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Integrability of Chern-Simons-Higgs and Abelian Higgs Vortex Equations in a Background Metric

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Abstract

We reformulate the equations for Chern-Simons-Higgs and Abelian Higgs vortices in a certain form of background metric. We apply Painlevé analysis to determine integrability of the equations, and find explicit solutions for cylindrically symmetric Chern-Simons-Higgs vortices for a specific choice of the metric.

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1. Introduction

Several authors [1] have recently shown that in a $(2+1)$-dimensional Chern-Simons-Higgs theory (i.e. a theory with a Higgs field interacting with an Abelian gauge field with dynamics governed by the pure Chern-Simons term*), with a certain form of ϕ^6 potential, there exist vortex-type finite-energy solutions, that can be obtained by solving a set of a first order differential equations. Similar results have been known for some time in the usual Abelian Higgs model, with a certain form of ϕ^4 potential [3]. In neither the Abelian Higgs case nor the Chern-Simons-Higgs case have exact vortex solutions been found; in both cases however, the equations describing the vortices look very similar to known integrable equations (we will elaborate on our understanding of the word 'integrable' shortly). It is natural to ask if the vortex equations are integrable, and, if not, if we can modify the theory in some simple way to produce integrable equations, and hence obtain solutions. In this paper we will show that both the Abelian Higgs and Chern-Simons-Higgs models can be reformulated in a background metric of the form $g_{\mu\nu} = diag(1, -b, -b)$, where b is some function of the space variables x_1, x_2. For the Chern-Simons-Higgs model we will show that for a special choice of b the equation describing cylindrically symmetric vortices becomes integrable, and we can extract solutions; furthermore the metric we use is physically acceptable in some sense. For the Abelian Higgs model we show that the equation describing general, not necessarily cylindrically symmetric, solutions can be made integrable for suitable choices of b. However, if we choose such a b which is cylindrically symmetric, the solutions to the equations for cylindrically symmetric vortices do not satisfy the necessary boundary conditions. For both Abelian Higgs and Chern-Simons-Higgs cases we present evidence that the usual equations in flat space are non-integrable.

In this paper we meet the concept of integrability, both for ordinary differential equations (the equations for cylindrically symmetric vortices) and partial differential equations (the equations for general vortex solutions). In both cases, we will use as our working definition of integrability the Painlevé property. A differential equation is said to possess

* The Chern-Simons term is non-dynamical. However it has been shown [2] that when a gauge field in three dimensions with Chern-Simons action is coupled to a Higgs field it becomes dynamical.

Painlevé property if its solutions have no movable singularities other than poles. (Solutions of differential equations in general display two types of singularities: singularities can emerge because of singularities in coefficient functions of the equation; these will be at specified points and are called 'fixed' singularities. Singularities can also emerge from the choice of boundary conditions, and these can typically occur anywhere, and are therefore called movable singularities.) For second order ODEs (satisfying certain conditions) there is a classification of equations with Painlevé property into 50 types [4], 44 of which can be integrated in terms of standard functions, and 6 others, the canonical forms of which are known as the Painlevé equations. The notion of integrability for second order ODEs is essentially the ability to solve the ODE, and we see that Painlevé property coincides with this notion provided we are willing to add the Painlevé transcendents (solutions of the Painlevé equations) to our vocabulary of standard functions. For our work here, in the Abelian Higgs case, when we have integrability we obtain a special case of the third Painlevé equation, which can be solved and has a rational function as its solution. In the Chern-Simons-Higgs case we obtain a variant of the third Painlevé equation, which requires substantial analysis, which is presented in appendix A, where also the cylindrically symmetric vortex equations for arbitrary b are tested to see when they have Painlevé property. We find, in both the Abelian Higgs and Chern-Simons-Higgs cases, that the equations are non-integrable for the flat metric.

For PDEs, the Painlevé property was introduced as a test for integrability in [5]; there exist more standard notions of integrability, such as the existence of an infinite number of conservation laws, or solvability by inverse scattering techniques, and remarkably the Painlevé property seems to predict these other properties, which are substantially harder to verify directly. In appendix B we perform Painlevé analysis on the PDEs of interest to us here; we introduce a technique to handle the arbitrary function that occurs in the PDE we wish to test for Painlevé property. We find that the general vortex equation is non-integrable for any b for the Chern-Simons-Higgs case, but is integrable for a large class of b's for the Abelian Higgs case.

Possibly the most significant of our results is the fact that for a particular metric we can exhibit exact solutions for Chern-Simons-Higgs vortices. This may be just a piece of luck. On the other hand it is suggestive that while the usual (flat metric) vortex equations are non-integrable, maybe in physical instances where they occur, they occur coupled to some other field in such a way that they become integrable. For applications of gauge field-Higgs systems to cosmic strings the result we have obtained is definitely attractive; for applications to high T_C superconductors it is necessary to think of how to couple the relevant system to another type of background field, but this is certainly not unimaginable.

The rest of this paper is arranged as follows: section 2 deals with Chern-Simons-Higgs vortices, section 3 with Abelian Higgs vortices, and section 4 suggests some avenues for further development. As mentioned above, appendices A and B deal with all the necessary Painlevé analysis for ODEs and PDEs respectively. In both appendices we go into substantial detail, as the calculations have various subtleties involved. In conclusion to the introduction we mention a few other points of relevance to the question of integrability in the Abelian Higgs and Chern-Simons-Higgs models. First, it is known for both models that the form of potential required to allow first order equations to be written for the vortices is exactly that which permits one to write a supersymmetric generalization of the theory [6]. Supersymmetry has for some time been believed to be strongly related to integrability. Second, a non-relativistic limit of the Chern-Simons-Higgs theory has recently been considered [7]; a gauged non-linear Schrödinger equation is obtained for which explicit solutions can be found, and which is in some sense integrable.

2. Chern-Simons-Higgs Vortices

We look for stationary, classical solutions of the theory defined by the action

$$S = \int d^3x \left(\frac{k}{4} \epsilon^{\mu\nu\rho} F_{\mu\nu} A_\rho + \sqrt{g} g^{\mu\nu} D_\mu \phi \overline{D_\nu \phi} - \frac{e^4}{k^2} \sqrt{g} |\phi|^2 (|\phi|^2 - m)^2 \right) \tag{2.1}$$

Here ϕ is a complex scalar field, A_μ a vector field, and $g_{\mu\nu}$ is a background metric of form $diag(1, -b, -b)$ where b is some currently undetermined function of x_1, x_2. We write $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu = \partial_\mu + ieA_\mu$. A bar denotes complex conjugation. e, k, m are parameters (e^2, k, m all having the dimensions of mass). For stationary solutions the A_0 equation of motion gives

$$A_0 = \frac{-kF_{12}}{2be^2|\phi|^2} \tag{2.2}$$

The energy momentum tensor is

$$T_{\mu\nu} = D_\mu\phi\overline{D_\nu\phi} + D_\nu\phi\overline{D_\mu\phi} + \left(\frac{e^4}{k^2}|\phi|^2(|\phi|^2-m)^2 - g^{\rho\sigma}D_\rho\phi\overline{D_\sigma\phi}\right)g_{\mu\nu} \tag{2.3}$$

and, specfically, for a stationary solution with A_0 given by (2.2) we obtain an expression for the energy

$$E = \int d^2x\sqrt{g}\, T_{00} = \int d^2x\Big(|D_1\phi|^2 + |D_2\phi|^2 + \frac{k^2F_{12}^2}{4be^2|\phi|^2} + \frac{e^4}{k^2}b|\phi|^2(|\phi|^2-m)^2\Big) \tag{2.4}$$

Using the result

$$|D_1\phi|^2 + |D_2\phi|^2 = |(D_1 \pm iD_2)\phi|^2 \mp eF_{12}|\phi|^2 + total\ derivatives \tag{2.5}$$

we obtain, assuming the total derivatives integrate to zero,

$$E = \int d^2x\Big(|(D_1\pm iD_2)\phi|^2 + \left|\frac{kF_{12}}{2\sqrt{b}e\phi} \mp \frac{e^2}{k}\sqrt{b}\,\overline{\phi}(|\phi|^2-m)\right|^2\Big) \mp m\int d^2x\ eF_{12} \tag{2.6}$$

We interest ourselves in solutions for which $|\phi|^2 \to m$ at spatial infinity. For finite energy $|D_i\phi|$ must tend to zero at spatial infinity, and thus we are led to the asymptotic form

$$\begin{aligned}\phi &\to \sqrt{m}e^{i\xi(\theta)}\\ eA_i &\to -\partial_i\xi\end{aligned} \tag{2.7}$$

For single valuedness $\xi(\theta)$ must satisfy

$$\xi(\theta+2\pi) = \xi(\theta) - 2\pi n \tag{2.8}$$

for some integer n , which is a topological invariant of solutions. By standard arguments, and providing the gauge fields satisfy certain smoothness conditions, we can write

$$n = \frac{1}{2\pi}\int d^2x\ eF_{12} \tag{2.9}$$

Thus (2.6) gives us a lower bound on the energy of solutions $E \geq 2\pi m|n|$, with equality if and only if the first order equations

$$(D_1 \pm iD_2)\phi = 0 \tag{2.10a}$$

$$eF_{12} \mp \frac{2e^4}{k^2} b|\phi|^2(|\phi|^2 - m) = 0 \tag{2.10b}$$

are satisfied (with the upper signs for negative n and the lower signs for positive n). Setting $\phi = \sqrt{m}he^{i\omega}$ in (2.10a) we deduce that (away from the zeros of h)

$$eA_i = -\partial_i \omega \mp \epsilon_{ij}\partial_j \ln h \tag{2.11}$$

Using this in (2.10b) we obtain

$$\nabla^2 \ln h = \frac{2e^4 m^2}{k^2} bh^2(h^2 - 1) \tag{2.12}$$

Writing $\gamma = 2e^2 m/k$ and $\chi = h^2$ this becomes

$$\nabla^2 \ln \chi = \gamma^2 b \chi(\chi - 1) \tag{2.13}$$

In appendix B we use Painlevé analysis to show that this is non- integrable for any choice of the function b. But if we restrict to the case of cylindrical symmetry, i.e. where b, χ depend only on the radial coordinate $r = \sqrt{x_1^2 + x_2^2}$, we obtain

$$\frac{1}{r}\left(\frac{r\chi'}{\chi}\right)' = \gamma^2 b \chi(\chi - 1) \tag{2.14}$$

where the prime denotes differentiation with respect to r. Painlevé analysis of this, presented in appendix A, suggests that this is 'partially integrable' (in the sense explained in the appendix) if we choose

$$b = \frac{p^2}{\gamma^2 r^2 (1 + (\frac{r}{\Lambda})^{-p})^2} \tag{2.15}$$

for some constants p, Λ. For this choice we find, in appendix A, a one parameter family of solutions to (2.14)

$$\chi = \frac{(\frac{r}{\Lambda})^{-p}}{B + \ln(1 + (\frac{r}{\Lambda})^{-p})} \tag{2.16}$$

where B is a constant. For $p > 0$, these solutions have bad singularities at $r = 0$ and hence are not suitable for physical vortex solutions (note though that we can satisfy the boundary condition $\chi \to 1$ at spatial infinity if we take $B = 0$). For $p < 0$, the solutions (2.16) do not have suitable behavior at spatial infinity. We will make some comment on the metrics given by (2.15) later, despite our inability to find solutions satisfying the necessary boundary conditions for them. The case where we can find exact solutions is, from appendix A,

$$b = \frac{p^2}{\gamma^2 r^2} \tag{2.17}$$

For this choice of metric, (2.14) becomes completely integrable, with general rational solution

$$\chi = \frac{\mu}{(r^{\lambda} + \alpha)(r^{-\lambda} + \beta)} \tag{2.18}$$

where the constants $\mu, \alpha, \beta, \lambda$ satisfy

$$\mu = \frac{2\lambda^2}{p(p+\lambda)} \qquad \alpha\beta = \frac{p-\lambda}{p+\lambda} \tag{2.19a}$$

or

$$\mu = \frac{2\lambda^2}{p(p-\lambda)} \qquad \alpha\beta = \frac{p+\lambda}{p-\lambda} \tag{2.19b}$$

In the above it is sufficient to take $\lambda > 0$, and we take $p > 0$. We require $\alpha, \beta \geq 0$ for a well-behaved positive solution, and it follows that $\lambda \leq p$. For a physical vortex solution we require $\chi \to 1$ at spatial infinity. The general solution satisfying this condition is

$$\chi = \frac{1}{1 + \alpha r^{-p}} \tag{2.20}$$

(for arbitrary α), which behaves as r^p at $r = 0$. Naively one might argue that this is acceptable if and only if p is an even integer. The reason for this would be the requirements of smoothness of the gauge field at the origin, and single-valuedness of $e^{i\omega}$. If $\chi \sim r^p$ at $r = 0$ then by choosing $\omega = \pm\frac{1}{2}p\ \theta = \pm\frac{1}{2}p\ tan^{-1}(\frac{x_2}{x_1})$, we obtain from equation (2.11) a gauge field sufficiently smooth at the origin. Single-valuedness of $e^{i\omega}$ then dictates that $p/2$ must be an integer. We can at once identify $|n| = p/2$ for the above solutions, and this can be confirmed by calculating the flux directly, using equation (2.10b) to obtain F_{12} in terms of χ. In fact it is not correct to insist upon single-valuedness of $e^{i\omega}$, because in our metric $r = 0$ is a singular point (we will explain this point further later). In the above p becomes arbitrary. The flux quantization law (2.9) is lost. So we conclude, that for the metric given by (2.17) there exist cylindrically symmetric Chern-Simons-Higgs vortex solutions of the first order system we have been considering, with $|\phi| \to \sqrt{m}$ at spatial infinity; the solutions have flux of magnitude $\pi p/e$, and $|\phi|(r) = (1+\alpha r^{-p})^{-1/2}$, where α is an arbitrary parameter.

In fact we can say somewhat more than this. Although we arrived at equations $(2.10a, b)$ by considerations only applicable to vortices for which $|\phi| \to \sqrt{m}$ at spatial infinity, it can in fact be checked that any stationary solution of equations (2.2), $(2.10a, b)$ gives a solution of the equations of motion for the action (2.1), which we write down here for reference:

$$\frac{k}{2}\epsilon^{\mu\nu\rho}F_{\mu\nu} + ie\sqrt{g}g^{\rho\mu}(\phi\overline{D_\mu\phi} - \overline{\phi}D_\mu\phi) = 0 \tag{2.21a}$$

$$g^{\mu\nu}D_\mu D_\nu\phi + \frac{e^4}{k^2}\phi(|\phi|^2 - m)(3|\phi|^2 - m) = 0 \tag{2.21b}$$

For an arbitrary stationary solution of (2.2), $(2.10a, b)$ we can compute the energy momentum tensor in terms of the magnitude of the scalar field (for arbitrary b) to give

$$\begin{aligned} T_{ij} &= 0 \\ T_{0i} &= \frac{e^2}{k}(|\phi|^2 - m)\epsilon_{ij}\partial_j|\phi|^2 \\ T_{00} &= \frac{2}{b}(\nabla|\phi|)^2 + \frac{2e^4}{k^2}|\phi|^2(|\phi|^2 - m)^2 \end{aligned} \tag{2.22}$$

(In fact, whereas we have used the methods of [3b] to obtain first order equations, it is possible to derive these equations from the requirement $T_{11} = T_{22} = 0$ [3c].) Returning now to our special 'integrable' case we see that all the solutions in (2.18) give rise to solutions of the equations of motion of our action, not just the special case (2.20); these are all likely to be finite energy solutions as for all of them χ tends to vacuum values, 0 or 1, at $r = \infty$. We can compute for these solutions the energy, and the angular momentum using (2.22)*, and the flux, using $(2.10b)$ (the solutions also carry charge, related to the

* In a background metric, care has to be taken with the definition of the conserved quantities. In our case, since the metric only depends on r, we can still define energy and angular momentum, which are given by

$$\begin{aligned} E &= \int d^2x\sqrt{g}T_{00} \\ J &= \int d^2x\sqrt{g}(x^1T^0_2 - x^2T^0_1) \end{aligned}$$

flux in the usual way for Chern-Simons electrodynamics [1]). We expect the relation that we found earlier, $E = me|\Phi|$ (Φ being the flux).

It is convenient to deal with the solutions in three groups. First we take those of form (2.20), that is the solutions of (2.18) with $\beta = 0$. These have the properties $\chi \to 0$ at $r = 0$ and $\chi \to 1$ at $r = \infty$. We find

$$\begin{aligned} E &= m\pi p \\ J &= \frac{\pi p^2 k}{4e^2} \\ |\Phi| &= \frac{\pi p}{e} \end{aligned} \tag{2.23}$$

which we note are identical to the results of [1] if we identify p as an even integer (with the provisos that for us p is a quantity fixed by the choice of a specific metric, and also we have a 1-parameter family of such solutions). Next we look at solutions of (2.18) with $\alpha = 0$, which have form

$$\chi = \frac{1}{1 + \beta r^p} \tag{2.24}$$

(where β is arbitrary). These satisfy boundary conditions $\chi \to 1$ at $r = 0$ and $\chi \to 0$ at $r = \infty$. Simple calculations yield

$$\begin{aligned} E &= m\pi p \\ J &= \frac{-\pi p^2 k}{4e^2} \\ |\Phi| &= \frac{\pi p}{e} \end{aligned} \tag{2.25}$$

The condition $E = me\Phi$ is satisfied. A quantization law that p should be an even integer would follow if we compactified and required smoothness of the gauge field at spatial infinity, but this is not necessary, so p is arbitrary. The third set of solutions of (2.18) are those for which $\alpha\beta \neq 0$, i.e.$\lambda < p$. These satisfy boundary conditions $\chi \to 0$ at $r = 0, \infty$. The maximum value of χ attained is

$$\chi_{max} = \frac{\lambda^2/p^2}{1 + \sqrt{1 - (\lambda^2/p^2)}} \tag{2.26}$$

which, for $\lambda < p$, is between 0 and 1. Straightforward computations yield the values

$$\begin{aligned} E &= 2\pi m\lambda \\ J &= 0 \\ |\Phi| &= \frac{2\pi\lambda}{e} \end{aligned} \tag{2.27}$$

These results seem a little surprising at first, since they do not reduce to the results in (2.23) or (2.25) when we set $\lambda = p$. This is because the general solution (2.18) does not converge to the solutions (2.20) and (2.24) in the limits $\beta \to 0$ and $\alpha \to 0$. Looking together at the three types of solutions we have, we see that as we increase p in (2.17), a new family of solutions of the first and second type appear, and as p is increased further they 'become' a family of solutions of the third type. Looking at the values of $E, J, |\Phi|$ in (2.23), (2.25), (2.27), we see that in some sense the first and second type solutions superimpose to give third type solutions, with E, J, Φ values adding.

It remains to give some physical meaning to our results. As mentioned in the introduction, even though we introduced a background metric with only the goal of achieving integrability, our results are reasonable from the point of view of gravity as well. The metric (2.17) is the metric appropriate to a point mass at the origin, with mass $1/4G$ [8], and it is certainly reasonable to consider solutions in such a background. Alternatively, we can try to interpret the metric (2.17) as an approximation to the metric induced, via Einstein's equations, by our solutions. We cannot expect an exact solution of Einstein's equations, as the metrics we have been considering are static, while our solutions carry (angular) momentum, and also the energy density of our solutions is not all concentrated at the origin. The sense in which the metric (2.17) can be regarded as an approximation to that induced by our solutions is that by a choice of parameters we can make the energy of one of our solutions equal to $1/4G$. For first and second type solutions we set $m = 1/4\pi pG$ and for third type solutions we set $m = 1/8\pi\lambda G$. We note that the metrics (2.15) also have potential for physical interpretation: the Einstein tensor for these metrics has one non-vanishing component $G_{00} = \gamma^2 (r/\Lambda)^{-p}$, and they are appropriate for matter with total energy

$$\frac{1}{8\pi G} \int d^2x \sqrt{g} G_{00} = \frac{p}{4G} \tag{2.28}$$

For $p > 2$ the quantity $\sqrt{g} G_{00}$ vanishes at $r = 0$, and peaks at some radius depending on Λ, which is what we might expect for the energy density of a vortex solution in this metric.

The comments in the last paragraph also suggest a reason why equation (2.13) seems to be non-integrable for any choice of b. Maybe we should be looking for solutions allowing b to vary with the solution. Of course it is possible to regard (2.13) as an equation giving a b for any choice of χ, but this will not in general lead to a metric which has any physical meaning. On the other hand, we cannot expect to satisfy Einstein's equations, as we are using static metrics. But it may be possible to obtain something reasonable: for example one might hope to be able to find a multi-vortex solution in the background metric appropriate for point particles located at the vortex centers. We hope to investigate this at a later date.

We mention here another question that needs further investigation. For Abelian Higgs vortices there is an index theorem that counts the number of parameters of solutions of the relevant first order equations of a given flux [9]. For an n-vortex there are $2n$ parameters, which correspond [10] to the positions of the n centers. The index theorem goes through exactly for Chern-Simons-Higgs vortices [1], but is affected by introducing a metric of the type we use. One can attempt a naive modification of the calculation of [9], evaluating the adjoint of the relevant elliptic operator with respect to a norm $\int d^2x\ b$ (so, for instance, the adjoint of ∂_1 is $-\partial_1 - \partial_1(\ln b)$). We obtain the result that, if we start from a solution of $(2.10a, b)$ with flux Φ, the dimension of the space of variations of that solution that preserve $(2.10a, b)$ which are not gauge transformations, is $4+e|\Phi|/\pi$, the 4 being a contribution from the Euler invariant of our metric (2.17). Since we have lost flux quantisation, this number is not necessarily an integer; this presumably reflects the fact that we are attempting to use an index theorem on a noncompact manifold, without including boundary corrections. Nevertheless, the result of the 'naive' index theorem may well be approximately correct. The necessary corrections, and the physical interpretation of the parameters remain to be clarified.

To conclude this section we observe that the metric given by (2.17) can be written

$$ds^2 = dt^2 + \frac{p^2}{\gamma^2}(d(\ln r)^2 + d\theta^2) \tag{2.29}$$

so our solutions can also be interpreted as solutions of the Chern-Simons-Higgs model in flat space which are constant in one direction, and have finite energy per unit length in that direction. This also yields a better understanding of the reason not to impose single-valuedness of the fields: changing the r-coordinate to $\tilde{r} = \ln r$ shows that we are really working on a cylinder, and the fields can change by a gauge transformation as we go around the cylinder.

3. Abelian Higgs Vortices

Here the relevant action is

$$S = \int d^3x\sqrt{g}\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\phi\overline{D^\mu\phi} - \frac{e^2}{8}(|\phi|^2 - m)^2\Big) \tag{3.1}$$

We proceed exactly as in section 1, with a metric $g_{\mu\nu} = diag(1, -b, -b)$, to obtain first order equations of the form

$$(D_1 \pm iD_2)\phi = 0 \tag{3.2a}$$

$$eF_{12} \mp \frac{e^2b}{2}(|\phi|^2 - m) = 0 \tag{3.2b}$$

Solving $(3.2a)$ for the gauge field we obtain

$$\nabla^2 \ln\chi = e^2mb(\chi - 1) \tag{3.3}$$

where $\chi = |\phi|^2/m$. Unlike the Chern-Simons-Higgs case, there exist b's for which (3.3) is integrable, namely if e^2mb satisfies the Liouville equation

$$\nabla^2 \ln(e^2mb) = e^2mb \tag{3.4}$$

Then we find the solution of (3.3) to be

$$\chi = \frac{\tilde{\chi}}{e^2mb} \tag{3.5}$$

where $\tilde{\chi}$ itself satisfies the Liouville equation. In appendix B we show that this is the only case in which (3.3) has Painlevé property, suggesting it is the only case in which (3.3) is integrable. At the end of appendix B we give some positive solutions of the Liouville equation (we need both b and $\tilde{\chi}$ to be positive). Here we move on to cylindrically symmetric solutions. We need

$$\frac{1}{r}\left(\frac{r\chi'}{\chi}\right)' = e^2mb(\chi - 1) \tag{3.6}$$

Painlevé analysis of this is presented in appendix A; it possesses Painlevé property if and only if

$$b = \frac{8p^2}{e^2mr^2((\frac{r}{\Lambda})^p - (\frac{r}{\Lambda})^{-p})^2} \tag{3.7a}$$

where $p \neq 0$, or

$$b = \frac{2}{e^2 m r^2 \ln^2(r/\Lambda)} \tag{3.7b}$$

Here Λ and p are constants and we can refer to (3.7b) as the $p = 0$ case of (3.7a). We note that all the b's in (3.7) solve (3.4), so the ODE (3.6) is only integrable when the PDE (3.3) is, unlike the Chern-Simons-Higgs case. The general solution of (3.6) for these metrics is

$$\chi = \frac{m^2((\frac{r}{\Lambda})^p - (\frac{r}{\Lambda})^{-p})^2}{p^2((\frac{r}{\alpha})^m - (\frac{r}{\alpha})^{-m})^2} \tag{3.8}$$

where m, α are constants, and we understand the correct limit for the cases where m and/or p are 0. For a finite energy solution of our theory we need $\chi \to 1$ at $r = \infty$. This eliminates all non-trivial solutions except for the case $m = p = 0$, which can also be rejected on the grounds of the singularity it has at $r = \alpha$ if $\alpha \neq \Lambda$. So we reach the surprising result that, even though the Abelian Higgs case seems to have better integrability properties than the Chern-Simons-Higgs case, we cannot find explicit cylindrically symmetric solutions in the Abelian Higgs case. However for the general case there are good solutions; the case $b = 2/e^2 m x^2$ (corresponding to $A(z) = z$ and $C = 0$ in equation (B13)) has been studied before [11], as the Abelian Higgs action with this metric arises naturally in the context of $O(3)$-symmetric $SU(2)$ Yang-Mills fields on $\mathbf{R}^4$.

4. Discussion

We briefly outline here two possible extensions of this work that we think may be of interest. The spirit of this work is that by introducing the right metric we have actually succeeded in making the vortex equations easier to handle, and therefore it is plausible that other generalisations will also simplify matters. One possible generalisation is to consider a model with gauge field dynamics governed by a combination of the Maxwell and Chern-Simons terms; in [12] it has been shown that vortex type solutions exist in such models with a ϕ^4 potential, but it seems one cannot obtain solutions from first order differential equations for this case. Maybe a suitable modification of the potential, extrapolating between the ϕ^4 potential of the Abelian Higgs model and the ϕ^6 potential of the Chern-Simons-Higgs model, is necessary to obtain first order equations. So far our attempts in this direction have failed (we draw the reader's attention to ref. [13], where it is shown that first order equations can be obtained for a whole class of modifications of the Abelian Higgs model).

Another possible generalization is to look at the theory in a simple non-static metric, with a view that we might get closer to satisfying the Einstein equations. In light of the results of [8] it seems reasonable to try a metric of form $ds^2 = (dt + Ad\theta)^2 + b(dr^2 + r^2 d\theta^2)$, but we have not so far succeeded to write first-order equations for vortex solutions in this metric. Clearly there is much more work to be done here.

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Appendix A

In this appendix we apply Painlevé analysis to investigate the integrability properties of the second order ODEs

$$\frac{1}{r}\left(\frac{r\chi'}{\chi}\right)' = f\chi^2 + g\chi \qquad (A1)$$

$$\frac{1}{r}\left(\frac{r\chi'}{\chi}\right)' = f\chi + g \qquad (A2)$$

Here χ is a function of the independent variable r, and a prime denotes differentiation with respect to r. f and g are arbitrary functions of r, and we wish to determine for which f,g each of these equations have Painlevé property (we assume $f \neq 0$ in each case). In our main work we are interested in these equations in the case $f = -g$. The methods we use are described in [14]. In [4] there is a classification of 50 canonical types of second order ODE of the form

$$\chi'' = F(\chi', \chi, r) \qquad (A3)$$

(F an arbitrary function) with Painlevé property. For equation ($A2$) we find the condition that Painlevé property holds is

$$g = -\frac{1}{r}\left(\frac{rf'}{f}\right)' \qquad (A4)$$

In this case, making the substitution $\chi = \tilde{\chi}/rf$ the equation becomes

$$\tilde{\chi}'' - \frac{(\tilde{\chi}')^2}{\tilde{\chi}} + \frac{\tilde{\chi}'}{r} = \frac{\tilde{\chi}^2}{r} \qquad (A5)$$

This is a special case of the type XIII equation of [4]; in [15] it is shown how to obtain exact solutions of certain special cases of the type XIII equation, including this one (in general though, the type XIII equation is not integrable in terms of 'usual' functions, and defines new functions known as the third Painlevé transcendents). The general solution is

$$\tilde{\chi} = \frac{8n^2}{r\left((\frac{r}{\alpha})^n - (\frac{r}{\alpha})^{-n}\right)^2} \qquad (A6)$$

or (corresponding to the limit $n \to 0$)

$$\tilde{\chi} = \frac{2}{r \ln^2(r/\alpha)} \qquad (A7)$$

where α,n are constants. For equation $(A1)$ the condition for Painlevé property to hold is

$$gr^D = (\pm\sqrt{f} r^D)' \qquad (A8)$$

where D is an arbitrary constant. For the case $f = \alpha$ (α a constant) we obtain the equation

$$\chi'' - \frac{\chi'^2}{\chi} + \frac{\chi'}{r} = \alpha\chi^3 + \frac{\beta\chi^2}{r} \qquad (A9)$$

(where β is a constant), a further solvable [15] case of equation type XIII of [4]. However, the general case of equation $(A1)$, with condition $(A8)$ is *not* reducible to one of the canonical equations in [4]. We will explain this after presenting the actual Painlevé analysis.

The Painlevé analysis goes as follows. We work with equations $(A1)$ and $(A2)$ in the forms

$$\chi\chi'' - \chi'^2 + \frac{\chi\chi'}{r} = f\chi^4 + g\chi^3 \qquad (A1')$$

$$\chi\chi'' - \chi'^2 + \frac{\chi\chi'}{r} = f\chi^3 + g\chi^2 \qquad (A2')$$

For $(A1')$ seek a solution in the form

$$\chi = \sum_{n=0}^{\infty} \chi_n (r - r_0)^{n-1} \qquad (A10)$$

where $r_0, \chi_0, \chi_1, \ldots$ are constants. We expand f, g and $\frac{1}{r}$ in Taylor series around r_0:

$$f = \sum_{n=0}^{\infty} f_n (r - r_0)^n \qquad\qquad g = \sum_{n=0}^{\infty} g_n (r - r_0)^n \tag{$A11a$}$$

$$\frac{1}{r} = \sum_{n=0}^{\infty} \frac{(-1)^n}{r_0^{n+1}} (r - r_0)^n \tag{$A11b$}$$

(we assume r_0 is neither 0 nor a singular point of f or g). Substituting these expressions into $(A1')$ and comparing terms order by order in $(r-r_0)$, we find (from terms in $(r-r_0)^{-4}$ and $(r-r_0)^{-3}$) that we need to choose χ_0, χ_1 to satisfy

$$\chi_0^2 = \frac{1}{f_0} \tag{$A12a$}$$

$$2\chi_1 + f_1\chi_0^3 + g_0\chi_0^2 + \frac{\chi_0}{r_0} = 0 \tag{$A12b$}$$

Examining terms in $(r-r_0)^{-2}$ we find that χ_2 can be chosen arbitrarily (in the language of [14] there is a 'resonance' at $n = 2$, which is the only other resonance apart from $n = -1$), but a consistency condition emerges of the form

$$6f_0\chi_0^2\chi_1^2 \;+\; \left(4f_1\chi_0^3 + 3g_0\chi_0^2 + \frac{\chi_0}{r_0}\right)\chi_1 \;+\; \left(f_2\chi_0^4 + g_1\chi_0^3 - \frac{\chi_0^2}{r_0^2}\right) = 0 \tag{$A12c$}$$

Higher powers of $(r-r_0)$ just give defining relations for $\chi_3, \chi_4, \ldots$. Thus $(A10)$ furnishes a general solution to $(A1')$ with two arbitrary coefficients r_0, χ_2 if equations $(A12)$ all hold. Eliminating χ_0, χ_1 from $(A12c)$ using $(A12a, b)$ yield a relation between f_0, f_1, f_2, g_0, g_1 that must hold. After some algebra, and using the fact that if we regard f_0, g_0 as functions of r_0, then $f_1 = f_0'$, $f_2 = \frac{1}{2}f_0''$, $g_1 = g_0'$ (here prime denotes differentiation with respect to r_0), we can cast the consistency condition into the form

$$\frac{d}{dr_0}\left(\pm \frac{2g_0 r_0}{\sqrt{f_0}} + \frac{f_1 r_0}{f_0} \right) = 0 \tag{$A13$}$$

(the $\pm$ ambiguity arises from $(A12a)$). Integrating this and manipulating a little further yields

$$g r_0^D = \frac{d}{dr_0}(\pm\sqrt{f_0}\, r_0^D) \tag{$A14$}$$

Since this has to hold for all r_0 we obtain $(A8)$.

For equation $(A2')$ the analysis proceeds along the same lines, except that we use an expansion

$$\chi = \sum_{n=0}^{\infty} \chi_n (r - r_0)^{n-2} \tag{A15}$$

The resonances are again at $n = -1$ and $n = 2$. We obtain relations

$$\chi_0 = \frac{2}{f_0} \tag{A16a}$$

$$2\chi_1 + \left(f_1 \chi_0 + \frac{2}{r_0} \right) \chi_0 = 0 \tag{A16b}$$

$$5\chi_1^2 + 3\left(f_1 \chi_0 + \frac{1}{r_0} \right) \chi_0 \chi_1 + \left(g_0 + f_2 \chi_0 - \frac{2}{r_0^2} \right) \chi_0^2 = 0 \tag{A16c}$$

which are easily manipulated to give condition $(A4)$.

We now need to explain why equation $(A1)$, with the condition $(A8)$ that we have found, does not seem to fall, in general, into the classification of [4]. Writing $h = \pm\sqrt{f} r^D$, our current result is that the equation

$$\chi'' - \frac{\chi'^2}{\chi} + \frac{\chi'}{r} = h^2 r^{-2D} \chi^3 + h' r^{-D} \chi^2 \tag{A17}$$

has Painlevé property for an arbitrary function h and an arbitrary constant D. By changing the independent variable to $t = \ln r$, then the dependent variable to $Y = \chi h^{-1} e^{(D-1)t}$, and finally replacing the arbitrary function h by an arbitrary function $v = \dot{h}/h$ (a dot denoting differentiation with respect to t), the equation is brought to the form

$$\frac{d^2}{dt^2} \ln Y = Y^2 + vY + \dot{v} \tag{A18}$$

Now set $Y = \dot{q}/q$. One obtains a third order equation for q, which can be integrated once to yield

$$\ddot{q} = 2\frac{\dot{q}^2}{q} + v\dot{q} + C\frac{\dot{q}}{q} \tag{A19}$$

where C is a constant. Writing $r = 1/q$ this becomes

$$\ddot{r} = v\dot{r} + Cr\dot{r} \tag{A20}$$

For $C = 0$ this can be integrated to give

$$r = A + B \int dt \exp\left(\int v dt \right) \qquad (A21)$$

where A, B are constants. This two parameter family of solutions yields only a one parameter family of solutions of $(A17)$

$$\chi = \frac{r^{D-1}}{\alpha - \int \frac{h}{r} dr} \qquad (A22)$$

where α is a constant. To get the general solution of $(A17)$ it is necessary to solve $(A20)$ for a nonzero value of C: one nonzero value is clearly sufficient since for $C \neq 0$ one can make the substitution $r = -2s/C$ in $(A20)$ to obtain

$$\ddot{s} = -2s\dot{s} + v\dot{s} \qquad (A23)$$

Let us now perform Painlevé analysis on equation $(A23)$. Our considerations so far would indicate that $(A23)$ has Painlevé property for an arbitrary function v; but as we might expect, it only falls into the classification of [4] in the case where v is constant, when it is an equation of type V. On attempting to do Painlevé analysis for $(A23)$ with the obvious principal balance of -1, i.e. with an expansion of form

$$s = \sum_{n=0}^{\infty} s_n (t - t_0)^{n-1} \qquad (A24)$$

one indeed finds a consistency condition for Painlevé property to hold, that v should be constant. However there is another possible principal balance one might consider; with an expansion of form

$$s = \sum_{n=0}^{\infty} s_n (t - t_0)^{n+1} \qquad (A25)$$

it is easy to verify that Painlevé property holds for arbitrary v (with t_0, s_0 arbitrary). In the analysis of equations of the type of $(A23)$ given in [4], only the possibility of principal balance of -1 is considered. The significance of these results - namely, that equation $(A23)$ has Painlevé property for both possible principal balances if v is constant, and only for one principal balance otherwise - is not completely clear. In the context of Hamiltonian systems, where there is an accurate notion of the meaning of integrability, there is some indication that integrability requires Painlevé property to hold for all possible principal balances [16]. Clearly in our case too, there is something special about the case where v is constant (or equivalently h is some power of r), for in this case, it is straightforward to reduce our equation ($A17$) to equation ($A9$), which can be completely solved [15]. In the general case, however, we have only succeeded in exhibiting a one parameter family of solutions to ($A17$). It seems reasonable to suggest that ($A17$) is 'partially integrable' for arbitrary h, and 'fully integrable' when h is a power of r. However from the point of view of the Painlevé analysis of equation ($A1'$) that we have done, there seems to be *no* reason to distinguish these cases (note that both forms of expansions for s, ($A24$) and ($A25$), lead to simple pole singularities in the function Y of ($A18$)). We thus are led to suggest that applying Painlevé analysis directly to an equation may not be sufficient to reveal all its integrability properties, and that it might be necessary to look at different forms of the equation.

As mentioned above, the case of relevance in our main work is where $f = -g$. For ($A1$) in this case we find 'partial' Painlevé property when

$$f = \frac{n^2}{r^2(1 + (\frac{r}{\Lambda})^{-n})^2} \tag{A26}$$

where $n = 1 - D$ and Λ are constants; in this case the one parameter family of solutions is

$$\chi = \frac{(\frac{r}{\Lambda})^{-n}}{B + \ln(1 + (\frac{r}{\Lambda})^{-n})} \tag{A27}$$

where B is constant. There is also a degenerate case ($D = 1$) with

$$f = \frac{1}{r^2 \ln^2(r/\Lambda)} \tag{A28}$$

$$\chi = \frac{1}{B - \ln\ln(r/\Lambda)} \tag{A29}$$

This degenerate case is clearly not suited for use in our main work, as χ is only defined for $r/\Lambda > 1$. We find 'complete' Painlevé property for ($A1$) with $f = -g$ when

$$f = \frac{p^2}{r^2} \tag{A30}$$

The general rational solution is then given by (2.18) and (2.19) (there is also a 1 parameter family of solutions involving logarithms [15], which are of no importance for us because they possess singularities). For $(A2)$ with $f = -g$ we have Painlevé property when

$$f = \frac{8p^2}{r^2((\frac{r}{\Lambda})^p - (\frac{r}{\Lambda})^{-p})^2} \qquad (A31)$$

where $p \neq 0, \Lambda$ are constants, or the $p \to 0$ limit of this, obtained by using the result

$$\lim_{p \to 0} \left(\frac{x^p - x^{-p}}{2p} \right) = \ln x \qquad (A32)$$

The general solution of $(A2)$ is given, in this case, by (3.8) and appropriate limits thereof.

Appendix B

In this appendix we apply Painlevé analysis to investigate the integrability properties of the second order PDEs

$$\nabla^2 \ln \chi = f\chi^2 + g\chi \qquad (B1)$$

$$\nabla^2 \ln \chi = f\chi + g \qquad (B2)$$

Here χ is a function of the two independent variables x, y, and f, g are two arbitrary functions of x, y. We wish to determine for which f, g each of these equations has Painlevé property. In our main work we are interested in the case where $f = -g$. The method that we use here is described in [5], but we need to find a way to handle arbitrary functions within this method. We will explain our approach to this below, with some *a priori* justification, but first we present our results, which, since they are reasonable, provide *a posteriori* evidence of the validity of our approach.

The results can be summed up in the statement that $(B1)$ and $(B2)$ only have Painlevé property when they can be reduced to the Liouville equation. For equation $(B1)$ we find we need

$$\nabla^2 \ln f = 0 \qquad g = 0 \qquad (B3)$$

in which case by the substitution $\tilde{\chi} = 2f\chi^2$ the equation becomes

$$\nabla^2 \ln \tilde{\chi} = \tilde{\chi} \qquad (B4)$$

which is the Liouville equation. For equation $(B2)$ we find we need

$$\nabla^2 \ln f + g = 0 \tag{B5}$$

in which case by the substitution $\tilde{\chi} = f\chi$ we obtain $(B4)$. These results are certainly reasonable, as in the cases when we find Painlevé property to hold, the equations do indeed reduce to a known integrable equation, and furthermore we find no 'new' results, which is in accordance with the belief that integrable systems are very rare, and known examples may well exhaust the entire list.

To use the method of [5] on the equations $(B1)$ and $(B2)$, we expand χ in a formal series of the form

$$\chi = \sum_{n=0}^{\infty} \chi_n \phi^{n+\alpha} \tag{B6}$$

Here ϕ is an arbitrary function of x, y, and the χ_n are some functions to be determined by substituting $(B6)$ into the relevant differential equation and comparing terms order by order in ϕ. α is the principal balance, -1 for $(B1)$ and -2 for $(B2)$. It might seem to be necessary to expand the arbitrary functions f, g in some series in ϕ (see the methods used for ODEs in Appendix A). We claim that this is unnecessary, and that f, g can be treated, in effect, as pure zeroth order terms. The justification for this is that the series of the form $(B6)$ is purely formal, in that we do not impose any condition that the χ_n are functions only of a coordinate normal to ϕ; in other words, when from our comparing terms order by order procedure we obtain, say, an equation for χ_0, this equation is only true 'mod ϕ', that is we could add to our expression a multiple of ϕ. This freedom certainly can be used to remove most of the effect of replacing pure zeroth order functions f, g by some arbitrary series in ϕ (the only potential difficulty might arise in checking the resonance conditions). We conjecture therefore that it is sufficient in the Painlevé analysis to treat f, g as pure zeroth order. Note, though, that there is a simplification of the procedure of [5] which involves using a 'reduced ansatz' in $(B6)$, where ϕ is taken to be of the form, say, $x - t(y)$ (t an arbitrary function), and the χ_n are functions of y alone (this is more analogous to the procedure we used in appendix A for ODEs). If we were to use this approach we lose the freedom mentioned above, and it would be necessary to expand f, g in series of form

$$f = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n f}{\partial x^n}\bigg|_{x=t(y)} (x - t(y))^n \tag{B7}$$

(similarly for g).

Assuming the above, the analysis procedes as follows. We work with the equations in form

$$\chi(\chi_{xx} + \chi_{yy}) - (\chi_x^2 + \chi_y^2) = f\chi^4 + g\chi^2 \tag{B1'}$$

$$\chi(\chi_{xx} + \chi_{yy}) - (\chi_x^2 + \chi_y^2) = f\chi^3 + g\chi^2 \tag{B2'}$$

(we assume $f \neq 0$ in both cases). We use expansions of type $(B6)$ with $\alpha = -1$ for $(B1)$ and $\alpha = -2$ for $(B2)$. In both cases resonances occur at $n = -1$ and $n = +2$, in other words, when we compare terms order by order in ϕ the equations can be satisfied (by correct choice of the χ_n) for arbitrary ϕ with an arbitrary choice of χ_2, provided one consistency condition is satisfied. For equation $(B2')$ we obtain from terms in ϕ^{-6} and ϕ^{-5} the relations

$$\chi_0 = \frac{2}{f}(\phi_x^2 + \phi_y^2) \tag{B8a}$$

$$\chi_1 = \frac{-2}{f}(\phi_{xx} + \phi_{yy}) \tag{B8b}$$

The consistency condition that emerges from terms in ϕ^{-4} is

$$\begin{aligned} 0 = {} & \chi_1^2(\phi_x^2 + \phi_y^2) - 3\chi_0\chi_1(\phi_{xx} + \phi_{yy} + f\chi_1) - g\chi_0^2 \\ & + \chi_0(\chi_{0xx} + \chi_{0yy}) - (\chi_{0x}^2 + \chi_{0y}^2) \\ & + 2\phi_x(\chi_0\chi_{1x} - \chi_1\chi_{0x}) + 2\phi_y(\chi_0\chi_{1y} - \chi_1\chi_{0y}) \end{aligned} \tag{B8c}$$

Eliminating χ_0 and χ_1 from $(B8c)$ using $(B8a)$ and $(B8b)$ leads, after some algebra to exactly the condition $(B5)$.

For $(B1)$, from ϕ^{-4} and ϕ^{-3} terms we obtain

$$\chi_0^2 = \frac{1}{f}(\phi_x^2 + \phi_y^2) \tag{B9a}$$

$$2f\chi_0\chi_1 + g\chi_0 + (\phi_{xx} + \phi_{yy}) = 0 \tag{B9b}$$

The consistency condition from ϕ^{-2} terms is

$$\begin{aligned}0 =& 6f\chi_0^2\chi_1^2 + 3g\chi_0^2\chi_1 + \chi_0\chi_1(\phi_{xx} + \phi^{yy}) \\ &- \chi_0(\chi_{0xx} + \chi_{0yy}) + (\chi_{0x}^2 + \chi_{0y}^2) \\ &- 2\phi_x(\chi_0\chi_{1x} - \chi_1\chi_{0x}) - 2\phi_y(\chi_0\chi_{1y} - \chi_1\chi_{0y})\end{aligned} \tag{B9c}$$

Manipulation of these equations yields a consistency condition

$$\nabla^2 \ln f = \pm\left(\frac{g}{\sqrt{f}}\frac{\phi_x}{\sqrt{\phi_x^2 + \phi_y^2}}\right)_x \pm \left(\frac{g}{\sqrt{f}}\frac{\phi_y}{\sqrt{\phi_x^2 + \phi_y^2}}\right)_y \tag{B10}$$

(The $\pm$ ambiguity arises from ($B9a$)). This has to hold for all ϕ, so we obtain the conditions ($B3$).

As mentioned above, in our main work we are interested in the case $f = -g$. From our work here we conclude that there is no integrable case of ($B1$) with $f = -g$ except $f = g = 0$. For ($B2$) we conclude that

$$\nabla^2 \ln \chi = f(\chi - 1) \tag{B11}$$

is integrable if and only if f satisfies the Liouville equation ($B4$), and in this case the general solution of ($B11$) is $\chi = \tilde{\chi}/f$, where $\tilde{\chi}$ itself solves the Liouville equation ($B4$). The general solution of the Liouville equation ($B4$) is well-known to be

$$\tilde{\chi} = \frac{8A'(z)B'(\bar{z})}{(A(z) + B(\bar{z}))^2} \tag{B12}$$

where $z = x + iy$, $\bar{z} = x - iy$, and A, B are arbitrary functions. In our main work we will be interested in positive (real) solutions of the equation. One way to obtain such solutions is to choose B to be the same function as A, up to addition of a real constant C, and to insist A is a 'real' function, i.e. $\overline{A(z)} = A(\bar{z})$. Then we have a positive solution

$$\tilde{\chi} = \frac{2|A'(z)|^2}{(C + ReA(z))^2} \tag{B13}$$

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