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Hyperbolic Vortices and Some Non-Self-Dual Classical Solutions of $SU(3)$ Gauge Theory

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Abstract

Following a proposal of Burzlaff [Phys.Rev.D **24** (1981) 546], we find solutions of the classical equations of motion of an abelian Higgs model on hyperbolic space, and thereby obtain a series of non-self-dual classical solutions of four-dimensional $SU(3)$ gauge theory. The lowest value of the action for these solutions is roughly 3.3 times the standard instanton action.

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1.Introduction

There has been some recent interest in finding finite action, non-self-dual classical solutions in (Euclidean) four dimensional non-abelian gauge theory (on flat space), in the wake of the proof of Sibner, Sibner and Uhlenbeck [1] that such objects do indeed exist for gauge group $SU(2)$. For many years after the discovery [2] and subsequent development [3] of the instanton solutions in gauge theories, it was an open question as to whether these were the only finite action solutions (this is often known in the literature as the Atiyah-Jones conjecture, see [4]). Some progress in this direction was made by Bourguignon and Lawson [5], who proved (for certain gauge groups) that the only local minima of the Yang-Mills functional were given by instantons, so other solutions would have to correspond to saddle points. Furthermore, in [6] Taubes proved that in the two dimensional abelian Higgs theory with critical coupling, both in flat and in hyperbolic space, the only finite action solutions of the equations of motion were given by the solutions of the relevant self-duality equations; this result, in hyperbolic space, implied the non-existence of finite action, non-self-dual solutions in four dimensional $SU(2)$ gauge theory with "cylindrical symmetry", as introduced by Witten [7]. We now realise that this result cannot be generalized as we might have hoped. In addition to the proof of existence of finite action, non-self-dual solutions for group $SU(2)$ [1], a set of such solutions has been explicitly constructed by Sadun and Segert [8], following a proposal of Bor and Montgomery [9].

The significance of the non-self-dual solutions, to both physics and mathematics, is currently not clear. In physics, despite the fact that the non-self-dual solutions correspond to saddle points, and not minima, of the Yang-Mills functional, to do a correct semi-classical approximation by a saddle point evaluation of the path integral, it is certainly necessary to include a contribution due to non-self-dual solutions, and if it should be the case that there is a non-self-dual solution with action lower than the instanton action (this question is currently open, and of substantial importance), then such a contribution would even dominate. Unfortunately, it is questionable whether the semi-classical approximation can give a reliable picture of quantized gauge theories; it has been argued that in four dimensional gauge theory small quantum fluctuations around classical solutions can not be responsible for confinement, unlike in certain lower dimensional theories. But it may still be possible to extract some physics from the semi-classical approach. A first step in such a direction would be to obtain a good understanding of the full set of non-self-dual solutions and their properties.

In this paper, we pursue an old idea, due to Burzlaff [10], for obtaining a non-self-dual, "cylindrically symmetric" solution for gauge group $SU(3)$. If we write $\mathbf{R}^4 = \mathbf{R} \times \mathbf{R}^3$, and identify some $SU(2)$ (or $SO(3)$) subgroup of $SU(3)$, with generators that we will denote T^i, then we can look at the set of $SU(3)$ gauge potentials which are invariant under the action of the group generated by the sum of the T^i's and the generators of rotations on the $\mathbf{R}^3$ factor of $\mathbf{R}^4$ (we choose the T^i's and the $\mathbf{R}^3$ rotation generators to satisfy the same commutation relations). We call such potentials "cylindrically symmetric" (in analogy to the standard notion of cylindrical symmetry in $\mathbf{R}^3$, which involves writing $\mathbf{R}^3 = \mathbf{R} \times \mathbf{R}^2$ and requiring rotational symmetry on the $\mathbf{R}^2$ factor). Such potentials will be specified by a number of functions of two variables, the coordinate on the $\mathbf{R}$ factor of $\mathbf{R}^4$ (which we will denote x), and the radial coordinate of the $\mathbf{R}^3$ factor (which we will denote y). Clearly the equations of motion for such cylindrically symmetric potentials (if they are consistent) will reduce to equations on the space $\{(x, y) : y \geq 0\}$. In [10] Burzlaff gave an ansatz for a cylindrically symmetric $SU(3)$ potential that would give a finite action, non-self-dual solution, with vanishing topological charge density, for every finite action solution of the equations of motion in a particular two dimensional abelian Higgs model in hyperbolic space (which is just the space $\{(x, y) : y > 0\}$, equipped with a certain metric). Most of this paper is, therefore, devoted to the study of the abelian Higgs model in hyperbolic space with arbitrary couplings; using the ball model for hyperbolic space, we argue that there should exist radially symmetric vortex solutions for a range of values of the coupling constants. For the couplings of Burzlaff we find solutions by straightforward numerical techniques. We also perform numerical experiments for other couplings; it seems quite possible that the same model, with different couplings, may emerge when examining other ansätze for non-self-dual solutions. We make some brief comments on the resulting non-self-dual solutions we have found.

2.Hyperbolic Vortices

The standard two dimensional abelian Higgs model on a spacetime with (Euclidean) metric $g_{\mu\nu}$ is given by the action

$$S \;=\; \int d^2x\sqrt{g}\left(\frac{\kappa}{2}g^{\mu\nu}D_\mu\phi\overline{D_\nu\phi}+\frac{\mu}{4}g^{\mu\mu'}g^{\nu\nu'}F_{\mu\nu}F_{\mu'\nu'}+\frac{\lambda}{8}(|\phi|^2-1)^2\right) \tag{1}$$

Here ϕ is a complex scalar field, A_μ is an abelian gauge potential, $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ is the field strength, and D denotes a covariant derivative, $D_\mu\phi=(\partial_\mu-iA_\mu)\phi$. κ, λ and μ are coupling constants; since classically an overall factor in the action is irrelevant, we can without loss of generality set $\kappa=1$. For the case of flat space ($g_{\mu\nu}=\delta_{\mu\nu}$) we can make a scale transformation $x^\mu\to\xi x^\mu$, $A_\mu\to A_\mu/\xi$ to set μ to 1, to be left with one physical parameter λ.

For the case of flat space, the above action has been thoroughly studied. Since for finite action we need $|\phi|\to 1$ at infinity, we can define, for finite action configurations, an integer-valued topological invariant, the *vorticity*

$$n=\frac{1}{2\pi}\int_{circle\ at\ \infty} d\arg\phi \tag{2}$$

Furthermore, for finiteness of the scalar field kinetic energy term in the action, it follows that if $\phi\to e^{i\chi}$ at infinity, then A_μ must tend to the pure gauge configuration $\partial_\mu\chi$ towards infinity. From this follows the *flux-vorticity relation*

$$n=\frac{1}{2\pi}\int d^2xF_{12} \tag{3}$$

For further analysis it is convenient to separate the cases $\lambda=1$ and $\lambda\neq 1$. For $\lambda=1$ it is possible to find solutions to the second-order equations of motion by solving a first-order set of equations, the "self-duality" or "Bogomolnyi" equations [11,12]. One can establish the existence of a radially symmetric solution of these equations with arbitrary vorticity n, and then, by use of an index theorem, one can show that there is in fact a $2|n|-$parameter family of solutions with vorticity n [13]. More precisely, Taubes has shown that the parameter space of $n-$vortex solutions is exactly $\mathbf{R}^{2|n|}$ [14]. The action for all $n-$vortex solutions is the same, $S = |n|\pi$, and it is convenient to consider an $n-$vortex solution, for $n > 0$ ($n < 0$) as a superposition of $|n|$ $1-$vortices ($(-1)-$vortices) at $|n|$ arbitrary points on the plane. Finally, as mentioned in the introduction, Taubes [6] has shown that, for $\lambda = 1$, the solutions of the self-duality equations give *all* finite action solutions of the equations of motion.

For $\lambda \neq 1$, one has to attack the equations of motion directly. In [15] it was established that there is a radially symmetric solution to the equations of motion for any vorticity n, for (apparently) arbitrary λ, but [11] that for $\lambda > 1$, $n > 1$ these solutions were unstable (i.e. did not correspond to minima of the action). A detailed numerical study by Jacobs and Rebbi [16] revealed that for $\lambda > (<) 1$ the action for the radially symmetric 2-vortex was greater (less) than twice that for the 1-vortex and thus the solutions with $n > 1$ were unstable (stable). Their results show convincingly that for $\lambda \neq 1$ there are no solutions of the equations of motion corresponding to two 1-vortices at some non-zero, finite separation; for $\lambda > (<) 1$ the vortices will repel (attract). It seems reasonable to suggest from this that the only solutions of the equations of motion for $\lambda \neq 1$ are the radially symmetric ones, but for our purposes it is only important to note that as we go away from "critical" coupling, the radially symmetric solutions of the equations of motion *do* persist. Another result of [16] that we will see reproduced for hyperbolic vortices is that the action for the 1-vortex is an increasing function of λ.

We now turn to the hyperbolic case. There are several useful representations of hyperbolic space; in [7] and [10], hyperbolic space appears naturally in the upper half plane model, $\{(x, y) : y > 0\}$ with metric $g_{\mu\nu} = \delta_{\mu\nu}/y^2$. But in this model there is no concept of radial symmetry, so it is much easier for our purposes to work with the ball model, $\{(x^1, x^2) : r = \sqrt{(x^1)^2 + (x^2)^2} < R\}$ with the metric $g_{\mu\nu} = \delta_{\mu\nu}/h$ where

$$h = \frac{(R^2 - r^2)^2}{4R^2} \tag{4}$$

Here R is an arbitrary parameter. The two models of hyperbolic space are related by the conformal transformation

$$x^1 + ix^2 = R\left(\frac{iR - (x+iy)}{iR + (x+iy)}\right) \tag{5}$$

We note that the point $(0, R)$ in the upper half plane model maps to the origin in the ball model. Using the ball model, our action is simply

$$S \;=\; \int_{r<R} d^2x \left(\frac{1}{2}D_\mu\phi\overline{D_\mu\phi} + \frac{h\mu}{4}F_{\mu\nu}F_{\mu\nu} + \frac{\lambda}{8h}(|\phi|^2 - 1)^2\right) \tag{6}$$

A scaling transformation here, $x^\mu \to \xi x^\mu$, $A_\mu \to A_\mu/\xi$, $R \to \xi R$ *cannot* be used to remove one of the parameters λ, μ (though it does show us that the choice of R is arbitrary). So in the hyperbolic abelian Higgs model we have two coupling constants.

Another difference between the hyperbolic and flat space cases is that we cannot, in the hyperbolic case, write down an immediate flux-vorticity relation, simply by finiteness of the action arguments. We can still define vorticity, as since $h \to 0$ as $r \to R$, we need $|\phi| \to 1$ as $r \to R$; we therefore define

$$n = \frac{1}{2\pi}\int_{circle\ r=R} d\arg\phi \tag{7}$$

Unlike the flat case though, we have no *finiteness* reason to insist that $|D\phi| \to 0$ as we approach the spacetime boundary. However, to make our theory well-defined we need to specify some specific behavior for the fields at the boundary, and, specifically, we would like to choose behavior such that the surface term, that appears when we vary the action to obtain equations of motion, vanishes. For this, the obvious condition to impose is $|D\phi| \to 0$ as $r \to R$; the solutions we obtain are consistent with this. We then have the flux-vorticity relation (3).

We approach the action (6) as we do in the flat case. It is first useful to establish when we can write a set of self-duality equations. Using the identity

$$D_\mu\phi\overline{D_\mu\phi} = |(D_1 \pm iD_2)\phi|^2 \pm |\phi|^2 F_{12} \pm i\big(\partial_1(\phi\overline{D_2\phi}) - \partial_2(\phi\overline{D_1\phi})\big) \tag{8}$$

we can integrate by parts to write the action

$$S = \frac{1}{2}\int_{r<R} d^2x\bigg(|(D_1 \pm iD_2)\phi|^2 + \bigg(\sqrt{h\mu}F_{12} \pm \frac{1}{2\sqrt{h\mu}}(|\phi|^2-1)\bigg)^2 \pm F_{12}\bigg) \tag{9}$$

provided $\lambda\mu = 1$, which is the condition for self-duality. In this case we can at once write down the self-dual equations

$$\begin{aligned}(D_1 \pm iD_2)\phi = 0\\ F_{12} \pm \frac{\lambda}{2h}(|\phi|^2-1) = 0\end{aligned} \tag{10}$$

Here, and in all that follows, the upper sign is appropriate for positive n, and the lower sign for negative n.) If we write $\phi = fe^{i\omega}$, we can solve the first of these to obtain

$$A_\mu = \pm\epsilon_{\mu\nu}\partial_\nu \ln f + \partial_\mu\omega \tag{11}$$

and the other equation yields a single equation for f (ω is just the gauge degree of freedom), which we can write in the form

$$\nabla^2 \ln\bigg(\frac{\lambda f^2}{h}\bigg) = \bigg(\frac{\lambda f^2}{h}\bigg) + \bigg(\frac{2-\lambda}{h}\bigg) \tag{12}$$

In writing this we have exploited the fact that $\nabla^2 \ln h = -2/h$. We see straight away that the case $\lambda = 2$, $\mu = 1/2$ is very special; in this case we obtain the Liouville equation, an integrable equation. This is the case that Witten considered in [7], where he found explicitly $2|n|$ solutions of vorticity n. For general λ, however, Painlevé analysis suggests that (12) is not integrable [17]. For later reference let us write down the equations for a radially symmetric solution to the self-duality equations; the appropriate ansatz for a radially symmetric n-vortex is

$$\phi = f(r)e^{in\theta} \tag{13}$$

where θ is the usual polar coordinate. Equation (11) gives

$$A_\mu = -n\epsilon_{\mu\nu}x_\nu \frac{a(r)}{r^2}$$
$$a(r) = 1 - \frac{rf'}{|n|f} \tag{14}$$

and the second of equations (10) tells us

$$\frac{|n|a'}{r} = \frac{\lambda}{2h}(1 - f^2) \tag{15}$$

or, equivalently, f must satisfy equation (12), which reduces to

$$(\ln f)'' + \frac{(\ln f)'}{r} = \frac{\lambda}{2h}(f^2 - 1) \tag{16}$$

For the integrable case, $\lambda = 2$, we can write down the solutions to this equation satisfying the necessary boundary conditions

$$f = p\left(\frac{(r/R) - (R/r)}{(r/R)^p - (R/r)^p}\right) \tag{17}$$

where $p = |n| + 1$. It is straightforward to check that these solutions have the following asymptotic behaviors; near $r = 0$

$$f(r) \sim p\left(\frac{r}{R}\right)^{|n|}$$
$$a(r) \sim \frac{2}{p-1}\left(\frac{r}{R}\right)^2 \tag{18}$$

and near $r = R$

$$1 - f(r) \sim \frac{p^2 - 1}{6}\left(1 - \frac{r}{R}\right)^2$$
$$1 - a(r) \sim \frac{(p-1)^2(p+1)}{3}\left(1 - \frac{r}{R}\right) \tag{19}$$

We will later be able to use these as a check for the asymptotic behaviors for general λ, μ.

Let us now look at the action (6) for arbitary λ, μ. The equations of motion are

$$D_\mu D_\mu \phi + \frac{\lambda\phi}{2h}(1 - |\phi|^2) = 0$$
$$\mu\partial_\nu(hF_{\mu\nu}) + \frac{i}{2}(\overline{\phi}D_\mu\phi - \phi D_\mu\overline{\phi}) = 0 \tag{20}$$

We look for a radially symmetric $n-$vortex solution in the form

$$\begin{aligned} \phi &= f(r)e^{in\theta} \\ A_\mu &= -n\epsilon_{\mu\nu}x_\nu \frac{a(r)}{r^2} \end{aligned} \tag{21}$$

The equations of motion reduce to

$$\begin{aligned} f'' + \frac{f'}{r} - \frac{n^2 f(1-a)^2}{r^2} + \frac{\lambda f}{2h}(1-f^2) = 0 \\ a'' - \left(\frac{1}{r} - \frac{h'}{h}\right)a' + \frac{f^2}{\mu h}(1-a) = 0 \end{aligned} \tag{22}$$

(Note that these reduce to equations (2.18) in ref [16] if we set $h = 1$, and suitably redefine coupling constants.) At this point it is useful to introduce the variable $t = r/R$ to eliminate the constant R from the problem. Using a dot to denote differentiation with respect to t, we obtain

$$\begin{aligned} \ddot{f} + \frac{\dot{f}}{t} - \frac{n^2 f(1-a)^2}{t^2} + \frac{2\lambda}{(1-t^2)^2}(1-f^2) = 0 \\ \ddot{a} - \left(\frac{1}{t} + \frac{4t}{1-t^2}\right)\dot{a} + \frac{4f^2}{\mu(1-t^2)^2}(1-a) = 0 \end{aligned} \tag{23}$$

It is straightforward to compute the action density for the ansatz (21), and we obtain

$$\begin{aligned} S &= 2\pi \int_0^1 dt E(t) \\ E(t) &= \frac{t\dot{f}^2}{2} + \frac{n^2 f^2(1-a)^2}{2t} + \frac{\mu n^2(1-t^2)^2\dot{a}^2}{8t} + \frac{\lambda t(1-f^2)^2}{2(1-t^2)^2} \end{aligned} \tag{24}$$

We need to analyze the system (23) with the requisite boundary conditions. The first step is to write Frobenius-type expansions for the solutions of (23) near the points $t = 0$ and $t = 1$, both of which are singular points of (23). We obtain the following results: near $t = 0$ the nonsingular solutions of (23) have form

$$\begin{aligned} f &= At^{|n|}\left(1 + \sum_{q=1}^{\infty} f_q t^{2q}\right) \\ a &= Bt^2\left(1 + \sum_{q=1}^{\infty} a_q t^{2q}\right) \end{aligned} \tag{25}$$

Here A, B are some unspecified constants, and the f_q's and a_q's are constants determined by A, B. It is possible to write a recursion relation for f_q, a_q in terms of $A, B, f_1, a_1, ..., f_{q-1}, a_{q-1}$, but here we just give the first few coefficients explicitly

$$\begin{aligned}
f_1 &= -\left(\frac{Bn^2+\lambda}{2(|n|+1)}\right) \\
f_2 &= \frac{1}{8(|n|+2)}\left(\frac{(Bn^2+\lambda)^2}{|n|+1} - 4\lambda + n^2B(B-2) + \left(2\lambda + \frac{n^2}{\mu}\right)A^2\delta_{|n|1}\right) \\
a_1 &= 1 - \frac{A^2}{2\mu}\delta_{|n|1} \\
a_2 &= 1 - \frac{A^2}{6\mu B}\delta_{|n|2} - \frac{A^2}{6\mu B}\left(4 - B - \frac{Bn^2+\lambda}{|n|+1}\right)\delta_{|n|1}
\end{aligned} \tag{26}$$

Near $t = 1$ we find that solutions of (23) with $f(1), a(1)$ finite are given by series

$$\begin{aligned}
1 - f &= \alpha(1-t)^\zeta\left(1 + \sum_{q=1}^{\infty} g_q(1-t)^q\right) \\
1 - a &= \beta(1-t)^d\left(1 + \sum_{q=1}^{\infty} b_q(1-t)^q\right)
\end{aligned} \tag{27}$$

Here α, β are arbitrary constants, the g_q's and b_q's are defined by a recursion relation, and ζ, d are given as the positive roots of

$$\begin{aligned}
\lambda &= \zeta(\zeta-1) \\
\frac{1}{\mu} &= d(d+1)
\end{aligned} \tag{28}$$

Equations (28) are very pleasing. For Witten's case [7], $\lambda = 2$ and $\mu = 1/2$, so we have $\zeta = 2$ and $d = 1$. For Burzlaff's case [10], $\lambda = 2$ and $\mu = 1/6$, so we have $\zeta = 2$ and $d = 2$. Uhlenbeck [18] has shown that any solution of the Yang-Mills equations on $\mathbf{R}^4$ with finite action can be obtained (in a suitable gauge) from a smooth gauge field on S^4; thus if we are to obtain finite-action solutions of the Yang-Mills equations from either the Witten or Burzlaff ansätze, we need the coefficients ζ and d to be integers, and we see they are.

To summarize our problem, we see that we need to find solutions of (23), with f, a given by (25) (for some A, B) near $t = 0$, and by (27) (for some α, β) near $t = 1$. Intuitively this problem is solvable; essentially we just need to choose A, B, α, β in such a way that

$f, a, \dot{f}, \dot{a}$ are continuous. We use a straightforward numerical method to actually solve the problem. For a specific A, B we use the power series (25) with the coefficients (26) to obtain f, a up to $t = 0.002$. We then use the Runge-Kutta method (introducing extra dependent functions $g = \dot{f},\ b = \dot{a}$ to obtain a first order system), with a step length of 10^{-5}, to integrate up to $t = 1$. All work was performed with double precision arithmetic. There is an inherent instability as we approach $t = 1$, corresponding, roughly speaking, to the negative roots d, ζ of equations (28). For generic A, B the functions f, a will be unbounded as we approach $t = 1$. We label the functions f, a arising from the numerical integration with a "+" if they are monotonically nondecreasing, and with a "−" otherwise. Thus we can plot two curves in the $A, B-$plane corresponding to the values of A, B where f and a change from "+" to "−" behavior. The critical values of A, B required for the vortex solution, A_{crit}, B_{crit}, will be at the intersection of these two curves. Plots of the curves in the $A, B-$plane are shown in figure 1 for $n = 1$ for the Witten case $\lambda = 2$, $\mu = 1/2$, and similar plots are found for $n = 2, 3, 4$. We obtain the results

$$\begin{aligned} A_{crit} &= |n| + 1 \\ B_{crit} &= \frac{2}{|n|} \end{aligned} \tag{29}$$

as expected from (18).

In general we find we require very accurate values of A_{crit}, B_{crit} (accuracy of about one part in 10^9) to obtain "reasonable" vortex solutions. There is a useful method of checking the "reasonableness" of a vortex solution; the action density $E(t)$ defined in (24) is linear in t for $t \approx 0$, and for $t \approx 1$ goes as $(1 - t)^{2z}$, where

$$z = \min(d, \zeta - 1) \tag{30}$$

In general, because of the instability at $t = 1$, we will find the numerical $E(t)$ has a slight "tail", that is, instead of tending to zero in the expected way at $t = 1$, it will, after a point, display a slight increase. We have aimed to obtain A_{crit}, B_{crit} to an accuracy such that this "tail" affects the numerical approximation to the action by less than one part in 10^3. The algorithm performs correctly, to well within the required accuracy, for the Witten case.

It remains to give some results. First we check a self-dual case, $\lambda = 6$, $\mu = 1/6$. For any self-dual case, it is easy to check that for the $n-$vortex

$$\begin{aligned} B_{crit} &= \frac{\lambda}{|n|} \\ S &= \pi|n| \end{aligned} \tag{31}$$

We reproduce these results accurately, and we find the values for A_{crit} given in table 1. (Note that while we quote A_{crit} values to the accuracy necessary to make our numerical algorithm produce reasonable vortex solutions, it is possible that the A_{crit} of our numerical procedure is only the same as the real A_{crit} to a lower degree of accuracy.) In figure 2 we display the curves in the $A, B-$plane for this case, for $n = 1$, and in figure 3 we display the functions f, a for $n = 1, 2, 3, 4$ for both the Witten case and this case: note the difference in the behaviors at $t = 1$.

Now we move to the Burzlaff case, $\lambda = 2$, $\mu = 1/6$. We obtain the results in table 2. The $A, B-$plane plot, for $n = 1$, is shown in figure 4 and f, a plots, for $n = 1, 2, 3, 4$, are in figure 5. The $A, B-$plane plot shows an interesting feature: the curve marking the change of behavior of f apparently has a cusp at (A_{crit}, B_{crit}). This feature is reproduced for higher n, and we have found this feature in general for $\lambda\mu < 1$ (but it seems that the curve straightens as $\lambda\mu \nearrow 1$, and we have not noticed a cusp in plots for $\lambda\mu > 1$). One proviso is in order here: our numerical algorithm is not necessarily reliable anywhere but exactly at the vortex solution.

Finally we present results for $1-$ and $2-$vortices for $\lambda = 2$ and a range of values of μ. The results are summarized in table 3, where the superfix on A_{crit} etc. denotes the value of n. The main result here is that for $\lambda\mu < 1$ ($\lambda\mu > 1$) it seems that the $2-$vortex is stable (unstable) against decay into two $1-$vortices. This, and the fact that the action is an increasing function of $\lambda\mu$ are in accordance with the flat space results.

3.Non-Self-Dual $SU(3)$ Yang-Mills Solutions

We feel no need to reproduce verbatim the analysis of Burzlaff [10], save for one point that requires a little clarification. Witten's ansatz [7] for cylindrically symmetric $SU(2)$ gauge fields can be embedded into $SU(3)$ in two distinct ways. One uses the the $SU(2)$ subalgebra of $SU(3)$ with the generators $\sigma^i = (\lambda_1/2, \lambda_2/2, \lambda_3/2)$ and the other uses the $SU(2)$ subalgebra with generators $E^i = (\lambda_7, -\lambda_5, \lambda_2)$ (Here the λ_a's are the usual Gell-Mann matrices). Both σ^i and E^i satisfy the $SU(2)$ commutation relation

$$[T^i, T^j] = i\epsilon^{ijk}T^k \tag{32}$$

but we have different trace formulae

$$\begin{aligned} Tr(\sigma^i\sigma^j) &= \tfrac{1}{2}\delta^{ij} \\ Tr(E^iE^j) &= 2\delta^{ij} \end{aligned} \tag{33}$$

Because of this difference, when we use the E^i's for the embedding we obtain the hyperbolic space action (6) with $\lambda = 2$, $\mu = 1/2$, with a prefactor of 32π, as compared to a prefactor of 8π which we obtain when using the σ^i's. Burzlaff's construction for non-self-dual solutions gives the action (6) with $\lambda = 2$, $\mu = 1/6$, with a prefactor of 32π.

Having stated this we can at once give the main result of this paper: we have non-self-dual $SU(3)$ solutions with action given by $64\pi^2$ times the figures in the last column of table 2. This is in units where the standard instanton action is $8\pi^2$, so the lowest action of our solutions is roughly 3.3 times the instanton action. We would speculate that there exists a solution of the type we have looked at for any n, and we see that the action for the "$n-$solution" is less than n times the action of the basic solution. The asymptotic behavior for large n is clearly of interest. We note that the solution for $n = 4$ is almost as low as only three times the $n = 1$ action. The lowest action for a non-self-dual solution in $SU(2)$ found by Sadun and Segert [8] is roughly 5.4 times the instanton action. In $SU(4)$ gauge theory, it is clear that we can find a non-self-dual solution with action twice that of the instanton: we pick two commuting $SU(2)$ subalgebras and consider the potential which is composed of an instanton in one $SU(2)$ and an anti-instanton in the other $SU(2)$.

Let us investigate just briefly the geometry of our solutions. To do this we must revert to the upper half plane model of hyperbolic space. In section 2 we introduced Cartesian coordinates (x, y) on hyperbolic space in the upper half plane model, and (x^1, x^2) on hyperbolic space in the ball model, and we also used polar coordinates (r, θ) on the ball model. Let us now introduce polar coordinates (ρ, ϕ) on the upper half plane model $(0 < \rho < \infty,\ 0 < \phi < \pi)$ via

$$\begin{aligned} \rho &= \sqrt{x^2 + y^2} \\ \phi &= \tan^{-1}(y/x) \end{aligned} \tag{34}$$

ρ is the standard radial coordinate of $\mathbf{R}^4$ (that is, if the $\mathbf{R}^4$ coordinates are (y^1, y^2, y^3, y^4), then $\rho = \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2 + (y^4)^2}$). The action density of our solutions can be expressed in the ball model as a function of r alone, but in the upper half plane model it is a function of the two variables ρ, ϕ. Explicitly we have, for the Yang-Mills action,

$$\begin{aligned} YM &= 32\pi \int_0^R \int_0^{2\pi} d\left(\frac{r}{R}\right) d\theta \ E(r/R) \\ &= 32\pi \int_0^\infty \int_0^\pi d\left(\frac{\rho}{R}\right) d\phi \ JE(r/R) \end{aligned} \tag{35}$$

where J is the necessary Jacobian

$$J = \frac{s}{t}\left(\frac{1}{1 + s^2 + 2s\sin\phi}\right)^2 \tag{36}$$

where we have written $t = r/R$, as before, and we have introduced $s = \rho/R$. In terms of ρ, ϕ, we have

$$t = \frac{r}{R} = \sqrt{1 - \frac{4s\sin\phi}{1 + s^2 + 2s\sin\phi}} \tag{37}$$

The functions $E(t)$ for our solutions, with $n = 1, 2, 3, 4$ are displayed in figure 6; as mentioned above, for $t \approx 1$, $E(t)$ behaves as a multiple of $(1 - t)^2$. This completes all the necessary information to work out the large ρ behavior of the action density, that is the integrand of (the second part of) (35). We find that the action density drops off (for fixed ϕ) as $(\rho/R)^{-5}$. This is identical to the behavior of the standard instanton, except of course we must remember that our solutions do not have full "spherical" symmetry in $\mathbf{R}^4$. The reproduction of the instanton result here is essentially due to the fact that the parameter z of equation (30) is the same for the Witten case and the Burzlaff case. We note that our solutions, like standard instantons, have a scale parameter R associated with them. It seems, in fact, that there is an eight parameter family of our solutions (for each n), as opposed to a five parameter family for the standard instanton: we have in addition to the usual "center" and "scale" parameters, three extra parameters associated with the choice of the time axis, which we use to define the cylindrical symmetry. Possible subtleties could arise in this naive counting, however, due to gauge transformations. We can also consider the effect of the full conformal group on our solutions: applying special conformal transformations to our eight parameter family could generate up to a twelve parameter family of solutions (compare [19]); the form of the potentials for the solutions thus generated would, it seems, be messy, and the task of checking that these solutions are not gauge equivalent might be very tricky.

In conclusion, we just mention a few more points worthy of study, in addition to the various points that have been mentioned in passing above. It is important to examine the stability of our solutions as solutions of the Yang-Mills equations (as solutions of the hyperbolic abelian Higgs model it seems they correspond to genuine minima of the action). By virtue of the results of [5] they do not correspond to minima of the action functional, but to saddle points, and it is of interest to count (if it is finite) the number of small variations away from the solutions that reduce the action functional. Intuitively, the objects that we have found are potentially of more relevance to physics if this number is low. Another question that could produce interesting results is to generalize Burzlaff's ansatz to larger groups, using other $SU(2)$ embeddings, to see if we can obtain hyperbolic abelian Higgs models with other couplings. We expect the couplings to be such that the numbers ζ,d defined by equation (28) are integers.

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Table Captions

Table 1: A_{crit} values for $\lambda = \mu^{-1} = 6$

Table 2: Vortex solution data for $\lambda = 2,\ \mu = 1/6$

Table 3: $1-$ and $2-$vortex solution data for $\lambda = 2$ and various values of μ

Figure Captions

Figure 1: $A, B-$plane plot for $\lambda = 2$, $\mu = 1/2$, $n = 1$. (The label "+−" denotes that the function f has "+" behavior and the function a has "−" behavior in the marked region; the other labels are defined similarly.)

Figure 2: $A, B-$plane plot for $\lambda = 6$, $\mu = 1/6$, $n = 1$.

Figure 3: (a) $f(t)$ plots for $\lambda = 2$, $\mu = 1/2$, $n = 1, 2, 3, 4$.

(b) $a(t)$ plots for $\lambda = 2$, $\mu = 1/2$, $n = 1, 2, 3, 4$.

(c) $f(t)$ plots for $\lambda = 6$, $\mu = 1/6$, $n = 1, 2, 3, 4$.

(d) $a(t)$ plots for $\lambda = 6$, $\mu = 1/6$, $n = 1, 2, 3, 4$.

Figure 4: $A, B-$plane plot for $\lambda = 2$, $\mu = 1/6$, $n = 1$.

Figure 5: (a) $f(t)$ plots for $\lambda = 2$, $\mu = 1/6$, $n = 1, 2, 3, 4$.

(b) $a(t)$ plots for $\lambda = 2$, $\mu = 1/6$, $n = 1, 2, 3, 4$.

Figure 6: $E(t)$ plots for $\lambda = 2$, $\mu = 1/6$, $n = 1, 2, 3, 4$.

Tables

n	A_{crit}
1	3.13728895
2	6.80933129
3	12.40261138
4	20.30221717

Table 1: A_{crit} values for $\lambda = \mu^{-1} = 6$

n	A_{crit}	B_{crit}	$S/2\pi$
1	2.32258782	4.55248618	0.412
2	4.18191496	2.18876301	0.783
3	6.57417323	1.43867781	1.145
4	9.51117487	1.07160765	1.504

Table 2: Vortex solution data for $\lambda = 2,\ \mu = 1/6$

μ	A^1_{crit}	B^1_{crit}	A^2_{crit}	B^2_{crit}	$S^1/2\pi$	$S^2/2\pi$
1/6	2.3225878	4.5524862	4.1819150	2.1887630	0.412	0.783
0.4	2.0546757	2.3706522	3.1859077	1.1771755	0.481	0.954
0.5	2.0000000	2.0000000	3.0000000	1.0000000	0.500	1.000
0.75	1.9131831	1.4605004	2.7169828	0.7383847	0.534	1.085
1.0	1.8607006	1.1626815	2.5531907	0.5917803	0.557	1.145
1.5	1.7984541	0.8362130	2.3661313	0.4290016	0.589	1.227
5.0	1.6807265	0.2949309	2.0341395	0.1535178	0.665	1.421

Table 3: 1− and 2−vortex solution data

for $\lambda = 2$ and various values of μ