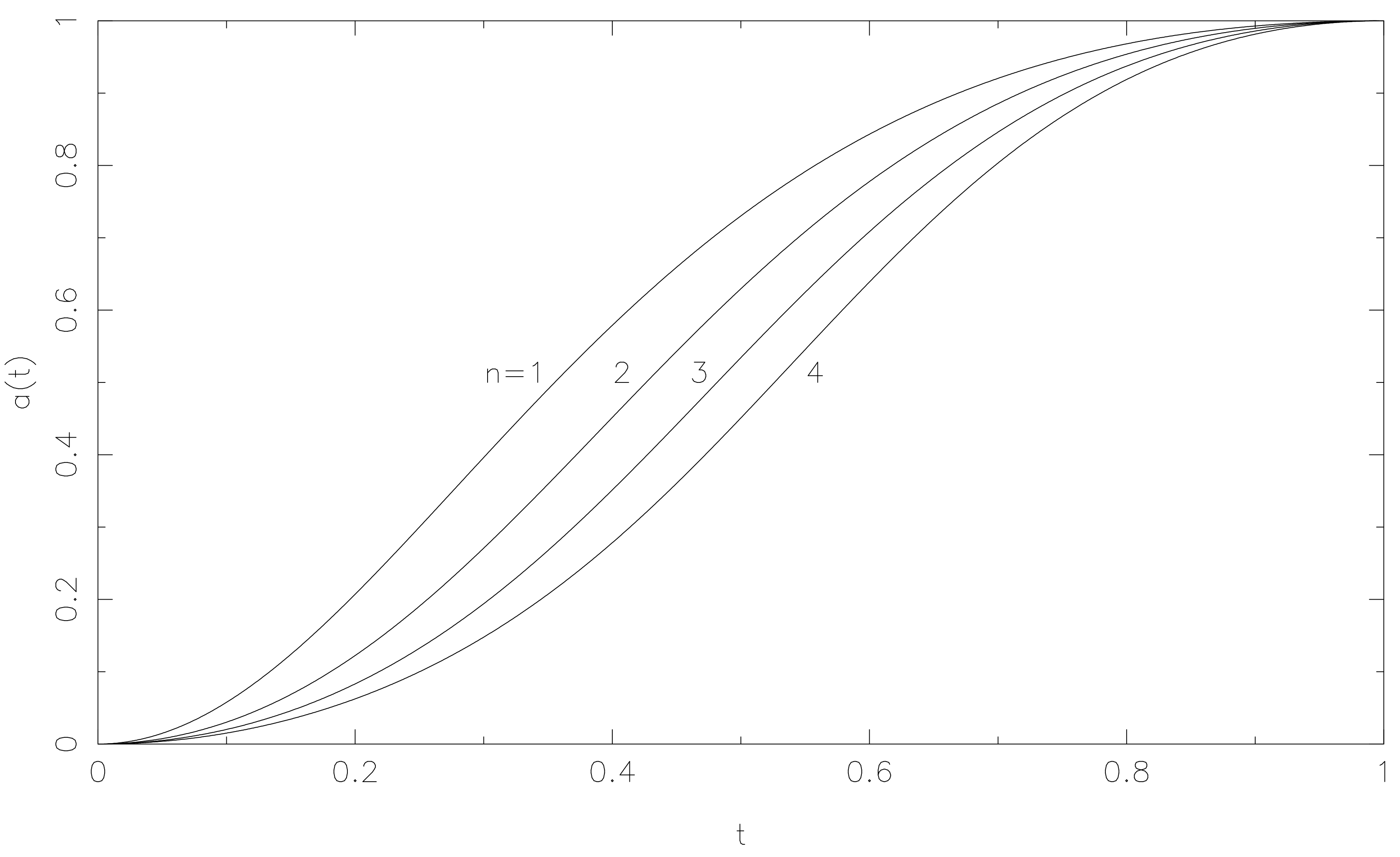


Figure 4: A,B−plane plot forlambda=2,mu=1/6,n=1

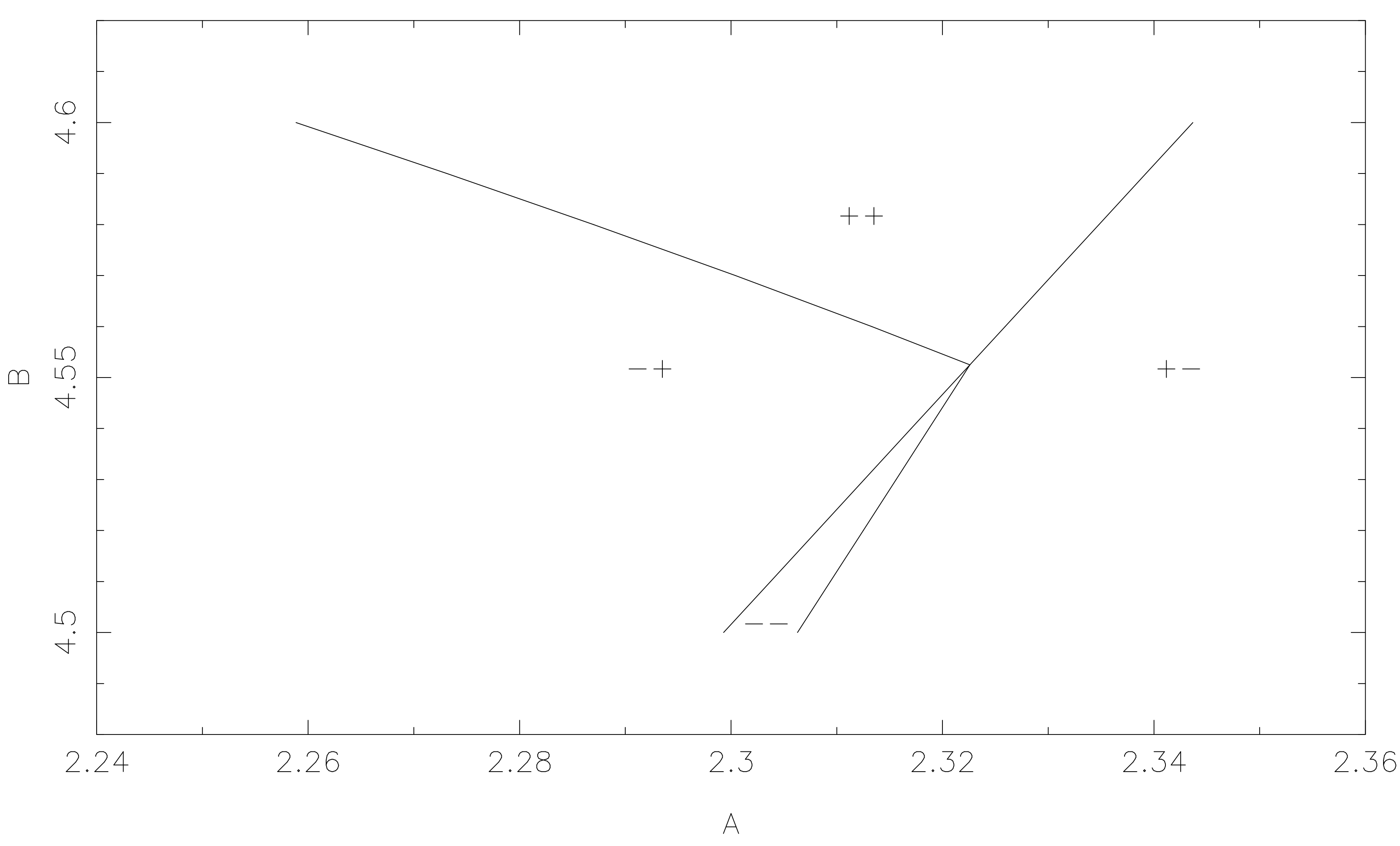


Figure 5a: f(t) for lambda=2,mu=1/6,n=1,2,3,4

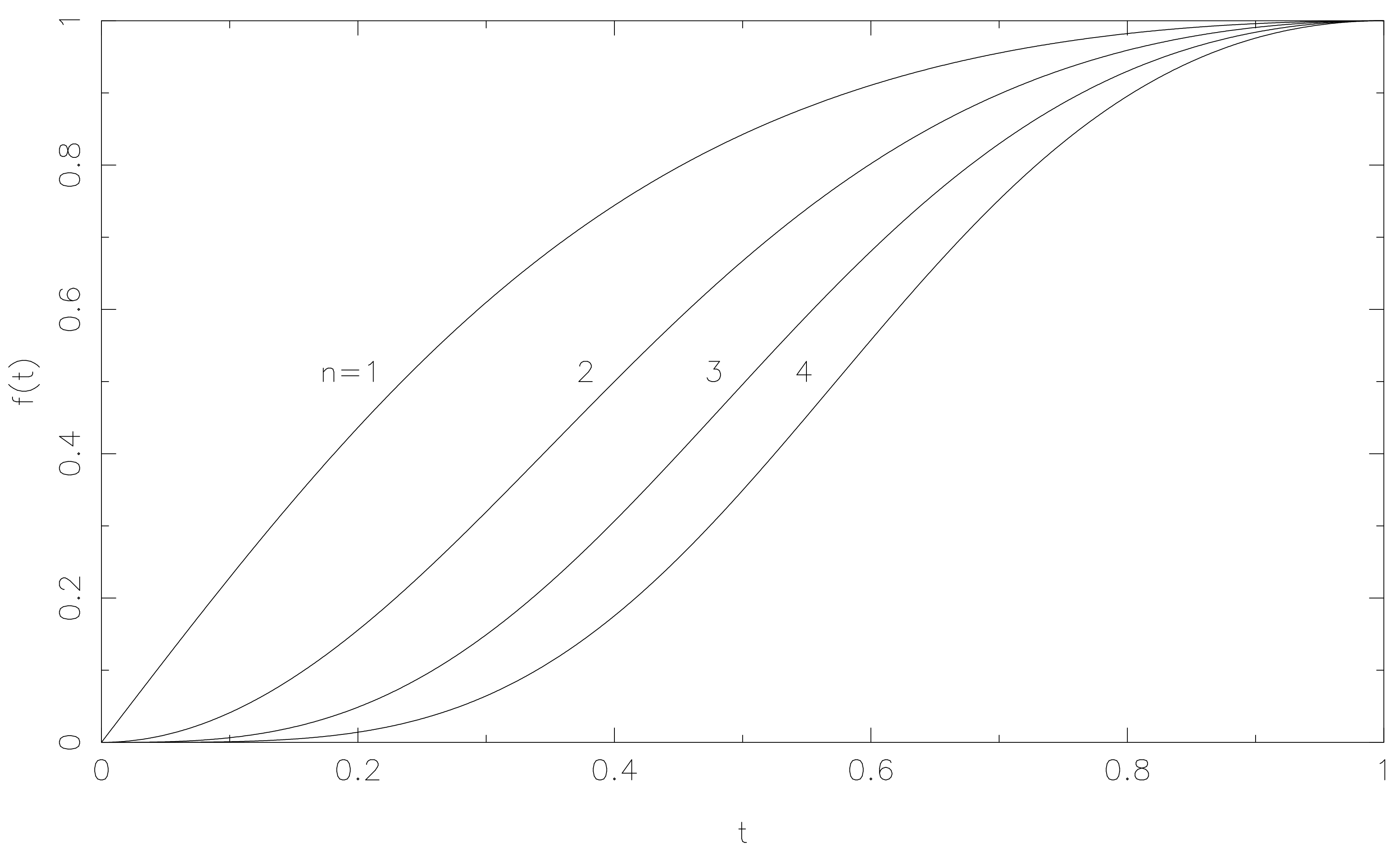


Figure 5b: a(t) for lambda=2,mu=1/6,n=1,2,3,4

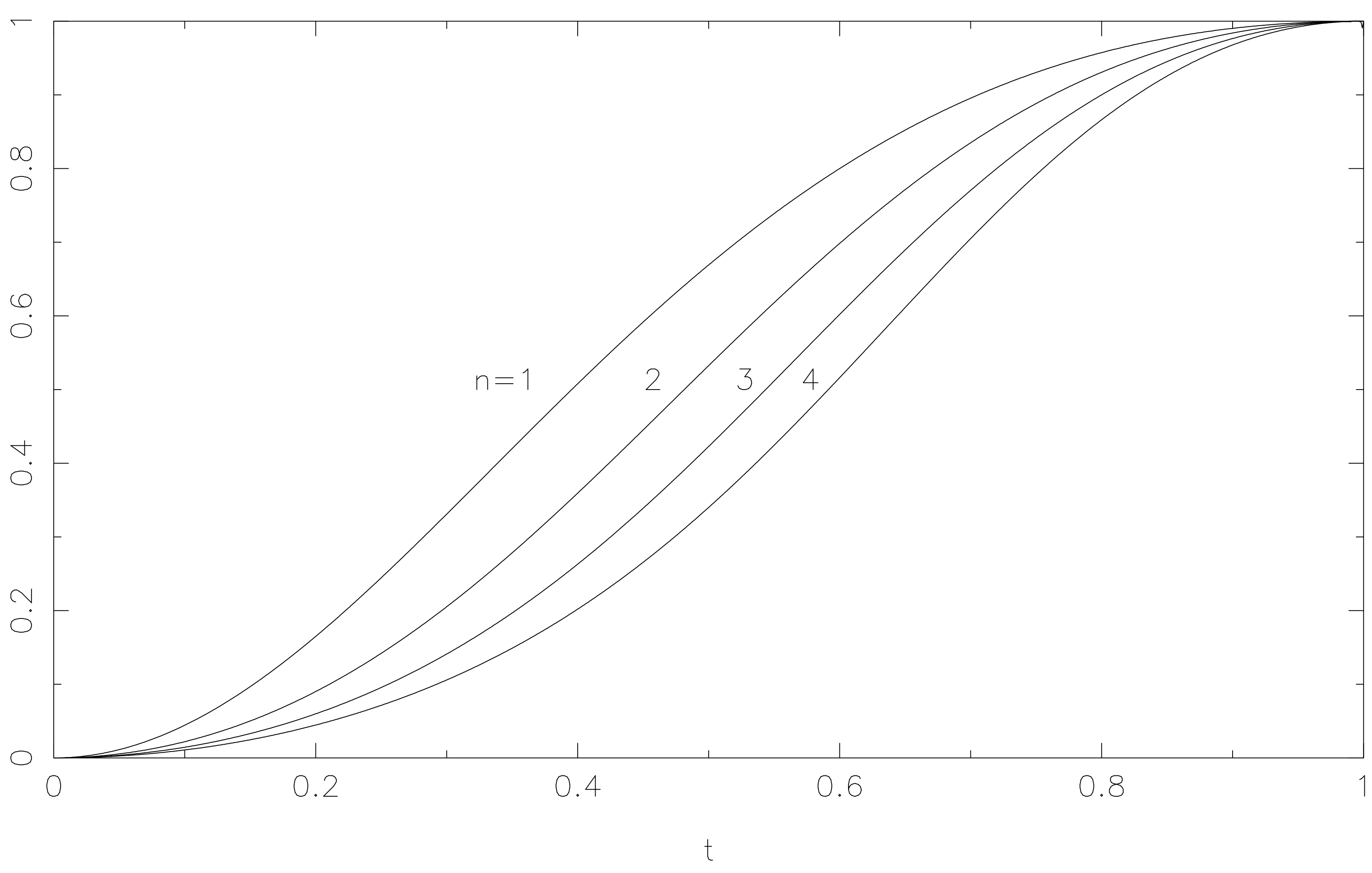
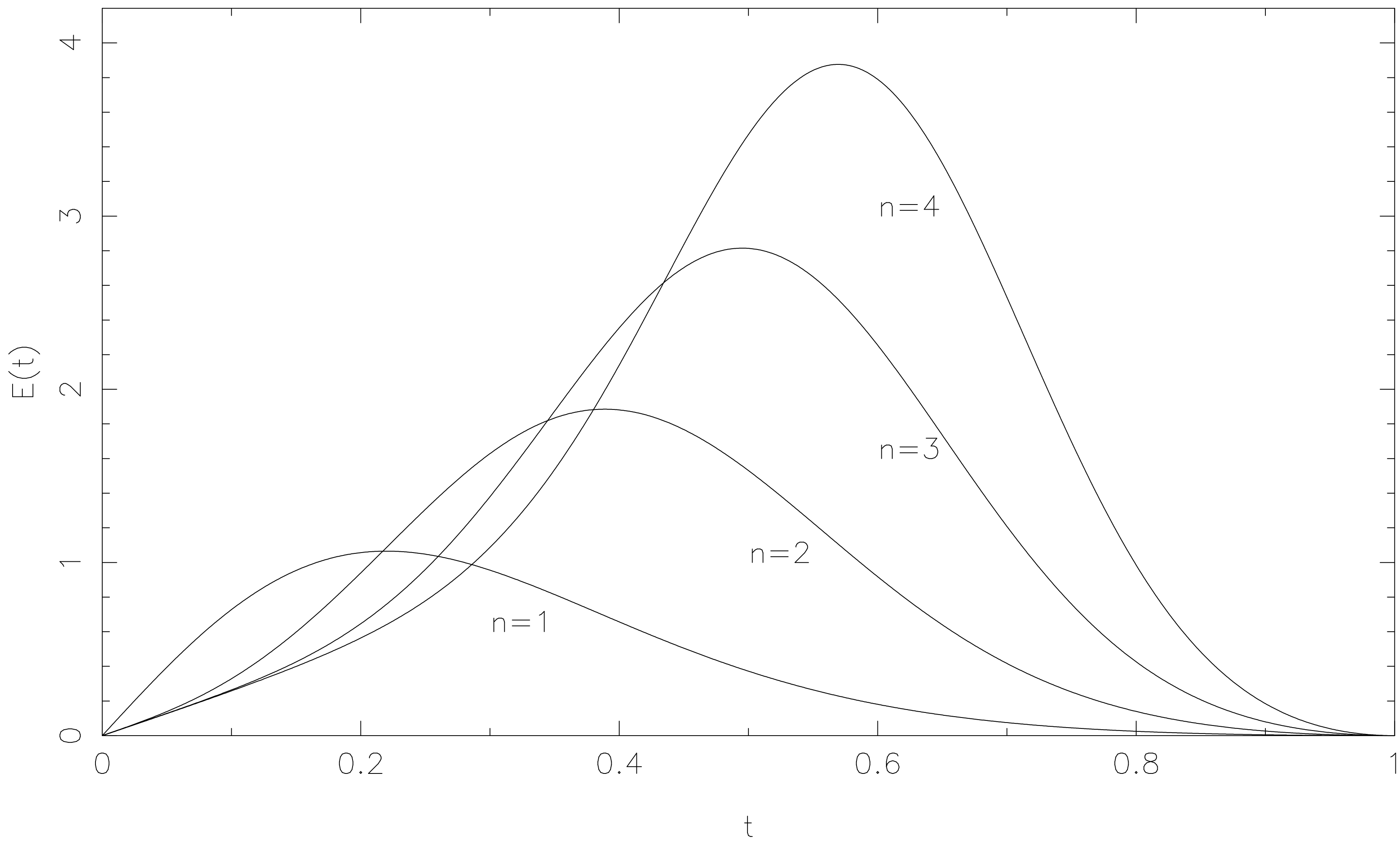


Figure 6: E(t) for lambda=2,mu=1/6,n=1,2,3,4



II. Kähler-Chern-Simons Theory and Symmetries of Anti-Self-Dual Gauge Fields

1. Introduction

Recently [1] we proposed a five-dimensional theory, referred to as Kähler-Chern-Simons theory (KCST), as a generalization to a $4+1$ dimensional setting of many features of three-dimensional Chern-Simons theory (3d CST) [2]. In this paper, we give a more elaborate presentation of the theory as well as several new results. The three-dimensional Chern-Simons theory, it is by now well known, gives an intrinsically three-dimensional quantum field theoretic interpretation of the Jones polynomials for links; the polynomials are essentially the correlation functions of the Wilson loop operators for the links. 3d CST is also very closely related to two-dimensional conformal field theory. In particular, 3d CST on $\Sigma \times R$, where Σ is a Riemann surface, is an exactly solvable theory with a finite dimensional Hilbert space which can be identified as the space of chiral blocks of a rational conformal field theory. The relevant chiral algebra is the current algebra of a Wess-Zumino-Witten (WZW) model, a Kac-Moody algebra, defined on Σ [3]. The construction of link invariants highlights the topological nature of 3d CST. The current algebraic aspects are, however, related to the fact that the reduced phase space is the space of flat gauge potentials on Σ modulo gauge transformations. The Narasimhan-Seshadri theorem [4] shows that, for gauge group $SU(N)$, this space is also the moduli space of stable, rank N holomorphic vector bundles of Chern class zero over Σ; holomorphic gauge transformations and chiral algebras are naturally defined in this case. It is as a generalization of the current algebraic features that we introduced KCST.

The four dimensional manifold in KCST, the analogue of the Riemann surface Σ in 3d CST, is a Kähler manifold. The analogue of the Narasimhan-Seshadri theorem is Donaldson's theorem [5] which relates the moduli space of holomorphic vector bundles to instanton moduli spaces. The equations of motion of KCST are thus, not surprisingly, the anti-self-duality conditions in four dimensions and the classical solutions are anti-self-dual instantons. Instantons or anti-self-dual (ASD) gauge fields have also been of interest recently for different but related reasons; this is in connection with integrable systems. There are two connections between integrable systems and conformal field theories. First, integrable theories describe a class of perturbations of conformal field theories away from criticality [6]. Second, the Poisson bracket algebras associated with certain integrable systems are classical analogues of the Virasoro algebra and the W_N algebras (the chiral algebras for conformal field theories with higher spin operators) [7]. The connection of ASD gauge theories with integrable systems is that ASD gauge theories are conjectured [8] to provide a unified description of all two-dimensional integrable systems. Systematic derivations of integrable systems by gauge and dimensional reduction of ASD gauge theories have given strong support to this conjecture [9],[10]. Virasoro and W_N symmetries emerge as the residual gauge symmetry of the ansätze for gauge and dimensional reduction. By virtue of this idea that ASD gauge theories are "master" integrable systems, and the known connection of integrable systems and conformal field theory, it is natural to look at ASD gauge theories for an extension of conformal field theoretic ideas to four dimensions. Quite apart from this, there are many hints that a four-dimensional Kähler manifold may be the natural setting for the study of W_N algebras [11]; in particular, the most appropriate description, from a geometric point of view, of W_∞ gravity in two dimensions may be in terms of ASD gravity on a four-dimensional Kähler manifold [12]. Thus it seems appropriate that we find a Lagrangian formulation of ASD gauge theory as a theory defined on a Kähler manifold, despite the fact that the notion of anti-self-duality exists on any Riemannian manifold.

From all the above we have ample motivation for the study of ASD gauge fields, especially in a Lagrangian and symplectic framework. We shall see that some of the symmetries of ASD gauge theories, known at the level of equations of motion, especially Bäcklund transformations, are realized as canonical transformations within KCST.

This paper is organized as follows. In section 2, we discuss the action and equations of motion of our theory. The (reduced) phase space is identified as the moduli space of ASD instantons, obtained by reduction of the space of gauge potentials on the Kähler manifold by the conditions of anti-self-duality. We discuss briefly the Hamiltonian version of the theory, and identify the symplectic form on the space of gauge potentials defined by the theory. In section 3 we discuss some of the properties of this symplectic form, specifically its gauge invariance, that it is an instance of the Donaldson μ-map [13], and its evaluation for specific instanton moduli spaces. We then look at the Poisson bracket algebra of our theory, and particularly the algebra of the anti-self-duality constraints. Classical and quantum symplectic reductions are discussed in section 4. We show that infinitesimal Bäcklund transformations are generated as canonical transformations by the anti-self-duality constraints. At the quantum level, the wave functions naturally involve a generalized form of the WZW action with an accompanying Polyakov-Wiegmann type factorization property. We examine briefly the chiral algebra associated with this action. In section 5, we specialize to the case of $\mathbf{R}^4$ as our Kähler manifold; we show that the infinitesimal transformations found in section 4 give the known "hidden symmetries" of the $\mathbf{R}^4$ ASD equations. We discuss reduction of the ASD equations to integrable systems such as KdV, putting known results into a unified framework, exploiting a critical insight from our theory. In section 6, in addition to some concluding discussion, we mention higher dimensional analogs of our theory.

2. Action, Equations of Motion and Phase Space

We define Kähler-Chern-Simons theory on a spacetime of form $M^4 \times \mathbf{R}$, where M^4 is a four (real) dimensional Kähler manifold, with a Kähler form denoted by ω. The action is taken to be

$$S = \int_{M^4 \times \mathbf{R}} \left[-\frac{k}{4\pi} Tr(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A) \wedge \omega + Tr\big((\Phi + \overline{\Phi}) \wedge \mathcal{F}\big) \right] \qquad (2.1)$$

A is the gauge potential; it is a locally defined 1-form on $M^4 \times \mathbf{R}$, with values in the Lie algebra of the gauge group G. We take G to be a compact semisimple Lie group; when needed we use a basis $\{T^i\}$ for the Lie algebra of G, with $Tr(T^i T^j) = -\frac{1}{2}\delta^{ij}$. $\mathcal{F}$ denotes the field strength, $\mathcal{F} = dA + A \wedge A$; we use F for the "magnetic field", i.e. that part of $\mathcal{F}$ which is a 2-form on M^4. Φ and $\overline{\Phi}$ are, respectively, locally defined, Lie algebra valued (2,0) and (0,2) forms on

M^4, which are also 1-forms on $\mathbf{R}$. Thus, if z^a, $a = 1,2$ denote local complex coordinates on M^4, and t denotes the coordinate on $\mathbf{R}$, then we can write locally

$$\begin{aligned} \Phi &= \phi \wedge dt, & \phi &= \tfrac{1}{2} \textstyle\sum_{a,b=1,2} \phi_{ab} dz^a \wedge dz^b, & \phi_{ab} &= -\phi_{ba} \\ \overline{\Phi} &= \overline{\phi} \wedge dt, & \overline{\phi} &= \tfrac{1}{2} \textstyle\sum_{\bar{a},\bar{b}=1,2} \overline{\phi}_{\bar{a}\bar{b}} d\bar{z}^{\bar{a}} \wedge d\bar{z}^{\bar{b}}, & \overline{\phi}_{\bar{a}\bar{b}} &= -\overline{\phi}_{\bar{b}\bar{a}} \end{aligned} \tag{2.2}$$

The behavior of the fields in the theory under gauge transformations is given by

$$\begin{aligned} A &\to A^u = uAu^{-1} - duu^{-1} \\ \Phi &\to \Phi^u = u\Phi u^{-1} \\ \overline{\Phi} &\to \overline{\Phi}^u = u\overline{\Phi} u^{-1} \end{aligned} \tag{2.3}$$

where u is a locally defined G-valued function on $M^4 \times \mathbf{R}$. By virtue of the fact that ω is closed, the action (2.1) is invariant under gauge transformations on $M^4 \times \mathbf{R}$ which are homotopic to the identity; in the case where M^4 is noncompact or has a boundary, it may be necessary to impose the vanishing of F at "infinity" or on the boundary (of course, in such a case, to completely define the theory it is necessary to give some boundary conditions, and we require that these should be compatible with gauge invariance of S). Invariance of e^{iS} under homotopically nontrivial gauge transformations can lead to quantization of k; we will discuss this quantization from another point of view later in the paper.

In local complex coordinates we write $\omega = \frac{i}{2} g_{a\bar{a}} dz^a \wedge d\bar{z}^{\bar{a}}$, and the Kähler metric is given by $ds^2 = g_{a\bar{a}} dz^a d\bar{z}^{\bar{a}}$. We remind the reader that the components of the Kähler form can be derived from a Kähler potential K, via $\omega = i\partial\bar{\partial}K$. On a Kähler 4-manifold, the notions of self-dual and anti-self-dual 2-forms become especially simple: a 2-form is anti-self-dual if it has no (2,0) or (0,2) part and its (1,1) part is perpendicular to ω (i.e. it vanishes upon taking the wedge product of it with ω).

The equations of motion of the theory from varying $\Phi, \overline{\Phi}, A_t$ are

$$F^{(2,0)} = F^{(0,2)} = 0 \tag{2.4a}$$

$$F \wedge \omega = 0 \tag{2.4b}$$

and express the fact that for all t the 2-form F on M^4 is anti-self-dual, i.e. that A_a,$A_{\bar{a}}$ is an instanton potential. The equations of motion from varying the spatial components of A yield, in the gauge $A_t = 0$,

$$\frac{k}{4\pi}\frac{\partial A_a}{\partial t} = -ig^{\bar{a}b}\overline{\nabla}_{\bar{a}}\phi_{ba} \tag{2.4c}$$

$$\frac{k}{4\pi}\frac{\partial A_{\bar{a}}}{\partial t} = +ig^{\bar{b}a}\nabla_a\overline{\phi}_{\bar{b}\bar{a}} \tag{2.4d}$$

Here $g^{\bar{a}a}$ is defined by $g_{b\bar{a}}g^{\bar{a}a} = \delta^a_b$, and we have denoted by ∇ and $\overline{\nabla}$ the gauge covariant versions of ∂ and $\bar{\partial}$ respectively, so, for instance, $\overline{\nabla}_{\bar{a}}\phi_{ba} = \partial_{\bar{a}}\phi_{ba} + [A_{\bar{a}}, \phi_{ba}]$. Since ∇, $\overline{\nabla}$ appear in the above equations acting, respectively, on (0,2) and (2,0) forms, we could replace both of them by the gauge covariant version of the exterior derivative $d = \partial + \bar{\partial}$. Furthermore, since on a Kähler manifold the Christoffel symbols $\Gamma^a_{b\bar{c}}$ and $\Gamma^{\bar{a}}_{\bar{b}c}$ vanish we can consider ∇, $\overline{\nabla}$ to be covariant with respect to both gauge and coordinate transformations. We note that we can write equations (2.4c,d) in coordinate independent form as, for instance,

$$\frac{k}{2\pi}\frac{\partial A^{(1,0)}}{\partial t} = -i * \overline{\nabla}\phi \tag{2.4e}$$

$$\frac{k}{2\pi}\frac{\partial A^{(0,1)}}{\partial t} = +i * \nabla\overline{\phi} \tag{2.4f}$$

where $*$ denotes the usual Hodge star operator. Equations (2.4c,d) give us the time evolution of the gauge potentials, or, rather, since the gauge potential is an instanton potential, and equations (2.4c,d) are gauge invariant under

t-independent gauge transformations (only these since we have fixed $A_t = 0$ gauge), equations (2.4c,d) give us the time evolution of the moduli of the instanton potential A_a,$A_{\bar{a}}$. We clearly need to check that the time evolution (2.4c,d) keeps us in the space of instanton potentials, and this will give us constraints on ϕ, $\overline{\phi}$. We will do this shortly.

First we make some other observations relevant to the action (2.1). We have not considered so far the equation of motion that could be obtained by varying the Kähler form ω, or, rather, varying the Kähler potential K. This is in line with our comments in the introduction that our theory should be viewed as being defined in a background Kähler metric. The equation of motion obtained by variation of K is satisfied automatically if the time derivatives of A_a,$A_{\bar{a}}$ are zero, which, as we shall see shortly, is certainly the case when M^4 is compact without boundary. Thus, although we use a metric expicitly, many of the properties of the theory will be independent of the metric, i.e. "topological" within the class of Kähler metrics.

The case where M^4 is hyperkähler merits some special comments. We then have, in addition to the Kähler form ω, a closed (2,0) form ω^+ and a closed (0,2) form ω^-, related to each other by complex conjugation. Writing $\omega^3 = \omega$, $\omega^1 = \frac{1}{2}(\omega^+ + \omega^-)$, $\omega^2 = \frac{1}{2i}(\omega^+ - \omega^-)$, equations (2.4a,b) may be written

$$F \wedge \omega^i = 0, \qquad i = 1, 2, 3 \tag{2.5}$$

These, along with the equations $\partial A_a/\partial t = \partial A_{\bar{a}}/\partial t = 0$, can be derived by variation of the imaginary quaternionic action

$$S' = -\frac{k}{4\pi} \int_{M^4 \times \mathbf{R}} Tr(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A) \wedge \omega^i e_i \tag{2.6}$$

where $e_\mu = (1, e_i)$ are a basis of quaternions. In this action we regard ω^i as fixed. For the case $M^4 = \mathbf{R}^4$, the action (2.6) has the advantage over the action (2.1) that it does not require one to pick a specific Kähler structure (which breaks $SO(4)$ invariance). But imposing hyperkähler structure may, in general, be too restrictive. We note that we can also consider the action (2.6) for a Riemannian manifold M^4 which is endowed with quaternion-Kähler structure [14]; it would then be necessary to use quaternionic quantization techniques [15].

We return now to consider the question of when the time evolution (2.4c,d) keeps us within the space of instanton potentials. Computing the time derivative of equation (2.4b) using equations (2.4c,d) we find no constraint on ϕ or $\overline{\phi}$. But taking the time derivative of $F^{(2,0)} = 0$, we find that ϕ must satisfy

$$\nabla(*\overline{\nabla}\phi) = 0 \tag{2.7}$$

For M^4 compact, with no boundary, this implies straightaway that $\overline{\nabla}\phi = 0$, since in this case we can write

$$\int_{M^4} Tr(\overline{\phi} \wedge \nabla(*\overline{\nabla}\phi)) = \int_{M^4} Tr((\overline{\overline{\nabla}\phi}) \wedge *(\overline{\nabla}\phi)) \stackrel{\text{def}}{=} -||\overline{\nabla}\phi||^2 \tag{2.8}$$

where $|| \quad ||$ denotes the standard norm. Thus for M^4 compact with no boundary, the time evolution is trivial and the equations of motion reduce to

$$F^{(2,0)} = F^{(0,2)} = 0$$

$$F \wedge \omega = 0 \tag{2.9}$$

$$\frac{\partial A_a}{\partial t} = \frac{\partial A_{\bar{a}}}{\partial t} = 0$$

In fact it is possible to say a little more than this; for the case where M^4 is compact, without boundary, and has positive scalar curvature it has been shown by Itoh [16], using a Bochner-type argument, that the dimension of the solution space of (2.7) is zero. This result is of importance in computing the dimension of the instanton moduli spaces for such manifolds, using the index of the twisted Dolbeault complex. The complex dimension of the moduli space (if it is not empty) of q-instantons on such a manifold is [16] $4q - \frac{1}{4} dimG(\chi + \tau)$, where χ is the Euler number and τ is the signature of the manifold; furthermore the moduli spaces have Kähler structure [17]. These moduli spaces are exactly the phase spaces for our theory (at least for M^4 compact, without boundary, and with positive scalar curvature).

For more general M^4 one can have nontrivial solutions to (2.7). We shall see below that changes in $A^{(1,0)}$ of form $*\overline{\nabla}\phi$ are generated canonically by the action of $F^{(0,2)}$ (and similarly the action of $F^{(2,0)}$ generates changes in $A^{(0,1)}$ of the form $*\nabla\overline{\phi}$). The equation $F^{(0,2)} = 0$ shows that such transformations may be regarded as a different type of gauge symmetry. The time evolution of A_a and $A_{\bar{a}}$ is flow along such gauge directions, and is still therefore trivial in a larger sense.

Consider now the Hamiltonian version of the theory, treating the coordinate t as time. The action (2.1) immediately gives the following first class constraints on the canonical momenta

$$\pi_{A_t} = 0 \qquad \pi_\phi = 0 \qquad \pi_{\overline{\phi}} = 0 \tag{2.10}$$

We can eliminate these constraints by choosing the gauge fixing constraints

$$A_t = 0 \qquad \phi = 0 \qquad \overline{\phi} = 0 \tag{2.11}$$

but we have to further impose the equations

$$\frac{\partial \pi_{A_t}}{\partial t} = 0 \qquad \frac{\partial \pi_\phi}{\partial t} = 0 \qquad \frac{\partial \pi_{\overline{\phi}}}{\partial t} = 0 \tag{2.12}$$

as constraints, which give equations (2.4a,b). Time evolution is once again trivial.

One more ingredient is necessary to complete the picture of the classical phase space of the theory. The action (2.1) determines naturally a symplectic 2-form on the space $\mathcal{A}$ of all gauge potentials on M^4,

$$\Omega = \frac{k}{4\pi} \int_{M^4} Tr(\delta A \wedge \delta A) \wedge \omega \tag{2.13}$$

Here δ denotes the exterior derivative on $\mathcal{A}$, and we have suppressed the wedge product between forms on $\mathcal{A}$. So our theory essentially describes the symplectic reduction of the space $\mathcal{A}$, endowed with the symplectic form Ω of (2.13), by $F^{(2,0)}$, $F^{(0,2)}$ and $F \wedge \omega$. $F \wedge \omega$ is , with respect to the symplectic form Ω, the generator of usual gauge transformations, as we shall see later, so (2.4b) is the "Gauss law" of the theory.

3. Properties of the Symplectic Form and the Algebra of Constraints

Ω, being nondegenerate, cannot be gauge invariant on the full space $\mathcal{A}$, but the general theory of symplectic reductions [18] dictates that there should be gauge invariance on the subset of $\mathcal{A}$ where the moment map arising from gauge transformations vanishes, which is the set of potentials such that $F \wedge \omega = 0$. It is interesting to see this gauge invariance emerge explicitly, without appealing to the general theory of symplectic reductions. Let us take the case where M^4 is compact and without boundary first. The Lie algebra valued 1-form A is only defined locally on M^4, so to compute (2.13) we need to introduce a set of patches on M^4, and to sum the contributions to (2.13) from a set of patches that exactly cover M^4. Explicitly, let $\{B_p\}$ be a (sufficiently large) collection of closed sets that cover M^4, with $B_p \cap B_q = \partial B_p \cap \partial B_q \stackrel{\text{def}}{=} \partial B_{pq}$. Let A_p represent the gauge potential on B_p; we are given a set of G-valued transition functions h_{pq}, one for each pair p,q such that $p \neq q$ and $\partial B_{pq} \neq 0$ (these satisfy the usual relations). On ∂B_{pq} we have $A_p = h_{pq} A_q h_{pq}^{-1} - dh_{pq} h_{pq}^{-1}$; gauge transformations act via $A_p \to g_p A_p g_p^{-1} - dg_p g_p^{-1}$, where g_p is a Lie group valued function on B_p and on ∂B_{pq} we have $g_p = h_{pq} g_q h_{pq}^{-1}$*. Under gauge transformations we have $\delta A_p \to g_p\big(\delta A_p + D_{A_p}(g_p^{-1}\delta g_p)\big) g_p^{-1}$, where D_{A_p} denotes the gauge covariant derivative. With all this one finds, after integration by parts and a few other simple manipulations,

$$\begin{aligned}\sum_p \int_{B_p} Tr(\delta A_p \wedge \delta A_p) \wedge \omega \to & \sum_p \int_{B_p} Tr(\delta A_p \wedge \delta A_p) \wedge \omega \\ & + 2\delta \left(\sum_p \int_{B_p} Tr\big((g_p^{-1}\delta g_p) F(A_p)\big) \wedge \omega \right) \\ & + \sum_p \int_{\partial B_p} Tr\big(2\delta A_p (g_p^{-1}\delta g_p) + (g_p^{-1}\delta g_p) D_{A_p}(g_p^{-1}\delta g_p)\big) \wedge \omega \end{aligned} \tag{3.1}$$

On the subspace of $\mathcal{A}$ where $F \wedge \omega$ vanishes, the second term in this expression is clearly zero. For the third term there are two contributions from ∂B_{pq}, which is contained in both ∂B_p and ∂B_q, and these can be shown to cancel; thus this term is also zero. This establishes gauge invariance. For noncompact manifolds, or manifolds with boundary, it clearly is necessary to impose certain boundary

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* Gauge transformations are correctly defined as fiber-preserving automorphisms of the underlying principal bundle. The gauge potentials A_p are obtained from the bundle connection in terms of local trivializations. Gauge transformations can then be described in terms of local G-valued functions g_p with the prescribed overlap relation $g_p = h_{pq} g_q h_{pq}^{-1}$.

conditions to avoid contributions to the third term in (3.1).

Despite the fact that Ω is not gauge invariant on the whole space $\mathcal{A}$, it does define a (non-trivial) element of the second cohomology $H^2(\mathcal{A}/\mathcal{G})$, where $\mathcal{G}$ is the group of gauge transformations. This follows from the fact that if we define

$$\Omega^1(A,g) = \int Tr\big((g^{-1}\delta g)F(A)\big) \wedge \omega \tag{3.2}$$

(Ω^1 is the 1-form on $\mathcal{A}$ appearing in the second term in (3.1) above, and is in some sense the obstruction to gauge invariance) then $\Omega^1(A,g)$ obeys a cocycle condition

$$\Omega^1(A,hg) = \Omega^1(A,g) + \Omega^1(A^g,h) \tag{3.3}$$

Furthermore, the cohomology class of Ω in $H^2(\mathcal{A}/\mathcal{G})$ depends only on the cohomology class of ω in $H^2(M^4)$. This is easily seen if we write Ω as

$$\Omega = \frac{k}{4\pi} \int Tr(\tilde{F} \wedge \tilde{F}) \wedge \omega \tag{3.4a}$$

where

$$\tilde{F} = (d+\delta)A + A \wedge A \tag{3.4b}$$

Then we have, for $\omega \to \omega + d\alpha$,

$$\Omega \to \Omega + \delta\left[\frac{k}{4\pi} \int Tr(\tilde{F} \wedge \tilde{F}) \wedge \alpha\right] \tag{3.5}$$

Since $\int Tr(\tilde{F} \wedge \tilde{F}) \wedge \alpha$ is a 1-form on $\mathcal{A}/\mathcal{G}$, it follows that the cohomology class of Ω is not changed. Actually, Ω is an example of the Donaldson map from $H^2(M^4)$ to $H^2(\mathcal{A}/\mathcal{G})$. (This is the cohomology version of the Donaldson map, as described in [13]. We have an extra factor $2\pi k$, since our Ω is derived from an action, which is measured in units of 2π.)

With this understanding we can now discuss the quantization of k, for the case where M^4 is compact and without boundary. If k is an integer, then, assuming that ω represents an integer cohomology class of M^4, the requirement that k is an integer is exactly the requirement that $\Omega/2\pi$ represents an integer cohomology class of $\mathcal{A}/\mathcal{G}$. (Being Kähler, M^4 has nontrivial homology 2-cycles; this implies that $\pi_1(\mathcal{G})$ is nonzero, which implies that there are nontrivial homology 2-cycles in $\mathcal{A}/\mathcal{G}$.) The relevance of this to the quantum theory, is that in a geometric quantization of our theory we construct a line bundle, called the prequantum line bundle, on the phase space with curvature Ω; sections of this line bundle satisfying a certain polarization condition are the wave functions. The existence of the prequantum line bundle requires that the integral of $\Omega/2\pi$ over any nontrivial homology 2-cycle in the phase space should be an integer [19], i.e. $\Omega/2\pi$ must belong to an integral cohomology class, which, as stated above, means k must be an integer. Let us see this explicitly in a specific example, say $M^4 = S^2 \times S^2$. We consider a homology 2-cycle in $\mathcal{A}/\mathcal{G}$, parametrized by σ, τ, $0 \leq \sigma, \tau \leq 1$, given by

$$A(x^1, x^2, \sigma, \tau) = -\tau dgg^{-1} \tag{3.6}$$

where x^1, x^2 are coordinates of one of the component S^2's of M^4, and $g(x^1, x^2, \sigma)$, with $g(x^1, x^2, 0) = g(x^1, x^2, 1) = 1$, is a nontrivial element of $\pi_3(G) = \mathbf{Z}$. Integration of Ω over this 2-cycle gives

$$\int \Omega = 2\pi k Q[g] \int_{S^2} \omega \tag{3.7}$$

where $Q[g]$ is the winding number of g, i.e.

$$Q[g] = \frac{1}{24\pi^2} \int_{S^3} Tr(dgg^{-1})^3 \tag{3.8}$$

Since $Q[g]$ and $\int \omega$ are integers, k must be an integer. This example we have given here is, since we essentially ignore the second component S^2 in M^4 in construction of the nontrivial 2-cycle in $\mathcal{A}/\mathcal{G}$, a simple extension of an argument that can be given for the quantization of k in three dimensional Chern-Simons theory; for arbitrary M^4 we use the nontrivial homology 2-cycles in place of the component S^2's of the example. For noncompact M^4 there is no quantization of k.

Having now established various properties of Ω, we consider briefly the evaluation of Ω on specific instanton moduli spaces. For instanton solutions we have $F^{(0,2)} = 0$, allowing us to write locally $A^{(0,1)} = -\overline{\partial} U U^{-1}$, where U is $G^{\mathbf{C}}$-valued. It follows that on some patch B we can write $A = UA'U^{-1} - dUU^{-1}$ where A' is a $G^{\mathbf{C}}$ Lie algebra valued (1,0) form, $A' = (U^\dagger U)^{-1}\partial(U^\dagger U)$. We have $\delta A = U\big(\delta A' - D'_A(U^{-1}\delta U)\big)U^{-1}$; comparing with (3.1) and using the fact that if A is an instanton potential we must have $F(A') \wedge \omega = 0$, we deduce that the contribution to Ω from the patch B is simply given by

$$\Omega_B = \int_{\partial B} Tr\big(2\delta A'(U^{-1}\delta U) + (U^{-1}\delta U) D_{A'}(U^{-1}\delta U)\big) \wedge \omega \qquad (3.9)$$

We see at once that Ω can be calculated on an instanton moduli space by summing contributions from surfaces of patches. Note that here, unlike in the calculation leading to gauge invariance above, the contributions to Ω from boundaries of neighboring patches do not in general cancel; adopting the patching notation from earlier on in this section, it is easy to see that on ∂B_{pq} we have $U_p = h_{pq} U_q g_{pq}$, where the g_{pq}'s are some holomorphic $G^{\mathbf{C}}$ matrices, depending on the moduli of the instanton potential A. There is some freedom in choosing the matrices g_{pq}, arising from the freedom in choice of the matrices U_p, but it is insufficient freedom to set them all to the identity (if we could set them all to the identity the sum of all the contributions of type (3.9) would vanish). More precisely, the matrices g_{pq} define a holomorphic vector bundle on M^4, and the freedom we have in the choice of g_{pq}'s means that a specific instanton solution determines exactly an isomorphism class of holomorphic vector bundles. This is one part of Donaldson's theorem [5] that states that there is an isomorphism between moduli of (irreducible) $SU(N)$ instanton potentials on M^4 and moduli of (stable) holomorphic rank-N vector bundles on M^4.

We now turn to an examination of the Poisson bracket algebra of our theory. The canonical Poisson brackets following from (2.13) for the components of A (defined by writing $A = (A^i_a dz^a + A^i_{\bar{a}} d\bar{z}^{\bar{a}})T^i$) are

$$[A^i_a(x), A^j_{\bar{a}}(y)] = \frac{2\pi}{ik} g_{a\bar{a}} \delta^{ij} \frac{\delta^{(4)}(x-y)}{\det(g)} \tag{3.10}$$

Here $\det(g) = \det(g_{a\bar{a}})$. The basic structure of the theory is the symplectic reduction of $\mathcal{A}$ by the constraints $F^{(2,0)}, F^{(0,2)}, F \wedge \omega$, so central to the quantization procedure is the algebra of these functions. We introduce the following generators

$$\begin{aligned} E(\overline{\varphi}) &= -\frac{k}{2\pi}\int Tr(\overline{\varphi} \wedge F) = \frac{k}{2\pi}\int dV g^{\bar{a}a} g^{\bar{b}b} \overline{\varphi}^i_{\bar{a}\bar{b}} F^i_{ab} \\ \overline{E}(\varphi) &= -\frac{k}{2\pi}\int Tr(\varphi \wedge F) = \frac{k}{2\pi}\int dV g^{\bar{a}a} g^{\bar{b}b} \varphi^i_{ab} F^i_{\bar{a}\bar{b}} \\ G(\theta) &= \frac{-k}{2\pi}\int Tr(\theta\omega \wedge F) = \frac{-ik}{2\pi}\int dV g^{\bar{a}a} \theta^i F^i_{a\bar{a}} \end{aligned} \tag{3.11}$$

where $\theta, \varphi, \overline{\varphi}$ are, respectively, Lie algebra valued (0,0),(2,0) and (0,2) forms (which essentially serve as parameters for the transformations generated by $F \wedge$

$\omega, F^{(0,2)}, F^{(2,0)}$); the components of $\theta, \varphi, \overline{\varphi}$ are defined by

$$\begin{aligned} \theta &= \sum \theta^i T^i \\ \varphi &= \tfrac{1}{2} \sum \varphi^i_{ab} T^i dz^a \wedge dz^b, \qquad \varphi^i_{ab} = -\varphi^i_{ba} \\ \overline{\varphi} &= \tfrac{1}{2} \sum \overline{\varphi}^i_{\bar{a}\bar{b}} T^i d\bar{z}^{\bar{a}} \wedge d\bar{z}^{\bar{b}}, \qquad \overline{\varphi}^i_{\bar{a}\bar{b}} = -\overline{\varphi}^i_{\bar{a}\bar{b}} \end{aligned} \tag{3.12}$$

In equation (3.11) the volume element is given by $dV = \frac{1}{4} d^2 z d^2 \bar{z} \det(g)$. We find the following Poisson brackets of these generators with the components of A:

$$\begin{aligned} [G(\theta), A^i_a(x)] &= -(\nabla\theta)^i_a(x) & [G(\theta), A^i_{\bar{a}}(x)] &= -(\overline{\nabla}\theta)^i_{\bar{a}}(x) \\ [E(\overline{\varphi}), A^i_a(x)] &= 0 & [E(\overline{\varphi}), A^i_{\bar{a}}(x)] &= i(*\nabla\overline{\varphi})^i_{\bar{a}}(x) \\ [\overline{E}(\varphi), A^i_a(x)] &= -i(*\overline{\nabla}\varphi)^i_a(x) & [\overline{E}(\varphi), A^i_{\bar{a}}(x)] &= 0 \end{aligned} \tag{3.13}$$

These tell us the facts that we have cited previously, that $F \wedge \omega$ is the generator, with respect to the symplectic form (2.13) on $\mathcal{A}$, of gauge transformations, and $F^{(0,2)}, F^{(2,0)}$ are, respectively, the generators of transformations of the form

$$\begin{aligned} A^{(1,0)} &\rightarrow A^{(1,0)} - i * \overline{\nabla}\varphi \\ A^{(0,1)} &\rightarrow A^{(0,1)} + i * \nabla\overline{\varphi} \end{aligned} \tag{3.14}$$

The symplectic form (2.13) on $\mathcal{A}$ is Lie-invariant with respect to flows on $\mathcal{A}$ generated by both these transformations and gauge transformations. The three constraints of our theory are moment maps corresponding to these three sets of transformations. This result generalizes both the observation of Donaldson [5] that $F \wedge \omega$ is the moment map corresponding to gauge transformations, and also the result [20] that the anti-self-dual equations on $\mathbf{R}^4$ (or for that matter any hyperkähler manifold) can be obtained via a hyperkähler reduction.

It remains to write down the algebra of the generators $E(\overline{\varphi}), \overline{E}(\varphi), G(\theta)$. We obtain

$$[G(\theta), G(\theta')] = G(\theta \times \theta') \tag{3.16a}$$

$$[G(\theta), E(\overline{\varphi})] = E(\theta \times \overline{\varphi}) \tag{3.16b}$$

$$[G(\theta), \overline{E}(\varphi)] = \overline{E}(\theta \times \varphi) \tag{3.16c}$$

$$[\overline{E}(\varphi), E(\overline{\varphi})] = \frac{ik}{2\pi} \int Tr(\overline{\varphi} \wedge \nabla * \overline{\nabla} \varphi) \tag{3.16d}$$

Here $\theta \times \theta' = f^{ijk}\theta^j \theta'^k T^i$ etc. $E(\overline{\varphi})$ and $E(\overline{\varphi'})$, and $\overline{E}(\varphi)$ and $\overline{E}(\varphi')$ evidently commute. We note that on compact M^4 without boundary, we can rewrite the right hand side of (3.16d) using (2.8). The algebra (3.16) is clearly of central importance in our theory. We should mention that while we obtained the symplectic structure (2.13) from our action (2.1), it is also that natural symplectic form to consider on $\mathcal{A}$ from the point of view of the connection of the moduli space of instantons with the moduli space of holomorphic vector bundles via Donaldson's theorem mentioned above. We thus, independently from the point of view of Kähler-Chern-Simons theory, expect the algebra (3.10),(3.13),(3.16) to play a significant role in the quantization of instantons, i.e. the construction of line bundles over the moduli spaces of holomorphic vector bundles on M^4.

In the quantum theory, $A, E, \overline{E}, G$ become operators, and the Poisson bracket relations (3.10),(3.13),(3.16) are replaced by commutators. Notice that right hand side of (3.16d) contains both the holomorphic and anti-holomorphic components of A, and hence one has to take care with operator ordering in (3.16d).

4. Symplectic Reductions, Classical and Quantum

The $G(\theta), \overline{E}(\varphi)$ operators have a closed Poisson bracket algebra, given by (3.16a,c). We attempt a two-stage (classical) symplectic reduction of $\mathcal{A}$, first setting $\overline{E}(\varphi)$ (or equivalently $F^{(0,2)}$) to zero, and then setting $G(\theta)$ (or equivalently

$F \wedge \omega$) to zero. The $\overline{E}(\varphi) = 0$ subspace of $\mathcal{A}$ consists of potentials locally of the form

$$(A_a, A_{\bar{a}}) = (A_a, -\partial_{\bar{a}} U U^{-1}) \tag{4.1}$$

where U is $G^{\mathbf{C}}$-valued. The flow on this subspace generated by $\overline{E}(\varphi)$ is given by

$$\begin{aligned} &U \to U \\ &A_a \to A_a' = A_a - i(*\overline{\nabla}\varphi)_a \end{aligned} \tag{4.2}$$

We need a gauge fixing condition that will restrict us to the orbit space of the flow (4.2). We note that under an infinitesimal change of the form (4.2) we have

$$F^{(2,0)} \to F^{(2,0)} - i\nabla * \overline{\nabla}\varphi \tag{4.3}$$

Thus it follows that provided there are no solutions to equation (2.7), the condition $F^{(2,0)} = 0$ (or equivalently $E(\overline{\varphi}) = 0$) will be a good gauge fixing condition. Consistently with this, we note that if there are no solutions of (2.7), then the right hand side of equation (3.16d) can be regarded as an invertible inner product on the space of Lie algebra valued (2,0)-forms φ; the invertibility of this inner product is exactly the criterion for $E(\overline{\varphi}) = 0$ to be a good gauge fixing for flows generated by $\overline{E}(\varphi)$. We are of course, only considering infinitesimal flows at this stage. Note that, even if there are no solutions to (2.7), the gauge fixing $F^{(2,0)} = 0$ could in principle suffer from a Gribov ambiguity. The solution to $F^{(2,0)} = 0$ satisfying the appropriate reality conditions is

$$A_a = (U^\dagger)^{-1} \partial_a U^\dagger \tag{4.4}$$

The phase space after reduction by $\overline{E}(\varphi)$ is thus given by the space of U's, U being a locally defined $G^{\mathbf{C}}$-valued function. Gauge transformations act on this space via $U \to gU$.

Continuing with discussion of the case where equation (2.7) has no solutions, let us proceed to the second stage reduction, by setting $G(\theta)$ to zero and dividing out by gauge transformations. The condition $G(\theta) = 0$ is easily seen to be

$$g^{\bar{a}a}\partial_{\bar{a}}(J^{-1}\partial_a J) = 0 \tag{4.5}$$

where $J = U^\dagger U$. J is gauge invariant. (4.5) gives the instanton equations in the so-called J-formulation. The reduced phase space now (i.e. the solutions of (4.5)) is the moduli space of instantons.

Consider now the case of manifolds where there are nontrivial solutions to equation (2.7), and impose $\overline{E}(\varphi) = 0$. Imposing $E(\overline{\varphi}) = 0$ will not be a good gauge fixing for the flow generated by the $\overline{E}$'s. More specifically, if A_a solves $E = 0$ then so will A'_a defined by (4.2) provided $\nabla * \overline{\nabla}\varphi = 0$. Another way to view this is as follows: if $\nabla * \overline{\nabla}\varphi = 0$, then locally we can write $*\overline{\nabla}\varphi = \nabla\sigma$ for some Lie algebra valued function σ, and it is straightforward to check that for such φ we have

$$[\overline{E}(\varphi), E(\overline{\varphi})] = -iE(\sigma \times \overline{\varphi}) \tag{4.6}$$

So the flow generated by $\overline{E}(\varphi)$ for these φ leaves the condition $E = 0$ invariant, i.e. in the space $\overline{E} = 0$, some of the flows generated by $\overline{E}$ are tangential to the subspace $E = 0$; in other words, whereas in the compact case $E = 0$ was a good gauge fixing for the symmetries generated by $\overline{E}$ on the space $\overline{E} = 0$, in the noncompact case there is an infinitesimal Gribov ambiguity.

Now instanton solutions are defined by $E = \overline{E} = 0$ and $G = 0$. From what we have said above, for the first two conditions we have two solutions, both with $A_{\bar{a}} = -\partial_{\bar{a}}UU^{-1}$. If we take the first to be the "real" solution, i.e.

$A_a = (U^\dagger)^{-1}\partial_a(U^\dagger)$, then the second is given by

$$\begin{aligned} A'_a &= (U^\dagger)^{-1}\partial_a(U^\dagger) - i(*\overline{\nabla}\varphi)_a \\ &= (U^\dagger)^{-1}\partial_a(U^\dagger) - i(\nabla\sigma)_a \\ &\stackrel{\text{def}}{=} (V^\dagger)^{-1}\partial_a(V^\dagger) \end{aligned} \tag{4.7}$$

where $V^\dagger = U^\dagger e^{-i\sigma}$ (φ and σ are infinitesimal). If $U^\dagger, U$ solves $F \wedge \omega = 0$, then so will $V^\dagger, U$. Thus $\overline{E}$-flow on the $\overline{E} = 0$ subspace may be regarded as generating new instanton solutions from old ones via solutions of $\nabla * \overline{\nabla}\varphi = 0$. This will be made more precise for the case of $\mathbf{R}^4$ in the next section. The new solution will clearly not satisfy reality conditions; but we can also consider the analogous E-flow on the space $E = 0$, which will use solutions of $\overline{\nabla} * \nabla\overline{\varphi} = 0$ to generate new instanton solutions. Taking the right combination of these flows we can generate new real solutions from old real solutions. Here we shall just write equations (4.7) in a more useful form. The quantities relevant for the condition $G = 0$ are $J = U^\dagger U$ and $J' = V^\dagger U$. We find

$$(J^{-1}\partial J) - (J'^{-1}\partial J') = iU^{-1}(\nabla\sigma)U \tag{4.8}$$

Note that equation (4.8) implies that if J satisfies (4.5) so does J' (use the identity $\overline{\partial}(U^{-1}\alpha U) = U^{-1}(\overline{\nabla}\alpha)U$). Thus (4.8) is an infinitesimal Bäcklund transformation for (4.5).

We now turn to the subject of quantum reductions. We can consider quantization of our theory in two ways. We can quantize the whole space $\mathcal{A}$, endowed with the symplectic form Ω, and then impose the constraints (2.4) by restricting to the set of wave functions annihilated by the appropriate operators. Alternatively, we can directly quantize the reduced phase space, i.e. the subset of $\mathcal{A}$

defined by equations (2.4). We shall consider the first method, and at the end of this section make some brief comments on the second method. For quantizing the space $\mathcal{A}$, a natural choice of polarization for the wave functions is the holomorphic polarization, i.e. the wave functions are functionals of $A^i_{\bar{a}}$ with the action of A^i_a given by

$$A^i_a \Psi(A^i_{\bar{a}}) = \frac{2\pi}{k} g_{a\bar{a}} \frac{\delta \Psi}{\delta A^i_{\bar{a}}} \tag{4.9}$$

The scalar product is given by

$$\langle \Psi_1 | \Psi_2 \rangle = \int e^{-\tilde{K}} \Psi_1^* \Psi_2 d\mu(A) \tag{4.10}$$

where $d\mu(A)$ is the Liouville measure on $\mathcal{A}$ given by Ω, and $\tilde{K}$ is the Kähler potential for Ω, i.e.

$$\tilde{K} = \frac{k}{2\pi} \int dV g^{\bar{a}a} A^i_a A^i_{\bar{a}} \tag{4.11}$$

We perform a first stage of reduction by requiring

$$\overline{E}(\varphi) \Psi(A^i_{\bar{a}}) = 0 \tag{4.12}$$

This implies that the wavefunctions have support only on configurations for which $F^{(0,2)} = 0$. For the simplest case, this means $A_{\bar{a}} = -\partial_{\bar{a}} U U^{-1}$ for some *global* G-valued function U on M^4 (in this case, of course, the instanton number is zero). We can now consider Ψ's to be functionals of U. The scalar product becomes

$$\langle \Psi_1 | \Psi_2 \rangle = \int [dU] e^{-\tilde{K}(U^\dagger, U)} \Psi_1^*(U) \Psi_2(U) \tag{4.13}$$

where $[dU]$ is defined as the product over M^4 of the Haar measure on $G^{\mathbf{C}}$. The second stage of reduction is now performed by imposing gauge invariance, i.e.

$$G(\theta) \Psi(U) = 0 \tag{4.14}$$

Using the definition of $G(\theta)$ from (3.11) and (4.9), we see that this is equivalent to

$$\Psi(e^{\theta}U) = \exp\left\{-\frac{k}{\pi}\int dV g^{\bar{a}a} Tr(\partial_a\theta\partial_{\bar{a}}UU^{-1})\right\}\Psi(U) \tag{4.15}$$

for infinitesimal θ. The solution to this is given by $\Psi = e^{\mathcal{S}(U)}$, where

$$\mathcal{S}(U) = \frac{k}{2\pi}\int_{M^4} dV g^{\bar{a}a} Tr\big(\partial_a U\partial_{\bar{a}}U^{-1}\big) + \frac{ik}{12\pi}\int_{M^5} Tr\big((U^{-1}dU)\wedge(U^{-1}dU)\wedge(U^{-1}dU)\big)\wedge\omega \tag{4.16}$$

This is an analogue of the Wess-Zumino-Witten (WZW) action which appears in the wave functions for three dimensional Chern-Simons theory. In this expression M^5 is taken to be $M^4\times[0,1]$; we identify one boundary component of M^5 (say $\lambda = 1$, where λ is the coordinate on $[0,1]$) with our space M^4, and extend U into M^5 in such a way that it tends to some fixed function U_0 on the other component of the boundary ($\lambda = 0$). Depending on what M^4 is, the set of $G^{\mathbf{C}}$-valued functions on M^4, i.e. the set of U's, may fall into distinct homotopy classes. Then we would need to specify a set of fixed functions U_0 on $\lambda = 0$, one in each homotopy class, in order to define $\mathcal{S}(U)$, and we define the extension of U into M^5 so that it tends to the appropriate U_0 on $\lambda = 0$. Note that (4.15) only gives the behavior of the Ψ's under homotopically trivial gauge transformations. The behavior of Ψ's under homotopically non-trivial transformations can involve additional phase factors (the same for all states), in a way analogous to the θ-vacua of QCD [21].

Before we go on to look at (4.16) as an action, we note that in a more general case than the one we have considered, when $F^{(0,2)} = 0$ cannot be solved in terms of a globally defined function U, but rather we have $A_{\bar{a}} = -\partial_{\bar{a}}UU^{-1}$

where U is only locally defined, we still expect a factor of the form $e^{\mathcal{S}(U)}$ in the wave functions, where $\mathcal{S}$ is now some refined notion of the functional (4.16) (to construct this we could exploit, for example, [22], where it is shown how to construct the usual two dimensional WZW functional for a Riemann surface with boundary). In the general case, the gauge invariance condition will not determine a unique wave function either; the residual freedom has to do with the degrees of freedom of the reduced phase space or moduli. The quantization of the latter will complete the identification of the wave functions. We will not consider these issues here (apart from some comments at the end of this section), but content ourselves with some discussion of the functional (4.16) as an action.

(4.16) clearly can be used as an action for a field theory on any Kähler manifold M^4. The action satisfies a Polyakov-Wiegmann type formula

$$\mathcal{S}(U_1U_2) = \mathcal{S}(U_1) + \mathcal{S}(U_2) + \Gamma(U_1, U_2) \tag{4.17a}$$

$$\Gamma(U_1, U_2) = -\frac{k}{\pi}\int_{M^4} dV g^{\bar{a}a} Tr\big(U_1^{-1}(\partial_a U_1)(\partial_{\bar{a}} U_2)U_2^{-1}\big) \tag{4.17b}$$

By use of this formula we see that the normalization integral in (4.13) will involve $e^{\mathcal{S}(J)}$. From (4.17) it is clear that the variational equation for $\mathcal{S}(J)$ is equation (4.5). Thus we may think of $\mathcal{S}(J)$ as an action for anti-self-dual gauge theory in the J-formulation*, something that has been sought in the past [23]. Note that if we choose any parametrization for the group G, the action (4.16) can be

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* J in equation (4.5), of course, is not globally defined, but on the intersection of patches we have $J_p = g^\dagger_{pq} J_q g_{pq}$ where g_{pq} is holomorphic. From (4.17b) we see that $\mathcal{S}(U)$ is unaffected by multiplication on the left(right) by antiholomorphic(holomorphic) matrices. Thus $\mathcal{S}(J)$ can be defined for all instanton numbers.

expressed as an integral over M^4 [24]; so for instance we can obtain Pohlmeyer's action [25] for Yang's equations [26].

Equation (4.17) shows that the transformations $U \to hU$, $U \to U\tilde{h}$, where h is antiholomorphic and $\tilde{h}$ is holomorphic, are gauge symmetries of the action (4.16). (Obviously such symmetries also exist for $\mathcal{S}(J)$; this has been noticed before at the level of the equations of motion.) In the case of the usual WZW action, these are, of course, the Kac-Moody symmetries. One can then consider a Hamiltonian quantization using the holomorphic coordinate z as the time coordinate [24]. An analogous quantization can be carried out in our case, using, say, z_1 as the time variable (of course, this procedure is only meaningful for four manifolds for which one can define a time coordinate consistently, e.g. $M^4 = \Sigma \times \mathbf{R}^2$, where Σ is a Riemann surface). The symplectic 2-form, on the space $\mathcal{G}^{(3)}$ of G-valued functions of $z_2, \bar{z}_1, \bar{z}_2$, given by (4.17) is then

$$\Omega^{(3)} = \frac{k}{4\pi} \int d^2\bar{z} dz_2 \det(g) g^{\bar{a}1} Tr\big(\xi \partial_{\bar{a}} \xi + 2\xi\xi(\partial_{\bar{a}} U)U^{-1}\big) \tag{4.18}$$

where $\xi = \delta U U^{-1}$ is a 1-form on $\mathcal{G}^{(3)}$, and we suppress the wedge product between forms on $\mathcal{G}^{(3)}$. The basic current density of interest is

$$I = \frac{k}{4\pi} \det(g) g^{\bar{a}1} (\partial_{\bar{a}} U) U^{-1} \tag{4.19}$$

The Poisson bracket algebra of these quantities is determined from (4.18) to be

$$[I^i(z_2, \bar{z}_1, \bar{z}_2), I^j(z_2', \bar{z}_1', \bar{z}_2')] = f^{ijk} I^k(z_2, \bar{z}_1, \bar{z}_2) \delta^{(3)}(z - z') - \frac{k}{4\pi} \det(g) g^{\bar{a}1} \partial_{\bar{a}} \delta^{(3)}(z - z') \delta^{ij} \tag{4.20}$$

The antiholomorphic symmetries $U \to hU$ are generated by $Q(\bar{z}_1, \bar{z}_2) = \int dz_2 I$. The algebra of Q's is obtained from (4.20) as

$$[Q^i(\bar{z}), Q^j(\bar{z}')] = f^{ijk} Q^k(\bar{z}) \delta^{(2)}(\bar{z} - \bar{z}') - \frac{k}{4\pi} \delta^{ij} C(\bar{z}, \bar{z}') \tag{4.21a}$$

$$C(\bar{z},\bar{z}') = \int dz_2 \det(g) g^{\bar{a}1} \partial_{\bar{a}} \delta^{(2)}(\bar{z}-\bar{z}') \tag{4.21b}$$

This algebra is obviously similar to the Kac-Moody algebra. However it is very limited in its utility in solving the theory defined by (4.16); this is because, unlike its two dimensional analogue, the solution space of equation (4.5) is not given just by some finite dimensional space of solutions, up to multiplication on the left and right by antiholomorphic and holomorphic matrices respectively.

Finally in this section, we note that the quantization of the reduced phase space can be carried out in a relatively straightforward way by calculating Ω on a specific instanton moduli space. The Hilbert space will be characterized by k and by q, the instanton number. As mentioned in section 2, the complex dimension of the moduli space (on a suitable manifold) is $4q - \frac{1}{4} dim G(\chi + \tau)$; furthermore the moduli space has finite volume, for compact M^4. Thus the number of states (dimension of the Hilbert space) will be finite. For noncompact M^4 this is not the case.

It would be interesting to provide a specific example of the reduced phase space quantization, but as far as we know, no one has succeeded in writing down all anti-self-dual instantons for any compact Kähler manifold, for any value of q for which instantons are known to exist, even for $G = SU(2)$, which is the only case we will consider in this paragraph. For the case of $\mathbf{C}P^2$, where all self-dual 1-instantons are known [27], there are no anti-self-dual 1-instantons [28], and the complex dimension of the anti-self-dual q-instanton moduli space for $q \geq 2$ is $4q - 3$. By virtue of Donaldson's theorem relating moduli spaces of instantons and moduli spaces of holomorphic vector bundles, we can identify at least the topology of the 2-instanton moduli space, which must be isomorphic [17] to the complement in $\mathbf{C}P^5$ of the hypersurface

$$z_0z_1z_2 + 2z_3z_4z_5 - z_0z_5^2 - z_2z_3^2 - z_1z_4^2 = 0 \tag{4.22}$$

Here $z_0, ..., z_5$ are homogeneous coordinates for $\mathbf{C}P^5$. For the case of the 4-torus $(S^1)^4$, some anti-self-dual 1-instantons are known [29], and the complex dimension of the q-instanton moduli space is $4q$. The 1-instanton moduli space is conjectured to be isomporphic to the product of a 4-torus and a K3 surface [30]; if this is true, finding all 1-instantons on the torus would provide us with a Kähler structure on a K3 surface. The closest we can get, for now, to an explicit calculation, is to consider the evaluation of Ω for the subset of solutions of the $\mathbf{R}^4$ anti-self-dual equations that are S^4 q-instantons. The real dimensions of these moduli spaces are well-known to be $8q - 3$, which is odd, so Ω is degenerate (which is not in contradiction with anything we have said). The evaluation for 1-instantons can be found in [1]; since we have shown above that Ω can be calculated as a sum of surface terms, we suspect it is possible to compute Ω for $q > 1$ explicitly, but have not, as of yet, succeeded in doing this.

5. Analysis on R^4

$\mathbf{R}^4$ is an interesting example of M^4, since as mentioned in the introduction, the study of instantons on $\mathbf{R}^4$ is relevant to the study of integrable systems. We shall look at reductions to integrable systems shortly. Before this, however, we explian how some of the general analysis of KCST can be carried farther in the special case M^4=$\mathbf{R}^4$.

Consider the (classical) symplectic reduction of $\mathcal{A}$ by $\overline{E}(\varphi)$ and $G(\theta)$. Equation (2.7) now has solutions, or, equivalently, we can find Lie algebra valued

0-forms and (2,0) forms σ and φ such that

$$*\overline{\nabla}\varphi = \nabla\sigma \tag{5.1}$$

The flow generated by $\overline{E}(\varphi)$ for such φ will generate new instanton solutions from old. Note that equation (5.1) implies

$$\begin{aligned} *\nabla * \overline{\nabla}\varphi &= 0 \\ *\overline{\nabla} * \nabla\sigma &= 0 \end{aligned} \tag{5.2}$$

Thus both φ and σ satisy the covariant Laplace equation. On $\mathbf{R}^4$ we have a covariantly constant (2,0) tensor ϵ_{ab}, and one can write $\varphi_{ab} = f\epsilon_{ab}$; f and σ satisfy the same equation. So we might try choosing f to be a multiple of σ, i.e. we might look for solutions of (5.1) with

$$\varphi_{ab} = \tfrac{1}{2}\lambda\sigma\epsilon_{ab} \tag{5.3}$$

where λ is a complex constant. (For a general hyperkähler manifold M^4 we might look for solutions of (5.1) with $\varphi = \lambda\sigma\omega^+$, in the notation of section 2.) From the definition of $V^\dagger$ in equation (4.7) we see that

$$J^{-1}J' = U^{-1}e^{-i\sigma}U \approx 1 - iU^{-1}\sigma U \tag{5.4}$$

Using (5.3),(5.4) we can rewrite (4.8)

$$J^{-1}\partial_a J - J'^{-1}\partial_a J' = -\lambda g^{\bar{a}b}\epsilon_{ba}\partial_{\bar{a}}(J^{-1}J') \tag{5.5}$$

This is exactly the form of the inifinitesimal Bäcklund transformations and the associated infinite dimensional symmetry of the ASD equations on $\mathbf{R}^4$ [31]. We see these are indeed generated as canonical transformations by $\overline{E}(\varphi)$. One may regard equation (4.7) as giving the analogue of the Bäcklund transformation on a general Kähler manifold M^4.

Equation (5.2a) is of course the same as (2.7); it is the variational equation obtained by varying $||\overline{\nabla}\varphi||^2$, the functional defined in (2.8). It is clear that the only way we can have a solution to (5.2a) with $\overline{\nabla}\varphi$ nonvanishing, is to have a nonzero surface integral of the form

$$\int_{\partial M^4} Tr(\overline{\varphi} \wedge *\overline{\nabla}\varphi) \tag{5.6}$$

(here $\overline{\varphi}$ is the hermitean conjugate of φ). The asymptotic behavior of $J^{-1}J'$ on $\mathbf{R}^4$ will thus be non-trivial.

We now turn to the consideration of reductions of the ASD equations on $\mathbf{R}^4$ that give rise to integrable systems. In [9] Mason and Sparling showed that in a certain reduction to two dimensions, the $SL(2,\mathbf{C})$ ASD equations yielded both the KdV and non-linear Schrödinger (NLS) equations, and these were (up to gauge transformations) essentially the only reductions of this kind. In [10], Bakas and Depireux, realised that the Mason-Sparling reduction to KdV could be gauge transformed into a particularly simple form, and by taking an ansatz of this form for larger gauge groups, found many more integrable equations arising as dimensional reductions of the ASD equations. In the context of this paper, it would be appropriate to fully realise these reductions in a symplectic framework, but we do not do this here. We shall, however, reconsider these reductions with a particular insight from our work. In [9],[10] it is apparent that the equations $F^{(0,2)} = 0$ and $F \wedge \omega = 0$ play a different role from the equation $F^{(2,0)} = 0$. This distinction is natural, of course, in our symplectic framework, and furthermore, from our viewpoint it is more natural to gauge fix after imposing $F^{(0,2)} = 0$ and

$F \wedge \omega = 0$ (these are moment maps in our presentation). Following this procedure, at least partially, we show that the Bakas-Depireux ansätze are actually *gauge choices*; this increases the significance of their results substantially.

We introduce complex coordinates w, z on $\mathbf{R}^4$. We dimensionally reduce by restriciting to potentials that in some gauge are independent of $\bar{w}$. We clearly still have the freedom to do gauge transformations that are independent of $\bar{w}$, under which we have

$$A_{\bar{w}} \rightarrow u A_{\bar{w}} u^{-1} \tag{5.7}$$

where $u(w, z, \bar{z})$ is the gauge transformation matrix. Exploiting this freedom, we can put $A_{\bar{w}}$ into a canonical form. For $G = SL(N, \mathbf{C})$, the possible canonical forms are just the possible Jordan normal forms for a traceless $N \times N$ matrix function of the variables $w, z, \bar{z}$. For $SL(2, \mathbf{C})$ we have two possible forms

$$A_{\bar{w}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \textit{or} \quad \kappa \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.8}$$

and for $SL(3, \mathbf{C})$ we have four possible forms

$$A_{\bar{w}} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} \kappa & 0 & 0 \\ 1 & \kappa & 0 \\ 0 & 0 & -2\kappa \end{pmatrix} \quad \textit{or} \quad \begin{pmatrix} \kappa & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -\kappa - \lambda \end{pmatrix} \tag{5.9}$$

Here κ, λ are arbitrary functions of $w, z, \bar{z}$. Each canonical form gives rise to a different type of reduction: the two $SL(2, \mathbf{C})$ choices give KdV and NLS type equations, as found in [9]; equations based on the first two $SL(3, \mathbf{C})$ choices are considered in [10]. Each canonical form also has associated with it a set of residual gauge transformations u, which leave it invariant; for the first $SL(2, \mathbf{C})$ form we can take

$$u = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \tag{5.10}$$

for some function γ of $w, z, \bar{z}$. Thus for each reduction we will obtain a whole gauge equivalence class of integrable systems, and this is the notion of gauge equivalence described in [32].

To proceed further let us look at a specific example; we will look at the first $SL(2,\mathbf{C})$ form, i.e. the KdV type reduction, but it is straightforward to work out any particular example. We parametrize the remaining potentials suitably

$$A_w = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \quad A_z = \begin{pmatrix} d & e \\ f & -d \end{pmatrix} \quad A_{\bar{z}} = \begin{pmatrix} g & h \\ j & -g \end{pmatrix} \tag{5.11}$$

All the entries in these matrices are functions of $w, z, \bar{z}$. We will *not* exploit the remaining gauge freedom at this juncture. Instead, we impose $F_{\bar{w}\bar{z}} = F_{w\bar{w}} + F_{z\bar{z}} = 0$ in line with our comments above. The potentials then must take the form

$$A_w = \begin{pmatrix} (j_z - f_{\bar{z}} - 2dj)/2 & d_{\bar{z}} - je \\ c & -(j_z - f_{\bar{z}} - 2dj)/2 \end{pmatrix} \quad A_z = \begin{pmatrix} d & e \\ f & -d \end{pmatrix} \quad A_{\bar{z}} = \begin{pmatrix} 0 & 0 \\ j & 0 \end{pmatrix} \tag{5.12}$$

where $e_{\bar{z}} = 0$. We see we are left with unfixed functions c, d, e, f, j with $e_{\bar{z}} = 0$. e is unchanged by the residual gauge transformations, and it actually emerges that any choice of e will give us an integrable system. The choice $e = 0$ gives a trivial system, so we will look at $e = 1$. Having fixed e, we have four functions left, one of which corresponds to the gauge degree of freedom γ of equation (5.10), and the remaining three of which will be "almost fixed" by imposing $F_{wz} = 0$ (these equations do not fix e in any way). Under gauge transformation with u given by (5.10), we find

$$\begin{aligned} c &\to c - \gamma_w + \gamma(j_z - f_{\bar{z}} - 2dj) + \gamma^2(j - d_{\bar{z}}) \\ d &\to d - \gamma \\ f &\to f + 2\gamma d - \gamma^2 - \gamma_z \\ j &\to j - \gamma_z \end{aligned} \tag{5.13}$$

We consider three possible gauge choices, $j = 0$ (Mason-Sparling gauge), $d = 0$ (Bakas-Depireux gauge) and $f = 0$ (MKdV gauge). In these three gauges we obtain, respectively, the following equations by imposing $F_{wz} = 0$ (and making suitable choices of integratiom coefficients):

$$\begin{aligned}(d_z)_w &= [\tfrac{1}{4}\partial_z^2 + 2(d_z) + (d_z)_z\partial_z^{-1}](d_z)_{\bar{z}} \\ f_w &= [\tfrac{1}{4}\partial_z^2 - f - \tfrac{1}{2}f_z\partial_z^{-1}]f_{\bar{z}} \\ d_w &= [\tfrac{1}{4}\partial_z^2 - d^2 - d_z\partial_z^{-1}d]d_{\bar{z}}\end{aligned} \tag{5.14}$$

These are three dimensional versions [33] of the KdV, KdV and MKdV equations respectively. Further reduction to two dimensions by imposing $\partial_z = \partial_{\bar{z}}$, as in [9] [10], yields the standard equations. For gauge group $SL(N, \mathbf{C})$, for any N, in the case when $A_{\bar{w}}$ is chosen in the canonical form with exactly one non-zero entry, not on the leading diagonal, it is straightforward to define the gauge choice to reproduce the ansätze in [10]. We note that, in addition to clarifying some issues of gauge freedom in reductions of the ASD equations to integrable systems, we also inherit from our work on the ASD equations a full understanding of the hidden symmetries of the integrable systems we obtain (see also [34]).

6. Concluding Remarks

The emergence of the anti-self-dual equations in the symplectic reduction of the space of gauge potentials $\mathcal{A}$ by $F^{(0,2)}$ and $F\wedge\omega$ is perhaps the most important feature of Kähler-Chern-Simons theory. The algebra (3.10),(3.13),(3.16) of the gauge potentials and the generators $F^{(0,2)}$, $F^{(2,0)}$, $F\wedge\omega$ plays a crucial role in this picture. Within this framework, the previously known “hidden symmetries” of the instanton equations, related to Bäcklund transformations, can be understood as canonical transformations. Our discussion in section 5 shows that reduction of $\mathcal{A}$ by $F^{(0,2)}$ and $F \wedge \omega$ is also the most appropriate setting for the gauge and dimensional reductions of ASD gauge fields leading to two-dimensional integrable systems. In this context we expect the study of analogous reductions and subalgebras of (3.10),(3.13),(3.16) to shed light on how Virasoro and W_N symmetries emerge, and the role they play, in two-dimensional integrable systems.

As an obvious generalization of what we have presented here, we can consider a KCST theory on a Kähler manifold of arbitrary even (real) dimension M^{2d}, $d \geq 2$. The natural action to look at is

$$S = \int_{M^{2d}\times\mathbf{R}} \left[-\frac{k}{4\pi} Tr(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A) \wedge \omega^{d-1} + Tr\big((\Phi + \overline{\Phi}) \wedge \mathcal{F}\big) \right] \tag{6.1}$$

where now Φ and $\overline{\Phi}$ are, respectively, $(d,d-2)$ and $(d-2,d)$ forms on M^{2d}, as well as being 1-forms on $\mathbf{R}$. The equations of motion (for suitable M^{2d}) are just

$$F^{(2,0)} = F^{(0,2)} = 0 \tag{6.2a}$$

$$\omega^{d-1} \wedge F = 0 \tag{6.2b}$$

On $\mathbf{R}^{2d}$, these equations were studied, amongst others, as possible candidates for the appropriate higher dimensional extension of the ASD equations [35]. The main reason we have focused our attention in this paper on the $d = 2$ case, is that for $d > 2$ it seems these equations of motion are not integrable on $\mathbf{R}^{2d}$. They cannot be written as consistency conditions for the integrability of a set of linear equations [36], and in the J-formulation (obtained by solving (6.2a) to write $A_{\bar{a}} = -\partial_{\bar{a}} U U^{-1}$, $A_a = (U^\dagger)^{-1} \partial_a (U^\dagger)$, in which case (6.2b) becomes $g^{\bar{a}a} \partial_{\bar{a}} (J^{-1} \partial_a J) = 0$, where $J = U^\dagger U$), the equation of motion fails the Painlevé test [37]. Nevertheless, the set of solutions to (6.2) on an arbitrary Kähler manifold M^{2d}, might well merit study; we are not aware of any work on the relationship of the moduli spaces of solutions to (6.2) and the moduli spaces of holomorphic vector bundles on M^{2d}. In [9] it is shown that higher order equations in the KdV hierarchy can be obtained from equations (6.2) on $\mathbf{R}^{2d}$, for appropriate choices of d. Also we note that much of what we have said in this paper goes through for the action (6.1); particularly, in the procedure of quantization we find the obvious extension to higher dimensions of the WZW functional (4.16).

Certain remarks in the last paragraph also may help clarify the distinction between 3d CST and KCST. The equations (6.2) on $\mathbf{R}^2$ are completely solvable, but on $\mathbf{R}^4$ they are only "integrable". The notion of integrability, for partial differential equations, is a (currently) non-precise notion, which reflects a degree of solvability, falling just short of the notion of complete solvability, which we take to mean the ability to write down explicitly the most general solution. This reinforces to us the possibility that KCST is the appropriate arena to discuss the host of phenomena now known that are generalizations, in one sense or another, of conformal field theory.

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III. The Self-Dual Yang-Mills Equations as a Master Integrable System

1. Introduction

The self-dual Yang-Mills (SDYM) equations first arose in physics in the context of the semiclassical approximation to Yang-Mills theories [1] in Euclidean space; solutions of the first order SDYM equations are also solutions of the full second order Yang-Mills equations, and in fact provide all the local minima of the Yang-Mills action [2]. Thus, determining all solutions to the SDYM equations was considered an important problem, which found a solution in the ADHM construction [3]. The combined efforts of mathematicians and physicists yielded a substantial body of knowledge about the SDYM equations and their solutions; amongst the findings [4] were an infinite dimensional symmetry group of the space of solutions, an infinite number of "conserved currents" for the equations, and the fact that the equations could be viewed as consistency conditions for a set of linear equations. By virtue of these results, the SDYM equations were afforded the status of an "integrable system"; it was probably the first known four-dimensional integrable system, and remains until today one of few [5].

Despite our relatively good understanding of the SDYM equations, to date little reliable information has been obtained from the semiclassical approximation to Yang-Mills theories, and its validity remains in question. The SDYM equations, however, have produced very useful results in mathematics, in the study of the topology of 4-manifolds. The SDYM equations can be formulated on an arbitrary Riemannian 4-manifold, and the equations contain explicit metric dependence. The space of solutions (of fixed instanton number) of the SDYM equations depends on the metric, but the *topology* of the space of solutions should, intuitively, be insensitive to slight variations in the metric, and therefore it appears that we can study the topology of 4-manifolds by studying the topology of the "moduli spaces" (i.e. spaces of solutions of the SDYM equations of fixed instanton number). This approach has been remarkably fruitful, and the interested reader is referred to [6].

In physics, the SDYM equations (specifically on $\mathbf{R}^4$ with the flat metric) have recently become of interest again for an entirely different reason. As mentioned above, the SDYM equations are one of the few known examples of four-dimensional integrable systems (moreover, the SDYM equations can of course be formulated for arbitrary gauge group, so they actually provide a large number of four-dimensional integrable systems). Ward [7] has conjectured that *all* two-dimensional integrable systems are contained in the SDYM equations, and can be obtained from them by suitable reduction (there also exist 3-dimensional integrable systems, such as the Kadomtsev-Petviashvili (KP) and Davey-Stewartson (DS) equations, but for reasons we shall present below these are not thought to be contained in the SDYM equations for any finite dimensional gauge group). Substantial evidence for Ward's conjecture has been found by two groups, Mason and Sparling (MS) [8] and Bakas and Depireux (BD) [9]. MS systematically considered reduction of the $SL(2,\mathbf{C})$ SDYM equations to two dimensions by imposing one null and one timelike symmetry. They found that there were essentially two such reductions, which yielded the Korteweg-De Vries equation (KdV) and the non-linear Schrödinger equation (NLS). BD considered reductions of the SDYM equations to two dimensions by imposing either one null and one timelike symmetry or two null symmetries; by taking suitable ansätze for the gauge fields they found a number of equations in the generalized KdV hierarchies, and also some new equations, which they dubbed “fractional KdV” type equations. All this will be presented in detail below.

The main purpose of this paper is to give a method of systematic reduction of the SDYM equations by imposing just one null symmetry (clearly all of the equations of MS and BD will be contained in the equations we find). A brief presentation of the scheme, and some results for gauge group $SL(2,\mathbf{C})$, has already been given in [10], of which this paper is essentially a more detailed version, with a number of new results. It will become apparent that the BD ansätze for reduction of the SDYM equations are actually *gauge choices* for certain types of reduction, substantially increasing the importance of their results.

The structure of this paper is as follows. In section 2 we give all the necessary background material on the SDYM equations, and also on the generalized KdV hierarchies and the Drinfeld-Sokolov [11] notion of gauge equivalence for the KdV hierarchies. In section 3 we explain our reduction method, and work out the case of gauge group $SL(2,\mathbf{C})$. We obtain KdV, MKdV, Gardner KdV, NLS, Sine-Gordon and Liouville equations, and generalizations. In section 4 we present some general facts about the reduction for gauge group $SL(N,\mathbf{C})$ for arbitrary N. In section 5 we look at $SL(3,\mathbf{C})$ in detail. In section 6, we will make some concluding remarks, and briefly discuss different possible approaches to the problem of obtaining the hierarchies of integrable two-dimensional equations from the SDYM equations.

Before leaving the introduction, it is appropriate to make a few comments about the relevance of integrable systems in two-dimensional physics, or, more specifically, in conformal field theory and two-dimensional gravity. Integrable systems are related to conformal field theories in two, apparently different ways. First, certain perturbations of conformal field theories away from criticality give rise to "integrable field theories" (i.e. field theories where the equations of motion are exactly the two-dimensional integrable systems we will be seeing) [12]. Second, the Poisson bracket algebras associated with two-dimensional integrable systems are classical analogues of the Virasoro algebra and the W_N algebras and their extensions, the operator algebras of conformal field theory [13]. The relation of integrable systems with two-dimensional (quantum) gravity is also twofold: first, two-dimensional gravity is described by Liouville field theory [14], and second, the partition functions of topological gravity theories (or of the equivalent matrix models) are given in terms of certain "tau functions" which are solutions of certain hierarchies of integrable equations [15]. While all these facts are currently unrelated, it seems it may be useful to have a general framework of integrable systems within which to understand them, particularly since this framework, the SDYM equations, is quite possibly of relevance in physical four-dimensional theories.

Throughout this paper we will focus solely on the process of systematically reducing the SDYM equations to obtain integrable systems. In light of the comments of the previous paragraph, it is also very important to study the Poisson bracket algebras of the integrable systems we obtain, but we leave this for later consideration. BD have studied the algebra associated with the simplest fractional KdV equation in [9], and found it is the W_3^2 algebra of Polyakov and Bershadsky [16]. We will see that there exists a certain way to reduce the $SL(N,\mathbf{C})$ SDYM equations for *any* partition of N into integers: the partition 1^N gives a trivial system, the partition 2.1^{N-2} gives the integrable system associated with the W_N algebra, and the partition 3 for $N=3$ gives the integrable system associated with the W_3^2 algebra. This suggests that possibly there are other W-type algebras that have as of yet not been found for $N \geq 4$ (unless of course there are coincidences between the algebras associated with the integrable systems we find). It seems there may well be a relation between the new integrable systems we predict and some recent work of de Groot, Hollowood and Miramontes [17].

2. Background Material

The self-dual Yang-Mills equations are usually written in the form

$$\begin{aligned} F_{12} &= F_{34} \\ F_{13} &= F_{42} \\ F_{14} &= F_{23} \end{aligned} \tag{2.1}$$

Here the fundamental fields are Lie algebra valued gauge fields A_μ, $\mu = 1,2,3,4$, and the "field strengths" are defined by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. The equations (2.1) are invariant under gauge transformations

$$A_\mu \to A'_\mu = uA_\mu u^{-1} - \partial_\mu u u^{-1} \tag{2.2}$$

where u is a gauge group valued function; under this $F_{\mu\nu} \to F'_{\mu\nu} = uF_{\mu\nu}u^{-1}$.

Yang [18] observed that by defining complex coordinates suitably (via, for instance, $w = x^1 + ix^2$, $z = x^3 - ix^4$), and defining new gauge fields appropriately (via $A_z dz + A_w dw + A_{\bar{z}} d\bar{z} + A_{\bar{w}} d\bar{w} = A_\mu dx^\mu$), equations (2.1) can be written

$$F_{zw} = 0 \tag{2.3a}$$

$$F_{\bar{z}\bar{w}} = 0 \tag{2.3b}$$

$$F_{z\bar{z}} + F_{w\bar{w}} = 0 \tag{2.3c}$$

(here $F_{zw} = \partial_z A_w - \partial_w A_z + [A_z, A_w]$ etc.). We will work throughout with the equations in this form, using complex coordinates. Most of the time we will not be considering real gauge groups, and therefore we will treat $z, \bar{z}, w, \bar{w}$ as independent.

There are two useful simplifications of the system (2.3). We can, following [18], solve the equations (2.3a),(2.3b) to write

$$\begin{aligned} A_w &= -\partial_w D D^{-1} \qquad & A_z &= -\partial_z D D^{-1} \\ A_{\bar{w}} &= -\partial_{\bar{w}} E E^{-1} \qquad & A_{\bar{z}} &= -\partial_{\bar{z}} E E^{-1} \end{aligned} \tag{2.4}$$

where D, E are some gauge group valued functions (which are related if we impose some reality condition on the potentials). Under gauge transformations we find $D \to uD$, $E \to uE$; note that the gauge potentials are unchanged if we multiply D (E) on the right by a antiholomorphic (holomorphic) gauge group valued function, so there is some freedom of choice in D and E. Substituting equations (2.4) back into (2.3c) we find that solving the SDYM equations is essentially equivalent to solving the equation

$$\partial_{\bar{z}}(J^{-1}\partial_z J) + \partial_{\bar{w}}(J^{-1}\partial_w J) = 0 \tag{2.5}$$

where $J = D^{-1}E$. J is clearly gauge invariant. A solution of equations (2.3) defines a solution of (2.5) up to multiplication on the left by an antiholomorphic gauge group valued function and on the right by a holomorphic gauge group valued function; a solution of (2.5) determines a gauge equivalence class of solutions of (2.3) (the gauge degree of freedom arises from the freedom we have in choosing D, E to satisfy $J = D^{-1}E$). Equation (2.5) is known as the "J-formulation" of the SDYM equations. We should mention that equation (2.5) can be rewritten

$$\partial_z(J\partial_{\bar{z}}J^{-1}) + \partial_w(J\partial_{\bar{w}}J^{-1}) = 0 \tag{2.6}$$

To obtain the other simplified version of the system (2.3) we observe that equation (2.3b) allows us to choose a gauge in which $A_{\bar{z}} = A_{\bar{w}} = 0$. Equation (2.3c), in this gauge, reads

$$\partial_{\bar{z}}A_z + \partial_{\bar{w}}A_w = 0 \tag{2.7}$$

which we can solve, to give

$$\begin{aligned} A_z &= \partial_{\bar{w}}N \\ A_w &= -\partial_{\bar{z}}N \end{aligned} \tag{2.8}$$

for some Lie algebra valued function N. Substituting in the remaining equation (2.3a) we find we need

$$\partial_w\partial_{\bar{w}}N + \partial_z\partial_{\bar{z}}N + [\partial_{\bar{w}}N, \partial_{\bar{z}}N] = 0 \tag{2.9}$$

Writing $M = \partial_{\bar{w}}N$ (also a Lie algebra valued function), we can write this

$$\partial_w M = -(\partial_{\bar{w}}^{-1}\partial_z\partial_{\bar{z}}M + [M, \partial_{\bar{w}}^{-1}\partial_{\bar{z}}M]) \tag{2.10}$$

(understanding the inverse of $\partial_{\bar{w}}$ in some suitable fashion), or equivalently

$$M_w = -\mathcal{L}M_{\bar{z}} \tag{2.11a}$$

$$\mathcal{L} = \partial_{\bar{w}}^{-1}\partial_z + [M, \partial_{\bar{w}}^{-1} \quad] \tag{2.11b}$$

(here suffices denote derivatives). This formulation is due to Bruschi, Levi and Ragnisco [19], who also pointed out that the operator $\mathcal{L}$ defined in (2.11b) is a recursion operator [20] for the equation (2.11a) (and in fact for a large class of equations, including the equations $M_w = \mathcal{L}^n M_{\bar{z}}$, $n = 2, 3, 4, ...$ which can be viewed as "higher order" SDYM equations; we shall say more of this later). The integrability of the SDYM equations (in the sense of the existence of an infinite number of conserved quantities) is immediate once this recursion operator has been identified. We note that the SDYM equations in the form (2.11) can be regarded as an integrable evolution equation with w identified as the "time". To obtain local equations (i.e. not involving integration operators) we need to reduce the system by specifying the form of the $\bar{w}$ dependence. All this lends some (admittedly *a posteriori*) motivation for the approach we will take to reduction of the SDYM equations; we shall consider solutions of the SDYM with potentials independent of $\bar{w}$, and we shall obtain integrable evolution equations by solving equations (2.3b) and (2.3c) first, and then imposing (2.3a). We will reiterate this later, with examples.

Having said that the integrability of the SDYM equations can be seen with ease in what we will call the "M-formulation" (2.11), it is interesting to see the emergence of an infinite number of conserved quantities for the SDYM equations in J-formulation (2.5), for which we will give an argument which is a slightly modified version of that given by Chau* [4]. We will construct, by an inductive

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* We deviate from Chau's argument in the form of some covariant derivatives we will shortly define; Chau's argument is perfectly valid, but the conserved

procedure, an infinite number of "currents" $V_{\bar{w}}^{(n)}, V_{\bar{z}}^{(n)}$, $n = 1, 2, 3, ...$ and an infinite number of quantities $\xi^{(n)}$, $n = 0, 1, 2, ...$ such that

$$\partial_w V_{\bar{w}}^{(n)} + \partial_z V_{\bar{z}}^{(n)} = 0 \tag{2.12a}$$

$$\begin{aligned} V_{\bar{w}}^{(n)} &= \nabla_{\bar{w}} \xi^{(n-1)} \\ V_{\bar{z}}^{(n)} &= \nabla_{\bar{z}} \xi^{(n-1)} \end{aligned} \tag{2.12b}$$

Here we have defined some covariant derivatives

$$\begin{aligned} \nabla_{\bar{w}} &= \partial_{\bar{w}} + [J\partial_{\bar{w}} J^{-1}, \quad] \\ \nabla_{\bar{z}} &= \partial_{\bar{z}} + [J\partial_{\bar{z}} J^{-1}, \quad] \end{aligned} \tag{2.13}$$

Clearly once we have proved the existence of the currents, the quantities

$$Q^{(n)}(\bar{w}, \bar{z}) = \int dz V_{\bar{w}}^{(n)} \tag{2.14}$$

are an infinite number of quantities independent of w (which we have identified as time). We start the inductive proof by taking

$$\xi^{(0)} = \rho \tag{2.15a}$$

$$\begin{aligned} V_{\bar{w}}^{(1)} &= [J\partial_{\bar{w}} J^{-1}, \rho] \\ V_{\bar{z}}^{(1)} &= [J\partial_{\bar{z}} J^{-1}, \rho] \end{aligned} \tag{2.15b}$$

where ρ is an arbitrary constant element in the Lie algebra. These satisfy (2.12) by virtue of equation (2.6). For the inductive step, suppose we are given $V_{\bar{w}}^{(p)}, V_{\bar{z}}^{(p)}, \xi^{(p-1)}$ satisfying (2.12). Then we can define $\xi^{(p)}$ via

$$\begin{aligned} V_{\bar{w}}^{(p)} &= \partial_z \xi^{(p)} \\ V_{\bar{z}}^{(p)} &= -\partial_w \xi^{(p)} \end{aligned} \tag{2.16}$$

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quantities she constructs do *not* live in the Lie algebra of the gauge group, but rather in its universal enveloping algebra. Our argument will give conserved quantities in the Lie algebra.

exploiting (2.12a), and we define $V_{\bar{w}}^{(p+1)}, V_{\bar{z}}^{(p+1)}$ by

$$\begin{aligned} V_{\bar{w}}^{(p+1)} &= \nabla_{\bar{w}} \xi^{(p)} \\ V_{\bar{z}}^{(p+1)} &= \nabla_{\bar{z}} \xi^{(p)} \end{aligned} \tag{2.17}$$

We need to check that $V_{\bar{w}}^{(p+1)}, V_{\bar{z}}^{(p+1)}$ satisfy (2.12a), which is a straightforward calculation, using (2.6) again, and the fact $[\nabla_{\bar{w}}, \nabla_{\bar{z}}] = 0$. Thus in the J-formulation we also can see the infinite number of conserved quantities. Dolan, in [4], gives a construction of the infinite number of conserved quantities directly for equations (2.3) without reference to any simplified formulation of the SDYM equations. This is actually a straightforward extension of the proof above, which is a "gauge-fixed" version of Dolan's proof, in the gauge $A_w = A_z = 0$ (in our proof above we use covariant derivatives $\nabla_{\bar{w}}, \nabla_{\bar{z}}$ and ordinary partial derivatives ∂_w, ∂_z; in a general gauge it is necessary to repeat the analysis above with all derivatives covariant).

When we reduce the SDYM equations the conserved quantities above will furnish an infinite number of conserved quantities for the reduced equations (in general though, it seems to be a tricky issue to check that they are the same conserved quantities as those more usually constructed directly for the integrable systems we obtain; we will not consider this issue in this paper). Another structure that reduced systems will inherit from the SDYM equations is a Lax pair formulation [21]. The equations (2.3) can be viewed as the consistency conditions for the linear equations

$$\begin{aligned} (\nabla_w + \lambda \nabla_{\bar{z}})\psi &= 0 \\ (\nabla_z - \lambda \nabla_{\bar{w}})\psi &= 0 \end{aligned} \tag{2.18}$$

Here $\nabla_\mu = \partial_\mu + [A_\mu, \ \ ]$, ψ is a Lie algebra valued function,λ is the "spectral parameter", i.e. some complex parameter, and we wish (2.18) to be consistent for all λ. Equivalently, equations (2.3) can be compactly written

$$[\nabla_w + \lambda\nabla_{\bar{z}}, \nabla_z - \lambda\nabla_{\bar{w}}] = 0 \tag{2.19}$$

(expanding order by order in λ gives equations (2.3)). On reduction, by taking some suitable form for the gauge potentials, this will yield immediately a Lax pair formulation for the reduced integrable system. From (2.19) it is also possible to understand, at an intuitive level, the origin of the integrability of the SDYM equations. Equations (2.19) are invariant under gauge transformations (2.2) even with u depending on λ, provided the combinations $A_w+\lambda A_{\bar{z}}$, $A_z-\lambda A_{\bar{w}}$ transform into linear expressions of λ. A little calculation shows that this is a substantially larger freedom than the freedom of gauge transformations with u independent of λ. Also from (2.19) we can see why we do not expect to obtain the KP or DS equations from SDYM: the KP and DS equations are believed *not* to have a Lax pair formulation of the type (2.19).

This completes the background material we need on the SDYM equations. For completeness we also define, and describe the notion of gauge equivalence of, the generalized KdV hierarchies, the integrable systems we will be trying to obtain and to generalize in this paper. The rth equation in the N-KdV hierarchy can be written in the form

$$L_w = [(L^{r/N})_+, L] \tag{2.20}$$

Here L is an Nth order "monic" [22] differential operator in the variable z with coefficients that are functions of w and z, i.e.

$$L = \partial_z^N + u_2(w,z)\partial_z^{N-2} + u_3(w,z)\partial_z^{N-3} + \ldots + u_{N-1}(w,z)\partial_z + u_N(w,z) \tag{2.21}$$

The Nth root of L is defined as a "psuedodifferential operator"; to compute it we suppose it takes the form

$$L^{1/N} = \partial_z + v_{-1}\partial_z^{-1} + v_{-2}\partial_z^{-2} + \dots \tag{2.22}$$

where $v_{-1}, v_{-2}, ...$ are some functions of w, z; we raise this to the Nth power and compare with L term by term. To determine the form of the equation (2.20) it is only necessary to know some finite number of terms in $L^{1/N}$, as by the expression $(L^{r/N})_+$ in (2.20) we mean the part of the rth power of $L^{1/N}$ that involves non-negative powers of ∂_z. Usually simpler than this procedure is to just write

$$(L^{r/N})_+ = \partial_z^r + w_2\partial_z^{r-2} + w_3\partial_z^{r-3} + \dots + w_r \tag{2.23}$$

and to substitute into (2.20) and determine the functions $w_2, w_3, ...w_r$ by requiring that the right hand side of (2.20) is a differential operator of order $N-2$, which it must be for consistency of the equation. This determines the functions $w_2, w_3, ..., w_r$. For more details on this construction see for instance [22]. Note that the equation (2.20) is trivial when r is a multiple of N, and when $r = 1$ just gives (a number of copies of) the equation $u_w = u_z$.

More relevant, for us, is another presentation of the KdV hierarchies: the equation (2.20) can, it turns out, also be written in the form

$$\tilde{L}_w - A_z + [A, \tilde{L}] = 0 \tag{2.24}$$

where

$$\tilde{L} = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ u_N + \lambda & u_{N-1} & u_{N-2} & u_{N-3} & \dots & u_2 & 0 \end{pmatrix} \tag{2.25}$$

and A is an $N \times N$ matrix, the dependence of which upon λ determines r (apart from this input, A is fixed by requiring the consistency of (2.24)). Let us consider the simplest example: for $N = 2$, we have

$$\tilde{L} = \begin{pmatrix} 0 & 1 \\ u_2 + \lambda & 0 \end{pmatrix} \tag{2.26}$$

and we take

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \tag{2.27}$$

Using equation (2.24) we eliminate α, γ to get

$$(u_2)_w = -\tfrac{1}{2}\beta_{zzz} + 2\beta_z(u_2 + \lambda) + \beta(u_2)_z \tag{2.28}$$

If we now specify the λ dependence of β (u_2 is independent of λ), by taking $\beta = \sum_{p=0}^{s} \beta_p \lambda^p$, where the functions β_p are independent of λ, and we take $\beta_s = 1$ (from (2.8) it has to satisfy $(\beta_s)_z = 0$), then we find that u_2 must satisfy

$$(u_2)_w = (\tfrac{1}{4}\partial_z^2 - u_2 - \tfrac{1}{2}(u_2)_z \partial_z^{-1})^s (u_2)_z \tag{2.29}$$

This is the $r = 2s + 1$ equation in the standard ($N = 2$) KdV hierarchy (and despite appearances does not involve nonlocal operators): for $s = 1$ we have the standard KdV equation

$$(u_2)_w = \tfrac{1}{4}(u_2)_{zzz} - \tfrac{3}{2}u_2(u_2)_z \tag{2.30}$$

Note that the forms of $\tilde{L}$ we have taken in (2.24) to obtain the KdV equations are not unique. One of the first derivations of the $N = 2$ KdV hierarchy in this fashion is due to Ablowitz, Kaup, Newell and Segur [23], who used

$$\tilde{L} = \begin{pmatrix} \lambda & q_1 \\ q_2 & -\lambda \end{pmatrix} \tag{2.31}$$

where q_1, q_2 are some functions independent of λ, to obtain not only the $N = 2$ KdV hierarchy, but also a number of other integrable systems.

Before we go on to look at the notion of gauge equivalence for the KdV hierarchies, we are now in a position to appreciate a further piece of motivation for trying to find integrable systems by reductions of the SDYM equations. Equation (2.24) has the form of a zero-curvature equation, and it is known that certainly most, if not all, two-dimensional integrable systems arise from a zero-curvature equation. Clearly, every equation we will obtain from the SDYM equations can be also obtained from a single zero-curvature equation: this is just saying that every solution of the SDYM equations satisfies (2.3a). So why do we need to look at the SDYM equations as opposed to focusing our attention purely on a single zero-curvature equation? The problem with the latter approach is that there is no method to systematically generate suitable ansätze for the matrix $\tilde{L}$. In the SDYM approach we will see matrices like (2.25) appearing naturally. In the next paragraph we will see that in fact it is not necessary, because of gauge equivalence, to produce ansätze as specific as (2.25), but it still is necessary to give $\tilde{L}$ by hand a specific dependence on the spectral parameter, at least to obtain hierarchies. In SDYM reduction, if we reduce equations (2.3) by specifying $\bar{w}$ dependence, we will have a system that depends on three coordinates $w, z, \bar{z}$. It might have been hoped that $\bar{z}$ would play the role of the spectral parameter; this is not actually the case, but as we shall see the $\bar{z}$ coordinate does give us some information which renders the need for a spectral parameter almost superfluous.

The notion of gauge equivalence, for the KdV hierarchies, is attributed in [11] to Mikhailov. The essential observation is that one can replace the matrix

$\tilde{L}$ of (2.25) by a more general form

$$\tilde{L} = L' + \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ \lambda & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \tag{2.32}$$

where L' is an *arbitrary* $N \times N$ traceless lower triangular matrix (with entries functions independent of λ). L' has $\frac{1}{2}N(N+1) - 1$ degrees of freedom; but the form of $\tilde{L}$ and the equation (2.24) are invariant under gauge transformations of the form

$$\begin{aligned} \tilde{L} &\rightarrow S^{-1}\tilde{L}S + S^{-1}\partial_z S \\ A &\rightarrow S^{-1}AS + S^{-1}\partial_z S \end{aligned} \tag{2.33}$$

where S is a lower triangular matrix with 1's on the leading diagonal (i.e. an element of the Borel subgroup of the gauge group). Thus we have a $\frac{1}{2}N(N-1)$ parameter gauge invariance; the real number of free functions in $\tilde{L}$ is $\frac{1}{2}N(N+1)-1-\frac{1}{2}N(N-1) = N-1$. The form (2.25) is just a gauge fixing; another useful gauge fixing is to take L' in (2.32) to be diagonal. This gauge choice gives rise to the "modified KdV (MKdV) hierarchies". Solutions of the KdV hierarchies are related to those of the MKdV hierarchies by gauge transformation relations, known as Miura maps (in general these maps show how to obtain solutions of the KdV hierarchies from those of the MKdV hierarchies, but not vice-versa). In reducing the SDYM equations one of our main goals will be to reproduce, and generalize, this picture of gauge equivalence classes of equations.

3. The Reduction Scheme and the Case of $\mathbf{SL(2, C)}$

As we have already mentioned, we dimensionally reduce the SDYM equations by considering only potentials independent of $\bar{w}$. This is a gauge dependent statement, so more precisely we consider potentials that in some suitable gauge are independent of $\bar{w}$, and we note that we still retain the freedom to make gauge transformations (2.2) with u independent of $\bar{w}$. Under such transformations $A_{\bar{w}}$ transforms homogeneously, i.e.

$$A_{\bar{w}} \to u^{-1} A_{\bar{w}} u \tag{3.1}$$

This divides up the set of $A_{\bar{w}}$'s into equivalence classes, and we choose one representative for each equivalence class. Generically, it seems that each equivalence class gives rise to a specific two-dimensional integrable system, though at the moment we only have a very few explicit calculations. There is of course no guarantee that there exist solutions for $A_{\bar{w}}$'s in every equivalence class, and we shall see a case where restrictions do arise.

Having chosen a specific $A_{\bar{w}}$, there will still, in general, be some residual gauge invariance. Our general philosophy will be to avoid further gauge fixing until after solving at least some of equations (2.3), which as mentioned before we impose in the order (2.3b), (2.3c), (2.3a). To reproduce the Drinfeld-Sokolov picture we need to arrive at (2.3a) with sufficient gauge invariance intact. The best way to discuss the reduction scheme further is by specific examples, and we will study at length in this section the case of $SL(2, \mathbf{C})$.

In the case of $SL(2, \mathbf{C})$ the representative $A_{\bar{w}}$'s we need consider are given by

$$A_{\bar{w}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \kappa \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.2}$$

where κ is an arbitrary function of $w, z, \bar{z}$. In the first case we have a residual gauge invariance

$$u = \begin{pmatrix} 1 & 0 \\ \gamma & 0 \end{pmatrix} \tag{3.3}$$

where γ is an arbitrary function of $w, z, \bar{z}$. We parametrize the remaining potentials by

$$A_w = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \qquad A_z = \begin{pmatrix} d & e \\ f & -d \end{pmatrix} \qquad A_{\bar{z}} = \begin{pmatrix} g & h \\ j & -g \end{pmatrix} \tag{3.4}$$

All the entries in these matrices are functions of $w, z, \bar{z}$. Imposing (2.3b) gives us straight away that $g = h = 0$. Imposing (2.3c) gives

$$\begin{aligned} e_{\bar{z}} &= 0 \\ b &= d_{\bar{z}} - ej \\ a &= \tfrac{1}{2}(j_z - f_{\bar{z}} - 2dj) \end{aligned} \tag{3.5}$$

The last two equations of (3.5) (which of course are unaffected by residual gauge transformations) allow us to eliminate a and b leaving us with the functions c, d, e, f, j with e satisfying $e_{\bar{z}} = 0$. Ignoring the constrained function e for the moment, we have four degrees of freedom remaining, one of which is the gauge degree of freedom, and the other three of which will be constrained by imposing (2.3a). The gauge transformation rules for c, d, e, f, j are

$$\begin{aligned} &c \to c - \gamma_w + \gamma(j_z - f_{\bar{z}} - 2dj) + \gamma^2(j - d_{\bar{z}}) \\ &d \to d - \gamma \\ &e \to e \\ &f \to f + 2\gamma d - \gamma^2 - \gamma_z \\ &j \to j - \gamma_{\bar{z}} \end{aligned} \tag{3.6}$$

Choosing e to be some fixed constant (say 1) we recover a close analogue of the Drinfeld-Sokolov picture (A_z is the analogue of the matrix $\tilde{L}$ we had in section

2). We could opt to take e to be a parameter, and take the functions d, f to be independent of e and the others dependent on e, to recover the exact Drinfeld-Sokolov picture. This is ad hoc, and we will not do it, but will suffice ourselves with looking at the picture with e fixed to be 1, and we will now discuss different gauge choices. Note from (3.6) that fixing e does not restrict the residual gauge invariance.

The "KdV type" gauge choice involves setting d to zero, which we can clearly do, from (3.6), leaving no further gauge invariance. In this our forms for the gauge fields reproduce those used in [9]. Equation (2.3a) now yields

$$\begin{aligned} 2j_z &= f_{\bar{z}} \\ c &= \tfrac{1}{2}(f_{z\bar{z}} - j_{zz}) - jf \\ f_w &= c_z + f(j_z - f_{\bar{z}}) \end{aligned} \tag{3.7}$$

Eliminating c using the second of these equations, and using the first of these equations to rewrite j_z in terms of f wherever possible, we find the system

$$\begin{aligned} 2j_z &= f_{\bar{z}} \\ f_w &= \tfrac{1}{4} f_{zz\bar{z}} - f f_{\bar{z}} - f_z j \end{aligned} \tag{3.8}$$

which we can rewrite compactly as

$$f_w = [\tfrac{1}{4}\partial_z^2 - f - \tfrac{1}{2} f_z \partial_z^{-1}] f_{\bar{z}} \tag{3.9}$$

This form is highly reminiscent of (2.11). In fact this is a known equation, a three dimensional version of the standard KdV equation first studied by Calogero [24]. We can write it in compact notation $f_w = \mathcal{R} f_{\bar{z}}$, where $\mathcal{R}$ denotes the recursion operator of the KdV equation [20], and once again we point out the resemblance to (2.11). Although we have not obtained the usual KdV hierarchy, since from our equation (3.9) we can read off the KdV recursion operator we essentially have just the same information. We suspect, by virtue of (2.11), that in general our reductions will lead us directly to recursion operators.

Two more points should be made before we leave this gauge. First, if we consider a reduction of (3.9) to two dimensions, we might well opt to impose $u_{\bar{z}} = 0$, in which case (taking $\partial_z^{-1} 0 = -2$) we obtain $u_w = u_z$, the $r = 1$ equation in the standard KdV hierarchy. In [8] imposing $u_{\bar{z}} = u_z$ was proposed; this gives us the $r = 3$ equation in the hierarchy. It is no less ad hoc to impose $u_{\bar{z}} = \mathcal{R}^s u_z$ in which case (3.9) yields $u_w = \mathcal{R}^{s+1} u_z$, the $r = 2s+3$ equation in the hierarchy. In this ad hoc manner the whole hierarchy can be obtained, but it is more correct to see our reduction procedure as leading logically to the recursion operator which is the structure behind the entire hierarchy. Second, we note that our reduction has been systematic apart from the detail of the choice of e. Any choice for e (satisfying $e_{\bar{z}} = 0$) gives an integrable system. The general form of these systems is not particularly illuminating; the most natural choice for e is probably $e = 0$, which gives a trivial system. It is possible to study the general case, but since we wish to focus on systems that are relevant to two-dimensional physics, we shall content ourselves with the choice $e = 1$ (which we shall now carry through to other gauges too), and we shall make similar choices in later reductions without comment. It is always possible to work in full generality.

We now look at the "MKdV type" gauge, in which we set $f = 0$ (and we still take $e = 1$). From (3.6) the ability to choose this gauge is predicated on the existence of a solution to a Riccati equation. Keeping a and b in our calculations for the moment, we get the following system from imposing (2.3a) and writing down the last two equations of (3.5):

$$\begin{aligned} d_w &= a_z + c \\ c_z &= 2cd \\ b_z &= 2(a - bd) \\ j_z &= 2(a + jd) \\ d_{\bar{z}} &= b + j \end{aligned} \tag{3.10}$$

We can clearly solve the second of these by imposing $c = 0$, and we shall do this. In the "KdV type" gauge we had no arbitrary choices to make after setting $e = 1$; here we are making an extra choice, so we expect the relevant gauge transformation relations to provide us with a solution of (3.9) for every solution of (3.10) with $c = 0$ but *not* vice-versa. It is straightforward, once we have chosen $c = 0$, to eliminate a, b, j from (3.10) to obtain

$$d_w = [\tfrac{1}{4}\partial_z^2 - d^2 - d_z \partial_z^{-1} d] d_{\bar{z}} \tag{3.11}$$

This is a generalized version of the MKdV equation, which can be written $d_w = \tilde{\mathcal{R}} d_{\bar{z}}$ where $\tilde{\mathcal{R}}$ is the MKdV recursion operator. Setting $d_{\bar{z}} = 0$ (and choosing $\partial_z^{-1} 0 = -1$) we obtain $d_w = d_z$, and setting $d_{\bar{z}} = d_z$ we obtain the usual MKdV equation. Also in this latter case it can be shown that j is related to d by a Miura map, and satisfies the KdV equation. Before we look at the gauge transformation relation between (3.9) and (3.11), let us just look at some further possible reductions of (3.11) obtained taking all functions to be independent of w. Going back to (3.10) we see we can then make the choice $a = 0$, and equations

(3.10) reduce to the system

$$b_z = -2bd$$
$$j_z = 2jd \tag{3.12}$$
$$d_{\bar{z}} = b + j$$

If both b and j are zero this system is trivial. If b is zero and j is nonzero, we eliminate d to get $\partial_z\partial_{\bar{z}} \ln j = 2j$, the Liouville equation. If j is zero and b is nonzero, we similarly get $\partial_z\partial_{\bar{z}} \ln d = -2d$. If both b and j are nonzero we can satisfy the equations above with $b = j^{-1}$, and b satisfying $\partial_z\partial_{\bar{z}} \ln b = -2(b + b^{-1})$, or, writing $b = ie^{ix}(= j^{-1})$, $x_{z\bar{z}} = -4i \sin x$, the Sine-Gordon equation. Combining the different results in this paragraph, we observe that by suitable reduction of (3.10) the function j can be made to satisfy the KdV equation (in w, z) or a version of the Sine-Gordon equation (in $z, \bar{z}$); this is very reminiscent of a known relation of the MKdV and Sine-Gordon equations [11].

We can now discuss the gauge transformation between equations (3.9) and (3.11). Let us consider the gauge transformation (3.6) between the MKdV gauge, in which $f = 0$, and the KdV gauge (we will mark quantities in this gauge with primes), in which $d' = 0$. Straight away we see we need to choose $\gamma = d$ in (3.6), and we find $f' = d^2 - d_z$. We deduce that if d satisfies (3.11) then f' satisfies (3.9), and this is the Miura map, generalized for the three dimensional equations (3.9) and (3.11). Note we get a solution of KdV from a solution of MKdV, but not vice-versa, as we expect from considerations presented above.

There are two more gauges in which we will briefly look at the SDYM reduction given by the first choice of $A_{\bar{w}}$ in (3.2). We can choose $j = 0$ as a gauge choice, and this is the gauge used in [8]. Imposing (2.3a) we find (amongst others) the equation $d_{z\bar{z}} + 2dd_{\bar{z}} = 0$. If we solve this by setting $d_z + d^2 = 0$ then we find, after some calculation, that we need

$$(d_z)_w = [\tfrac{1}{4}\partial_z^2 + 2(d_z) + (d_z)_z\partial_z^{-1}](d_z)_{\bar{z}} \tag{3.13}$$

i.e. $-2d_z$ satisfies (3.9). We could have predicted that this equation would have arisen by just looking at the gauge transformation relations (3.6). Finally we consider the gauge choice $f = \epsilon d$, where ϵ is an arbitrary constant (and the MKdV gauge is the choice $\epsilon = 0$). The manipulation of the system in this gauge is a little complicated. It can be shown that

$$(\partial_z - \epsilon - 2d)(c - \epsilon a - \tfrac{1}{4}\epsilon^2 d_{\bar{z}}) = 0 \tag{3.14}$$

We make the choice of setting $c - \epsilon a - \frac{1}{4}\epsilon^2 d_{\bar{z}}$ to zero. The resulting equation is then

$$d_w = [\tfrac{1}{4}\partial_z^2 - d^2 - d_z\partial_z^{-1}d - \epsilon(d + \tfrac{1}{2}d_z\partial_z^{-1})]d_{\bar{z}} = 0 \tag{3.15}$$

This is a three dimensional version of the so-called Gardner KdV equation, which we can obtain from (3.15) by setting $d_{\bar{z}} = d_z$:

$$d_w = \tfrac{1}{4}d_{zzz} - \tfrac{3}{2}d^2 d_z - \tfrac{3}{2}\epsilon d d_z \tag{3.16}$$

Examination of the gauge transformation from this gauge to the regular KdV gauge tells us that if d solves (3.15) then $f' = d^2 + \epsilon d - d_z$ solves (3.9), which is a generalization of the standard "Gardner map" which relates solutions of the Gardner KdV (3.16) with solutions of the usual KdV equation (note that again the map is a one-way map, since in this last gauge we had to make more choices than in the KdV gauge). The Gardner map is considered of some importance in the theory of integrable systems, as it can be used to give a simple proof of the existence of an infinite number of conserved quantities for the KdV equation [25]. Generalizations of this are of considerable interest.

We now turn to consider the second possible choice for $A_{\bar{w}}$ in (3.2), which we shall treat in somewhat less detail, having made the general scheme we are using evident. The case $\kappa = 0$ is trivial, so we will assume $\kappa \neq 0$. The residual gauge transformations for this choice of $A_{\bar{w}}$ have the form

$$u = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{pmatrix} \tag{3.17}$$

where γ is a arbitrary function of $w, z, \bar{z}$. We parametrize the remaining potentials again using (3.4). Solving (2.3b) implies that $j = h = 0$ and $\kappa_{\bar{z}} = 0$. Here we are seeing a restriction on the form of $A_{\bar{w}}$ for which solutions of the SDYM equations exist. Solving (2.3c) yields

$$\begin{aligned} e_{\bar{z}} - 2eg - 2b\kappa &= 0 \\ f_{\bar{z}} + 2fg + 2c\kappa &= 0 \\ g_z + \kappa_w - d_{\bar{z}} &= 0 \end{aligned} \tag{3.18}$$

Under gauge transformation we find

$$\begin{aligned} d &\to d - \gamma_z \gamma^{-1} \\ e &\to e\gamma^2 \\ f &\to f\gamma^{-2} \end{aligned} \tag{3.19}$$

and the gauge $d = 0$ suggests itself. We restrict to the case where κ is a non-zero constant, and then from equation (3.18) we see we can choose $g = 0$, and eliminate b, c in terms of e, f. (2.3a) now yields the system

$$\begin{aligned} 2\kappa e_w &= e_{z\bar{z}} - 2\tilde{a}e \\ -2\kappa f_w &= f_{z\bar{z}} - 2\tilde{a}f \\ \tilde{a}_z &= (ef)_{\bar{z}} \end{aligned} \tag{3.20}$$

where we have written $\tilde{a} = 2\kappa a$. *A priori* the functions $e, f, \tilde{a}$ are all complex, but we can consistently take f to be plus or minus the complex conjugate of e if κ is pure imaginary. Then, setting $\kappa = ik$, we can write the system in the compact form

$$2ke_w = \left(-i\partial_z \pm 4ie\partial_z^{-1} Re(e^* \cdot)\right) e_{\bar{z}} \tag{3.21}$$

i.e. $2ke_w = \mathcal{R}' e_{\bar{z}}$, where $\mathcal{R}'$ is the recursion operator for the NLS equation, which can be recovered from (3.21) by the further reduction $e_{\bar{z}} = e_z$. Again, note the similarity in form of (3.21) to (2.11a).

There are other interesting reductions of (3.20). If we consider functions that are independent of w we can consistently take $e = f$ in (3.20), and obtain the system

$$\begin{aligned} e_{z\bar{z}} &= 2\tilde{a}e \\ \tilde{a}_z &= 2ee_{\bar{z}} \end{aligned} \tag{3.22}$$

We can use the second of these equations to get an expression for $e_{\bar{z}}$ in terms of $e, \tilde{a}$, and substituting this in the first equation we find we need

$$\tilde{a}_z e_z - e\tilde{a}_{zz} + 4\tilde{a}e^3 = 0 \tag{3.23}$$

This is a Bernoulli equation for e which we can solve in the usual manner to obtain

$$e^2 = \frac{\tilde{a}_z^2}{4\tilde{a}^2 + \lambda(\bar{z})} \tag{3.24}$$

Here $\lambda(\bar{z})$ is an arbitrary function of $\bar{z}$ arising as the integration constant. Making the substitution

$$\tilde{a} = \tfrac{1}{2} i\lambda^{1/2} \cos\psi \tag{3.25}$$

we have

$$e^2 = -\tfrac{1}{4}\psi_z^2 \tag{3.26}$$

and substituting back into the second of equations (3.22) we see we need ψ to satisfy

$$\psi_{z\bar{z}} = \nu(\bar{z}) \sin\psi \tag{3.27}$$

where $\nu(\bar{z}) = i\lambda^{1/2}$. Once again we find the Sine-Gordon equation (in fact a generalized version thereof) in coordinates $z, \bar{z}$. This reduction of SDYM to Sine Gordon is actually, up to gauge transformation, equivalent to that given by Ward in [7]. It would be interesting to see if in this reduction (or the previous reduction given for Sine-Gordon) the operator $\mathcal{L}$ of (2.11) relates to the Sine-Gordon recursion operator [20], but we will not undertake this here.

4. $\mathbf{SL(N, C)}$

The first step in our reduction scheme is to find the equivalence classes of $A_{\bar{w}}$'s under the relation (3.1). For gauge group $SL(N, \mathbf{C})$ this problem is solved by writing down all Jordan normal forms for traceless $N \times N$ matrices. If the eigenvalues of the matrix are λ_i, $i = 1, 2, ..., D$ say, with the eigenvalue λ_i occuring f_i times (we will have $\sum_i f_i = N$, and the tracelessness condition $\sum_i f_i \lambda_i = 0$), then the possible Jordan normal forms all have the form

$$A_{\bar{w}} = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_D \end{pmatrix} \tag{4.1}$$

where the A_i's are $f_i \times f_i$ blocks, each with form

$$A_i = \begin{pmatrix} A_{i1} & & & \\ & A_{i2} & & \\ & & \ddots & \\ & & & A_{in_i} \end{pmatrix} \tag{4.2}$$

and each A_{ip}, $i = 1, 2, ..., D$, $p = 1, 2, ...n_i$ is a matrix of form

$$A_{ip} = \begin{pmatrix} \lambda_i & 0 & 0 & \dots & 0 & 0 \\ 1 & \lambda_i & 0 & \dots & 0 & 0 \\ 0 & 1 & \lambda_i & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_i & 0 \\ 0 & 0 & 0 & \dots & 1 & \lambda_i \end{pmatrix} \tag{4.3}$$

Thus for $SL(3, \mathbf{C})$, for instance, we have 3 possible forms for $A_{\bar{w}}$:

$$A_{\bar{w}}^{(1)} = \begin{pmatrix} \kappa & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -\kappa-\lambda \end{pmatrix} \quad A_{\bar{w}}^{(2)} = \begin{pmatrix} \kappa & 0 & 0 \\ 1 & \kappa & 0 \\ 0 & 0 & -2\kappa \end{pmatrix} \quad A_{\bar{w}}^{(3)} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{4.4}$$

where κ, λ are arbitrary functions of $w, z, \bar{z}$. As we have previously mentioned, there is no guarantee that a particular form of $A_{\bar{w}}$ will yield any solutions of the SDYM equations.

We will focus on the systems obtained for $A_{\bar{w}}$'s that have all eigenvalues equal to zero. The motivation for this is simply that it is these that give integrable systems of interest in two-dimensional physics, and furthermore the calculations for these $A_{\bar{w}}$'s are in many respects more tractable than for others. Once we restrict to these $A_{\bar{w}}$'s, there is just one block in (4.1), and the different normal forms are classified by partitions of N into integers (or equivalently Young tableaux with N blocks), the partition telling us the sizes of the blocks in (4.2). For $SL(3, \mathbf{C})$ we have the three forms in (4.4), with $\kappa = \lambda = 0$. These correspond to the partitions 1^3, 2.1, 3 respectively. For $SL(4, \mathbf{C})$ we have five forms:

$$A_{\bar{w}}^{I} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad A_{\bar{w}}^{II} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_{\bar{w}}^{III} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad A_{\bar{w}}^{IV} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.5}$$

$$A_{\bar{w}}^{V} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

These correspond to partitions $1^4,\ 2.1^2,\ 2^2,\ 3.1,\ 4$ respectively. It is convenient, in fact, to conjugate these forms with suitable u's so that, if there are l non-zero entries, these lie in the bottom l rows of the matrix. Thus we replace, at this juncture, form $A_{\bar{w}}^{(2)}$ of (4.4) by

$$A_{\bar{w}}^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{4.6}$$

and forms $A_{\bar{w}}^{II}, A_{\bar{w}}^{III}, A_{\bar{w}}^{IV}$ of (4.5) by

$$A_{\bar{w}}^{II} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad A_{\bar{w}}^{III} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$A_{\bar{w}}^{IV} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{4.7}$$

The fractional KdV hierarchies of Bakas and Depireux [9] are the hierarchies obtained by taking an $A_{\bar{w}}$ that has one complete lower diagonal filled by 1's, with no other non-zero entries. It seems these hierarchies are related to the W_N^l algebras. For $N \geq 4$ we see that these are not the only methods of reduction of the SDYM equations corresponding to $A_{\bar{w}}$'s with all zero eigenvalues; if we can consistently solve SDYM with the choice $A_{\bar{w}}^{IV}$ of (4.7), and this does not coincide with any other reduction (which seems improbable), we will have an interesting new structure. The manipulations for any $N = 4$ calculation are long, but accessible with the use of symbolic manipulators. This particular problem will be investigated in a later paper.

Having chosen the forms of $A_{\bar{w}}$ that we shall use, we consider the actual reduction. We first impose (2.3b) which is just

$$[A_{\bar{w}}, A_{\bar{z}}] = 0 \tag{4.8}$$

This equation will give simple algebraic constraints on the entries of $A_{\bar{z}}$. Equation (4.8) also has another interpretation; the number of free entries in $A_{\bar{z}}$ tells us exactly the number of residual gauge degrees of freedom for our particular choice of $A_{\bar{w}}$. It is apparent that we have the residual gauge freedom to fix $A_{\bar{z}} = 0$; we call this an MS gauge, since this was used in [8]. MS gauges in general seem to be inconvenient for actual calculations, and of course we will not be able to reproduce the Drinfeld-Sokolov picture. Instead we take the following approach: we *partially* gauge fix $A_{\bar{z}}$, taking it to be strictly lower triangular. The residual gauge freedom after this will be exactly the intersection of the Borel subgroup of the gauge group with the group of gauge transformations that leave $A_{\bar{w}}$ invariant. The main need to partially gauge fix $A_{\bar{z}}$ at this stage is to make the task of imposing (2.3c) practical. We note that in addition to the trivial choice $A_{\bar{w}} = 0$ (corresponding to the partition 1^N), there is, for each N, another choice of $A_{\bar{w}}$ for which the residual gauge freedom will be the *whole* Borel subgroup; this is the choice corresponding to the partition 2.1^{N-2},

$$A_{\bar{w}} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix} \tag{4.9}$$

This choice of $A_{\bar{w}}$ gives the Drinfeld-Sokolov hierarchies.

We now make some comments on solving (2.3c), the next stage of the reduction. This will give us a variety of different types of relation, some of which will be constraints on the entries of A_z, and others that will give relations between entries of the different potentials (this will become clear in the $SL(3,\mathbf{C})$ examples to follow in the next section). It is the former that are particularly significant. In the particular case of the BD fractional KdV type reduction, we will be able to satisfy the relations of this type by choosing $(A_z)_{i,i+l} = 1$ for $i = 1, 2, ..., N-l$, and $(A_z)_{ij} = 0$ for $j > i+l$. The key step in handling the reductions is imposing these constraints on the entries of A_z before exploiting the residual gauge invariance, as without them it is very difficult to see how to exploit the gauge invariance constructively.

The final step of the reduction that we discuss in generality is the fixing, at this stage, of the residual gauge invariance. We specify, rather broadly, two types of gauge choice. In a "KdV type gauge" we use the residual gauge freedom to eliminate some of the remaining entries in A_z starting from the top right of the matrix. In an "MKdV type gauge" we start from the bottom left. There will in general be a vast choice of gauges (even within these guidelines), and it will be interesting to pick different gauges, and to work out the relevant Miura maps between them, thereby finding different representations for the relevant W-type algebras. For the Drinfeld-Sokolov hierarchies, i.e. the case (4.9), the notion of

KdV gauge is

$$A_z = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ u_N & u_{N-1} & u_{N-2} & u_{N-3} & \dots & u_2 & 0 \end{pmatrix} \tag{4.10}$$

(c.f. (2.25)); the notion of MKdV gauge is

$$A_z = \begin{pmatrix} q_1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & q_2 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & q_3 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & q_{N-1} & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & q_N \end{pmatrix} \tag{4.11}$$

with $\sum_{i=1}^{N} q_i = 0$. For the W_3^2-related hierarchy, we will see in section 5 that unfortunately this set of gauge choices is not large enough to find a Miura map that gives the free field representation [16] of the algebra; it would be interesting to see if this could be obtained from some more general gauge transformation.

5. $\mathbf{SL(3,C)}$

This section is divided into two subsections, one for each of the two choices of $A_{\bar{w}}$ we make.

5.1 $\mathbf{A_{\bar{w}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}}$

Imposing (2.3b) we find that $A_{\bar{z}}$ must take the form

$$A_{\bar{z}} = \begin{pmatrix} \delta & 0 & 0 \\ \alpha & -2\delta & 0 \\ \beta & \gamma & \delta \end{pmatrix} \tag{5.1}$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary functions of $w, z, \bar{z}$. We use some of the residual gauge invariance of $A_{\bar{w}}$ to choose

$$\delta = 0 \tag{5.2}$$

in accordance with our comments in section 4. The residual gauge invariance is now exactly the Borel subgroup of $SL(3, \mathbf{C})$, which we parametrize via

$$u = \begin{pmatrix} 1 & 0 & 0 \\ \epsilon & 1 & 0 \\ \zeta & \eta & 1 \end{pmatrix} \tag{5.3}$$

Here ϵ, ζ, η are arbitrary functions. We parametrize the remaining potentials via

$$A_w = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & -a-e \end{pmatrix} \qquad A_z = \begin{pmatrix} p & q & r \\ s & t & u \\ v & w & -p-t \end{pmatrix} \tag{5.4}$$

(we are using u in two different ways now, but it should be obvious when it is a matrix and when a single function). Imposing (2.3c) we find eight equations, three of which take the form

$$\begin{aligned} r_{\bar{z}} &= 0 \\ q_{\bar{z}} &= \gamma r \\ u_{\bar{z}} &= -\alpha r \end{aligned} \tag{5.5}$$

We solve these by choosing

$$r = 0 \tag{5.6a}$$

$$q = u = 1 \tag{5.6b}$$

in accordance with the contents of section 4. There is some freedom we are ignoring here, but other solutions of (5.5) could also be studied. We will make no further arbitrary choices of this kind for this $A_{\bar{w}}$, at least in KdV gauge.

For reference we write down the remaining equations obtained from (2.3c) (after making the choices in (5.6)):

$$\begin{aligned} \alpha &= p_{\bar{z}} - c \\ \gamma &= (p+t)_{\bar{z}} - c \\ \beta &= s_{\bar{z}} - (p_{\bar{z}} - c)_z - f + (p-t)(p_{\bar{z}} - c) \\ 2a + e &= \alpha w - \beta(2p+t) - \gamma s - v_{\bar{z}} + \beta_z \\ b &= -\beta - \gamma(p+2t) - w_{\bar{z}} + \gamma_z \end{aligned} \tag{5.7}$$

The first three of these give the entries of $A_{\bar{z}}$ in terms of the entries of A_z and the entries c, f of A_w; the last two equations (after substituting for α, β, γ) give two other entries of A_w in terms of entries of A_z and c, f.

It is hard to go further without a gauge choice, and we will first look at KdV gauge, where we set $p = t = v = 0$ (it is straightforward to analyze the gauge transformation relations to see that this is a valid gauge fixing). After eliminating α, β, γ we find that we have ten equations (the last two of (5.7) and eight from (2.3a)) determining v, w and the eight entries of A_w. We will not write these down in full here; after some manipulations we find

$$\begin{aligned} c_z &= -w_{\bar{z}}/3 \\ f_z &= -v_{\bar{z}}/3 \end{aligned} \tag{5.8}$$

Six of the other equations give a, b, e, d, g, h in terms of c, f, z, w. The remaining two equations give the evolution of v, w, and we find, after eliminating all the other variables (including c, f, using the ∂_z^{-1} operator to solve (5.8)), that we can write them in the form:

$$3v_w = [-\partial_z^3 - 3v + w\partial_z - v_z\partial_z^{-1}]v_{\bar z}$$
$$+ \tfrac{1}{3}[-2\partial_z^4 + 6v_z - 2w^2 + 6w_{zz} + 6w_z\partial_z + 4w\partial_z^2 + (3v_{zz} + 2w_{zzz} - 2ww_z)\partial_z^{-1}]w_{\bar z}$$
$$3w_w = [2\partial_z^2 - 2w - w_z\partial_z^{-1}]v_{\bar z}$$
$$+ [\partial_z^3 - 2w_z - w\partial_z - 3v - (2v_z + w_{zz})\partial_z^{-1}]w_{\bar z} \tag{5.9}$$

We can read off from this a 2×2 matrix recursion operator for the Boussinesq equation (it is not the usual recursion operator; we will explain this below). This can be used to construct the entire $N = 3$ KdV hierarchy. First we try setting

$$v_{\bar z} = w_{\bar z} = 0 \tag{5.10}$$

From (5.8) we see this means $c_z = f_z = 0$, and we will take c, f both to be constants; in effect we just take (5.9) with $v_{\bar z} = w_{\bar z} = 0$ but

$$\partial_z^{-1} v_{\bar z} = -3f$$
$$\partial_z^{-1} w_{\bar z} = -3c \tag{5.11}$$

We obtain the system

$$w_w = fw_z + c(2v_z + w_{zz})$$
$$v_w = fv_z + c(-v_{zz} - \tfrac{2}{3}w_{zzz} + \tfrac{2}{3}ww_z) \tag{5.12}$$

which reproduces, up to some trivial rescalings, equations (4.3),(4.4) in the second paper of [9]. For $f = 1, c = 0$ we obtain the $r = 1$ equation in the $N = 3$ KdV hierarchy, and for $f = 0, c = 1$ we have the $r = 2$ equation. In the latter case we can eliminate v to get a second order in time evolution equation for w,

$$w_{ww} = -\tfrac{1}{3}w_{zzzz} + \tfrac{2}{3}(w^2)_{zz} \tag{5.13}$$

We note that (5.12) (for $f=0, c=1$) is not the only possible first order system that leads to (5.13); for instance we could also have

$$\begin{aligned}\tilde{w}_w &= \tilde{v}_z \\ \tilde{v}_w &= -\tfrac{1}{3}\tilde{w}_{zzz} + \tfrac{2}{3}(\tilde{w}^2)_z\end{aligned} \tag{5.14}$$

This is the form used by Olver in [26]. It is related to our form by the simple substitutions $\tilde{v} = 2v + w_z, \tilde{w} = w$. Rewriting (5.9) in terms of $\tilde{w}, \tilde{v}$ we obtain:

$$\begin{aligned}3\tilde{v}_w &= [\partial_z^2 - \tilde{w} - \tfrac{1}{2}\tilde{w}_z\partial_z^{-1}]\tilde{v}_{\bar{z}} \\ &\quad - \tfrac{1}{2}[3\tilde{v} + 2\tilde{v}\partial_z^{-1}]\tilde{w}_{\bar{z}} \\ 3\tilde{v}_w &= -\tfrac{1}{2}[3\tilde{v} + \tilde{v}\partial_z^{-1}]\tilde{v}_{\bar{z}} \\ &\quad - \tfrac{1}{3}[\partial_z^4 + 4\tilde{w}^2 - 5\tilde{w}\partial_z^2 - \tfrac{15}{2}\tilde{w}_z\partial_z - \tfrac{9}{2}\tilde{w}_{zz} + (4\tilde{w}\tilde{w}_z - \tilde{w}_{zzz})\partial_z^{-1}]\tilde{w}_{\bar{z}}\end{aligned} \tag{5.9$'$}$$

From this we can read off the usual recursion operator, as given in [26].

The next step in the analysis of (5.9) is to see what we obtain when we set

$$\begin{aligned}w_{\bar{z}} &= w_z \\ v_{\bar{z}} &= v_z\end{aligned} \tag{5.15}$$

We obtain the system

$$\begin{aligned}3w_w &= w_{zzzz} + 2v_{zzz} - 4(wv)_z - (w^2)_{zz} \\ 9v_w &= -2w_{zzzzz} - 3v_{zzzz} + 6ww_{zzz} + 12w_zw_{zz} - 4w^2w_z + 6(wv_z)_z - 6(v^2)_z\end{aligned} \tag{5.16}$$

This is the same (up to some trivial rescalings) as equations (26),(27) in the first paper of [9]; it is the $r=4$ equation in the $N=3$ KdV hierarchy. It is apparent how to proceed to get higher order equations in the hierarchy: for instance for $r=5$ we would take (5.9) and impose

$$\begin{aligned}w_{\bar{z}} &= (2v + w_z)_z \\ v_{\bar{z}} &= (-v_z - \tfrac{2}{3}w_{zz} + \tfrac{1}{3}w^2)_z\end{aligned} \tag{5.17}$$

In [9], the $r = 5$ equation could not be obtained from $SL(3, \mathbf{C})$ SDYM. By this stage we understand in general the findings of [9]: it is clear that if we take our reduction scheme and impose further reduction by "$\partial_{\bar{z}} = 0$" we will obtain the $r < N$ equations of the Nth KdV hierarchy, and if we impose the reduction "$\partial_{\bar{z}} - \partial_z = 0$" we will obtain the $r = N+1$ equation. Larger r cannot be obtained, it would seem, by a "simple" reduction to two dimensions.

Let us proceed to look at the current example in MKdV gauge. In this gauge we choose $s = v = w = 0$ as the gauge fixing (this requires the existence of a solution to a certain nonlinear second order ODE, which we will assume), and it is easy to verify from equations (5.7) and (2.3a) in this gauge that we can consistently set $d = g = h = 0$ too. We are left with ten equations (five from (5.7) and five from (2.3a)) determining the ten quantities $\alpha, \beta, \gamma, a, b, e, f, g, p, t$. Handling these equations is tricky, so we will present the strategy for attack, which will be of use in MKdV gauges in other examples. The five equations of (5.7) can be rewritten

$$\alpha = p_{\bar{z}} - c \tag{5.18a}$$

$$\gamma = (p+t)_{\bar{z}} - c \tag{5.18b}$$

$$\beta + f = -\alpha_z + \alpha(p - t) \tag{5.18c}$$

$$2a + e = \beta_z - \beta(2p + t) \tag{5.18d}$$

$$b + \beta = \gamma_z - \gamma(p + 2t) \tag{5.18e}$$

and the five equations from (2.3a) are

$$p_w = a_z \tag{5.18f}$$

$$t_w = e_z \tag{5.18g}$$

$$a - e = b_z + b(p - t) \tag{5.18h}$$

$$b - f = c_z + c(2p + t) \tag{5.18i}$$

$$2e + a = f_z + f(2t + p) \tag{5.18j}$$

Equations (a),(b) give α, γ in terms of p, t, c. Using these, equations (c),(e),(i) give b, f in terms of p, t, c, β, and also give one consistency condition for p, t, c. Using the formulae found for b, f, equations (d),(h),(j) give a, e in terms p, t, c, β, and a further consistency condition for p, t, c, β. Thus we are left with equations (f) and (g) for the evolution of p, t, the right hand sides of which we can write in terms of p, t, c, β; we also have the two consistency condtions. It turns out that the consistency conditions can be solved for c, β using the ∂_z^{-1} operator; explicitly they are

$$\begin{aligned} 3c_z &= \big((2p + t)_z - (p^2 + tp + t^2)\big)_{\bar{z}} \\ 3(\beta - tc + tp_{\bar{z}})_z &= \big(t_{zz} - (2t + p)t_z - tp(t + p)\big)_{\bar{z}} \end{aligned} \tag{5.19}$$

It is possible to work out the analogue of (5.9), giving the time derivatives of p, t in terms of some operators acting on $p_{\bar{z}}, t_{\bar{z}}$. The result is very long (compare the length of equations (5.19) to their analogues in KdV gauge, equations (5.8)!) and not very illuminating. So here we shall just obtain the $r = 1$ and $r = 2$ equations of the hierarchy in this gauge, which we do by taking all the functions to be independent of $\bar{z}$. Equations (5.19) become simply

$$\begin{aligned} c_z &= 0 \\ \beta_z &= ct_z \end{aligned} \tag{5.20}$$

which we solve by taking c constant and

$$\beta = ct - \mu \tag{5.21}$$

where μ is a constant. The evolution equations yield

$$\begin{aligned} 3p_w &= 3\mu p_z + c[(p+2t)_{zz} + (p^2 - 2tp - 2t^2)_z] \\ 3t_w &= 3\mu t_z + c[-(2p+t)_{zz} + (t^2 - 2tp - 2p^2)_z] \end{aligned} \tag{5.22}$$

This is the MKdV version of (5.12); choosing $c = 0, \mu = 1$ gives the $r = 1$ equation, and choosing $c = 1, \mu = 0$ gives the $r = 2$ equation.

We conclude this section by giving the Miura map between MKdV and KdV gauges for this case, which just requires finding a u of form (5.3) such that

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ v & w & 0 \end{pmatrix} = u \begin{pmatrix} p & 1 & 0 \\ 0 & t & 1 \\ 0 & 0 & -p-t \end{pmatrix} u^{-1} - u_z u^{-1} \tag{5.23}$$

is consistent. We deduce that if p, t satisfies the MKdV equation for our case, then we can find a v, w that satisfies the KdV equation, given by

$$\begin{aligned} v &= p_{zz} - pp_z + pt_z - pt(p+t) \\ w &= p^2 - 2p_z - t_z + t(p+t) \end{aligned} \tag{5.24}$$

This relates solutions of (5.12) and solutions of (5.22) if we identify $\mu = f$ (it is apparent that under the relevant gauge transformation from MKdV gauge to KdV gauge c is left unchanged).

5.2 $\mathbf{A_{\bar{w}}} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \end{pmatrix}$

Imposing (2.3b) we find that $A_{\bar{z}}$ must take the form

$$A_{\bar{z}} = \begin{pmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ \beta & \alpha & 0 \end{pmatrix} \tag{5.25}$$

where α, β are arbitrary functions. This also informs us that the gauge transformations that leave $A_{\bar{w}}$ invariant have the form

$$u = \begin{pmatrix} 1 & 0 & 0 \\ \gamma & 1 & 0 \\ \delta & \gamma & 1 \end{pmatrix} \tag{5.26}$$

where γ, δ are arbitrary functions. All such u's lie in the Borel subgroup so we need no further gauge fixing at this stage. We parametrize A_w, A_z as in equation (5.4). Imposing (2.3c) gives, amongst other conditions, the constraints

$$\begin{aligned} r_{\bar{z}} &= 0 \\ u_{\bar{z}} &= -\alpha r - c \\ q_{\bar{z}} &= \alpha r + c \end{aligned} \tag{5.27}$$

We see we can therefore impose on A_z the constraints $r = 1$ and $u = -q$, which we shall do. Looking at gauge transformations, and using our general philosophy for KdV gauges outlined in section 4, we find we can use the two parameter gauge invariance to set $q = 0$ and to set $p = \epsilon t$ where ϵ is any constant. This explains the choice of the ansatz (5.1) in the second paper of [9]. We can choose ϵ as we wish; under a gauge transformation taking $\epsilon \to \epsilon'$ we find s, t, w are invariant and $v' = v + (1 + \epsilon + \epsilon')(\epsilon - \epsilon')t^2 - (\epsilon - \epsilon')t_z$. In [9] the choice $\epsilon = -1/2$ is used, which simplifies certain details of the calculations. The evolution equations (for t, s, w, v) that we obtain depend on the choice of ϵ, but for gauge invariance reasons we should expect them to depend on v, ϵ only through the gauge invariant combination $v + (\epsilon + \epsilon^2)t^2 - \epsilon t_z$. This can be checked, through some arduous calculations not fixing the parameter ϵ. We will fix $\epsilon = 0$, without any loss of generality.

We have six remaining conditions from (2.3c) and eight conditions from (2.3a) which determine $\alpha, \beta, t, u, v, w$ and the eight entries of A_w. It is straightforward but very lengthy to manipulate the equations. Two crucial "consistency conditions" emerge:

$$
\begin{aligned}
3b_z &= -(v + t_z + t^2)_{\bar{z}} \\
3c_z &= -(s + w)_{\bar{z}}
\end{aligned}
\tag{5.28}
$$

In the course of the manipulations it becomes apparent that it is helpful to replace the variable v by a new variable $x = v + t^2 - \frac{1}{2}t_z$. Note that this *not* just a gauge transformation (which, since we have chosen $\epsilon = 0$ amounts to choosing a new variable $v - \epsilon'(1+\epsilon')t^2 + \epsilon' t_z$). We obtain the following system of evolution equations:

$$
\begin{aligned}
3t_w =& -\tfrac{3}{2}(s+w)t_{\bar{z}} + (s-w)_{z\bar{z}} - t(s+w)_{\bar{z}} - t_z\partial_z^{-1}(s+w)_{\bar{z}} \\
& - (s-w)\partial_z^{-1}x_{\bar{z}} \\
3s_w =& \tfrac{3}{2}t_{zz\bar{z}} + \tfrac{9}{2}tt_{z\bar{z}} + \tfrac{9}{4}t_zt_{\bar{z}} + \tfrac{9}{2}t^2t_{\bar{z}} - \tfrac{3}{2}xt_{\bar{z}} - 3ss_{\bar{z}} \\
& - s_z\partial_z^{-1}(s+w)_{\bar{z}} - x_{z\bar{z}} - 3tx_{\bar{z}} - (\tfrac{3}{2}t_z + 3t^2 - x)\partial_z^{-1}x_{\bar{z}} \\
3w_w =& \tfrac{3}{2}t_{zz\bar{z}} - \tfrac{9}{2}tt_{z\bar{z}} - \tfrac{9}{4}t_zt_{\bar{z}} + \tfrac{9}{2}t^2t_{\bar{z}} - \tfrac{3}{2}xt_{\bar{z}} - 3ww_{\bar{z}} \\
& - w_z\partial_z^{-1}(s+w)_{\bar{z}} + x_{z\bar{z}} - 3tx_{\bar{z}} - (\tfrac{3}{2}t_z - 3t^2 + x)\partial_z^{-1}x_{\bar{z}} \\
3x_w =& \tfrac{3}{4}(w-s)_zt_{\bar{z}} - \tfrac{9}{4}(s-w)t_{z\bar{z}} + \tfrac{1}{2}(s+w)_{zz\bar{z}} + \tfrac{3}{2}t(s-w)_{z\bar{z}} \\
& - 2x(s+w)_{\bar{z}} - x_z\partial_z^{-1}(s+w)_{\bar{z}} - \tfrac{3}{2}(w+s)x_{\bar{z}} - \tfrac{1}{2}(w+s)_z\partial_z^{-1}x_{\bar{z}}
\end{aligned}
\tag{5.29}
$$

From these equations, as usual, a recursion operator, here a 4×4 matrix, can be read off. From (5.29) the entire hierarchy can be reproduced in the now usual manner. First we impose $t_{\bar{z}} = s_{\bar{z}} = w_{\bar{z}} = x_{\bar{z}} = 0$; we take $\partial_z^{-1}(s+w)_{\bar{z}}$ and $\partial_z^{-1}x_{\bar{z}}$ as constants to obtain

$$
\begin{aligned}
3t_w &= \lambda t_z + \mu(s-w) \\
3s_w &= \lambda s_z + \mu(\tfrac{3}{2}t_z + 3t^2 - x) \\
3w_w &= \lambda w_z + \mu(\tfrac{3}{2}t_z - 3t^2 + x) \\
3x_w &= \lambda x_z + \mu\big(\tfrac{1}{2}(s+w)_z\big)
\end{aligned}
\tag{5.30}
$$

Here λ, μ are constants. Taking $\lambda = 1, \mu = 0$ we obtain the trivial $r = 1$ flow, and taking $\lambda = 0, \mu = 1$ we obtain the $r = 2$ flow (as in the second paper of [9], equation (5.12)). Reducing (5.29) instead by taking $t_{\bar{z}} = t_z, s_{\bar{z}} = s_z, w_{\bar{z}} = w_z, x_{\bar{z}} = x_z$ we find the $r = 4$ flow

$$\begin{aligned}
3t_w &= (s-w)_{zz} - t(s+w)_z - \tfrac{5}{2}(s+w)t_z - (s-w)x \\
3s_w &= x^2 - x_{zz} - 3t^2x - 3(tx)_z + \tfrac{3}{2}t_{zzz} + \tfrac{9}{2}tt_{zz} + \tfrac{9}{4}(t_z)^2 + \tfrac{3}{2}(t^3)_z - (4s+w)s_z \\
3w_w &= -x^2 + x_{zz} + 3t^2x - 3(tx)_z + \tfrac{3}{2}t_{zzz} - \tfrac{9}{2}tt_{zz} - \tfrac{9}{4}(t_z)^2 + \tfrac{3}{2}(t^3)_z - (4w+s)w_z \\
3x_w &= \tfrac{1}{2}(s+w)_{zzz} - \tfrac{5}{2}\big(x(s+w)\big)_z + \tfrac{3}{2}t(s-w)_{zz} - \tfrac{9}{4}(s-w)t_{zz} - \tfrac{3}{4}t_z(s-w)_t
\end{aligned} \tag{5.31}$$

This agrees with equations (5.16)-(5.19) in the second paper of [9]. Finally let us reduce (5.29) by imposing

$$\begin{aligned}
t_{\bar{z}} &= s - w \\
s_{\bar{z}} &= \tfrac{3}{2}t_z + 3t^2 - x \\
w_{\bar{z}} &= \tfrac{3}{2}t_z - 3t^2 + x \\
x_{\bar{z}} &= \tfrac{1}{2}(s+w)_z
\end{aligned} \tag{5.32}$$

(i.e. imposing the $r = 2$ flow in the $\bar{z}, z$ plane); we find then the $r = 5$ flow in the w, z plane:

$$\begin{aligned}
3t_w &= (3t^2 - 2x)_z - 2(s^2 - w^2) \\
3s_w &= (s - 2w)_{zz} - 6t^2(s+w) + 2x(s+w) - 6tw_z - 3(s+w)t_z \\
3w_w &= (2s - w)_{zz} + 6t^2(s+w) - 2x(s+w) - 6ts_z - 3(s+w)t_z \\
3x_w &= \tfrac{3}{2}t_{zzz} + 6(t^3)_z - 6(xt)_z + 2(ws)_z - 2(w^2 + s^2)_z
\end{aligned} \tag{5.33}$$

This was found in the third paper of [9] (equation (26)). However, it was only derived after taking the lower flows and finding the relevant bihamiltonian structure. Here we have derived it using only data from the SDYM equations. (We emphasize again that imposing equation (5.32) as a reduction is ad hoc; we are just showing how to use the recursion operator, which is the object we have found from SDYM). It is worth mentioning that if we define $\Sigma = s+w, \Delta = s-w$ then (5.29) can be written in the form

$$3\begin{pmatrix} t_w \\ \Sigma_w \\ \Delta_w \\ x_w \end{pmatrix} = \begin{pmatrix} -\partial & -\frac{3}{2}\Delta & -\frac{3}{2}\Sigma & -\frac{3}{2}(t\partial + t_z) \\ \frac{3}{2}\Delta & -\frac{9}{2}(2t\partial + t_z) & 3(\partial^2 + 3t^2 - x) & -\frac{3}{4}(3\Sigma\partial + 2\Sigma_z) \\ \frac{3}{2}\Sigma & -3(\partial^2 + 3t^2 - x) & \frac{9}{2}(2t\partial + t_z) & -\frac{3}{4}(3\Delta\partial + 2\Delta_z) \\ -3t\partial & -\frac{3}{4}(3\Sigma\partial + \Sigma_z) & -\frac{3}{4}(3\Delta\partial + \Delta_z) & \frac{3}{2}(\frac{1}{2}\partial^3 - 2x\partial - x_z) \end{pmatrix}$$
$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{3}{2}\partial \\ -1 & 0 & 0 & 0 \\ 0 & \frac{3}{2}\partial & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} t_{\bar{z}} \\ \Sigma_{\bar{z}} \\ \Delta_{\bar{z}} \\ x_{\bar{z}} \end{pmatrix} \tag{5.34}$$

What we have done here is factorize the recursion operator $\mathcal{R}$ in the manner $\mathcal{R} = \Theta_2\Theta_1^{-1}$ where Θ_1, Θ_2 are local anti-self-adjoint operators. In general for a bihamiltonian system the recursion operator can be factored in this manner, and the two operators Θ_1, Θ_2 give the two hamiltonian structures. The factorization above reproduces the bihamiltonian structure given for this hierarchy in [9]; although such a factorization is not algorithmic, in the above case it could easily be guessed, and it is to be hoped that in other examples it might be too. (We did not mention this factorization in our work on the standard KdV equation in section 3, or our work on the $N = 3$ KdV in section (5.1) as it is standard, see for example [26]. MKdV equations are in general not bihamiltonian in the usual sense but it is still possible to factorize the recursion operator into two anti-self-adjoint operators, one of which is local, and gives the one hamiltonian structure that exists, and the other of which is *nonlocal*.)

We now have to tackle the problem of finding the notion of MKdV gauge for this choice of $A_{\bar{w}}$. Recall that the matrix A_z, after imposing the appropriate constraints from (2.3c) had the form

$$A_z = \begin{pmatrix} p & q & 1 \\ s & t & -q \\ v & w & -p-t \end{pmatrix} \tag{5.35}$$

We have a two parameter gauge invariance, which we used in KdV gauge to set $p = q = 0$. The most natural guess for an MKdV gauge is to use the gauge invariance to set v and one linear combination of s and w (say $s - w$ for definiteness) to zero (it is straightforward to check that this is possible). Let us do this; we could then find an MKdV equation for the remaining four functions p, q, s, t. For any solution of this MKdV we will have a solution of the equation (5.29) given by

$$\begin{aligned} t' &= q^2 + t \\ s' &= q(p-t) + s - q_z - q^3 \\ w' &= q(p+2t) + s - q_z + q^3 \\ x' &= p(p+t) + p_z - 2q^2 t - q^4 \end{aligned} \tag{5.36}$$

This is all very well, but it does not, it seems, give us the free field representation of the W_3^2 algebra. The (second) Poisson bracket algebra associated with system (5.29) is the W_3^2 algebra, where v is the Virasoro generator, t is the spin 1 operator, and s, w are the bosonic spin 3/2 operators (in general in SDYM reductions, as explained in [9], the entries on the same diagonal have the same spin, with the spin increasing as we go from top right to bottom left of the matrix). The free field representation [16] of the W_3^2 algebra has two bosonic scalars and two bosonic spin 1/2 operators. In our MKdV gauge, there will be operators of three different spins, since we have unfixed entries on three different diagonals. However, we note that there is a remarkable similarity between equation (5.36) and the free field representation of W_3^2, given for instance in [9], equations (5.2)-(5.6).

Now, we can consider for our system a more general gauge transformation which does not leave $A_{\bar{w}}$ invariant, but that lets us set $v = w = s = 0$ in A_z if we relax the condition that identified the $(2,3)$ entry of A_z with minus the $(1,2)$ entry. Then we have a form for A_z which has two entries on each of two diagonals, but it can be quickly be seen from the relevant Miura map that this will at best give us a representation for W_3^2 in terms of two spin 1 and two spin $1/2$ operators. Thus we have a puzzle to understand the correct notion of MKdV gauge for this hierarchy (should this be resolved, it would help us face the question of the free field representation for the W_N^l algebras in general, an interesting open problem).

6. Concluding Remarks

There is little that needs to be said in conclusion to the work above; the work presented is just to be considered a start of the task of systematically reducing the SDYM equations, and even for $SL(2,\mathbf{C})$ we have not been exhaustive in our study, but made a particular "choice" when it was called for. The examples of section 5 have shown us the power of looking at integrable systems as reductions of SDYM, and the prediction we have made, on the existence of integrable systems for all classes of $A_{\bar{w}}$, may well afford the most powerful tool yet for the classification of integrable systems, and certainly can provide us with many new examples.

As we have said now many times, we have, in our work, rendered the question of finding complete hierarchies within SDYM almost unnecessary, as we seem to have an algorithmic approach to finding the recursion operators associated with these hierarchies. The question still remains though as to whether there exists a reduction of SDYM to two dimensions which yields all the equations of a given hierarchy. In [9] a method of finding all the equations of the KdV hierarchies is proposed, but to find equations of the Nth KdV hierarchy for $r > N+1$ it is necessary to work with a larger gauge group than $SL(N, \mathbf{C})$. Using gauge group $SL(MN, \mathbf{C})$ it should be possible to obtain all equations of the $N-KdV$ hierarchy for $r \leq MN+1$. While this is a valid solution to the problem raised above, it is far more desirable to have a solution using only $N \times N$ matrices, and we certainly cannot rule out that such a solution might exist. We also should note that in [8], Mason and Sparling find a way to obtain every equation of the $N = 2$ KdV and NLS hierarchies by reduction from a certain higher dimensional analogue of the SDYM equations; for details on higher dimensional analogues of SDYM see [27].

One facet of this problem that seems not to have been considered in the literature is the fact, pointed out in section 2, that there actually exists a hierarchy of the SDYM equations, i.e. the generalization of equation (2.11a)

$$M_w = \mathcal{L}^n M_{\bar{z}} \qquad n = 1, 2, 3, 4, ... \tag{6.1}$$

($\mathcal{L}$ is defined in (2.11b)). It is possible to show that for $SL(2, \mathbf{C})$ this does indeed reproduce the KdV and NLS hierarchies for suitable ansätze for the matrix M. For NLS, the simpler of the two cases, the appropriate ansatz for M is

$$M = \begin{pmatrix} 0 & \beta e^{ik\bar{w}} \\ \beta^* e^{-ik\bar{w}} & 0 \end{pmatrix} \tag{6.2}$$

where β is a complex function of $w, z, \bar{z}$, and k is a real constant. A straightforward calculation shows that equations (6.1) for this ansatz require

$$\beta_w = \left[\frac{1}{ik} (\partial_z - 4\beta\partial_z^{-1} Re(\beta^* \cdot)) \right]^n \beta_{\bar{z}} \tag{6.3}$$

and we recognize in this the recursion operator of NLS raised to the nth power. This result is obviously interesting in its own right, but it may possibly be explained within the framework of [10]; in [10] we showed that on the subspace of gauge potentials defined by (2.3b) and (2.3c) there is a large gauge symmetry (generated by the two constraints (2.3b), (2.3c)). Imposing *any* further constraint (such as (2.3a) or (6.1)) to further restrict this space will still leave us with a space with some residual symmetry which we can identify, and this is one of the notions of integrability.

Whether the idea in the last paragraph should prove accurate or not, the idea of the SDYM equations as a "master integrable system" should not only help us find and classify integrable systems, but also understand better the idea of integrability. Integrable systems have already many applications in diverse branches of physics; but these are in general not in fundamental areas. We are used to the idea of symmetry playing an important role in determining the fundamental interactions, and it is to be hoped, even expected, that the beautiful hidden symmetries associated with integrable systems will find a home in this arena too.

Note Added in Proof

As this paper was nearing completion I recieved a number of articles of relevance to this work, which I have not had time to digest or refer to in the text. I have already mentioned [17]: in this paper a number of "new" hierarchies related to the Lie group $SL(3,\mathbf{C})$ are found, which are presumably the hierarchies associated with the choices of $A_{\bar{w}}$ in (4.4) for nonzero κ, λ. Checking this should be straightforward.

In [28], a detailed study of the hierarchy associated with the W_3^2 algebra is undertaken. The relationship of this hierarchy with the standard Boussinesq hierarchy (postualted in [9], and also examined in [17]) is clarified, and a version of the Miura map, which we have sought in section 5.2, is presented. Whether this is the final answer to the questions we raised in section 5.2 is unclear; moreover, the map given in [28] is derived exploiting the relationship of the Boussinesq and W_3^2 hierarchies, and still needs to be interpreted as a gauge transformation. [28] also contains some material showing that in general there will be certain relationships between all the W_N^l hierarchies for a given N.

In [29] and [30] different ways of constraining the $SL(N)$ Wess-Zumino-Witten model are considered. It is well-known that the $SL(N)$ WZW model can be constrained to give a Toda theory with a W_N symmetry [31]. It seems that there are other ways to constrain the WZW model, and specifically in [29] it is shown that there are distinct ways to do this for every embedding of $SL(2)$ in $SL(N)$, and such embeddings are classified precisely by partitions of N. This would seem to give extra strength to our hypothesis that there exists a generalization of W_N for every partition of N into integers, a result that emerged so simply in section 4. It seems of some interest to determine what physical theories are described by these algebras.

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