REGULARITY OF THE INVERSION PROBLEM FOR THE STURM-LIOUVILLE DIFFERENCE EQUATION IV. STABILITY CONDITIONS FOR A THREE-POINT DIFFERENCE SCHEME WITH NON-NEGATIVE COEFFICIENTS

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Abstract. Consider a three-point difference scheme

$$-h^{-2}\Delta^{(2)}y_n + q_n(h)y_n = f_n(h), \quad n \in Z = \{0, \pm 1, \pm 2, \dots\} \tag{1}$$

where $h \in (0, h_0]$, h_0 is a given positive number,

$$\Delta^{(2)}y_n = y_{n+1} - 2y_n + y_{n-1}, \quad f(h) \stackrel{\text{def}}{=} \{f_n(h)\}_{n\in Z} \in L_p(h),\ p \in [1, \infty),$$
$$L_p(h) = \{f(h) : \|f(h)\|_{L_p(h)} < \infty\}, \quad \|f(h)\|^p_{L_p(h)} = \sum_{n\in Z} |f_n(h)|^p h.$$

Assume that the sequence $q(h) \stackrel{\text{def}}{=} \{q_n(h)\}_{n\in Z}$ satisfies the a priori condition

$$0 \le q_n(h) < \infty \quad \forall\, n \in Z, \quad \forall\, h \in (0, h_0].$$

We obtain criteria for the stability of scheme (1) in $L_p(h)$, $p \in [1, \infty)$.

1. Introduction

In this paper we continue the study started in [4],[5],[3],[2]. We consider a difference scheme

$$-h^{-2}\Delta^{(2)}y_n + q_n(h)y_n = f_n(h), \quad n \in Z = \{0, \pm 1, \pm 2, \dots, \} \tag{1.1}$$

where $h \in (0, h_0]$, h_0 is a given positive number, $f(h) \stackrel{\text{def}}{=} \{f_n(h)\}_{n\in Z} \in L_p(h)$, $p \in [1, \infty)$ (for $p = \infty$ see [2]),

$$L_p(h) = \{f(h) : \|f(h)\|_{L_p(h)} < \infty\}, \quad \|f(h)\|^p_{L_p(h)} = \sum_{n\in Z} |f_n(h)|^p h.$$

Throughout the sequel, it is assumed that the sequence $q(h) \stackrel{\text{def}}{=} \{q_n(h)\}_{n\in Z}$ satisfies the a priori condition

$$0 \le q_n(h) < \infty \quad \forall\, n \in Z, \quad \forall\, h \in (0, h_0]. \tag{1.2}$$

Our main goal is to describe the class of sequences $q(h)$ such that the difference scheme (1.1) is stable in the space $L_p(h)$, $p \in [1,\infty)$ (see [10, Ch.5, §12], or, equivalently, such that the equation (1.1) is correctly solvable in $L_p(h)$ regardless of $h \in (0,h_0]$ (see [12, Ch.II, §3, n°3; Ch.II, §4, n°2]). The latter statement requires that we study conditions necessary and sufficient for the validity of the following assertions:

I) for every $h \in (0,h_0]$ and for any sequence $f(h) \in L_p(h)$, $p \in [1,\infty)$, there is a unique solution to (1.1) $y(h) \stackrel{\text{def}}{=} \{y_n(h)\}_{n\in Z} \in L_p(h)$.

II) for every $h \in (0,h_0]$ and for any sequence $f(h) \in L_p(h)$, $p \in [1,\infty)$ the solution $y(h) \in L_p(h)$ of (1.1) satisfies the inequality

$$\|y(h)\|_{L_p(h)} \le c(p)\|f(h)\|_{L_p(h)} \tag{1.3}$$

with an absolute constant $c(p) \in (0,\infty)$.

Here by a solution to (1.1) we mean any sequence $y(h) = \{y_n(h)\}_{n\in Z}$ satisfying equality (1.1) for all $n \in Z$. We impose no additional requirement to $q(h)$ (see, for example, Definition 2.2 in Section 2). Thus validity or non-validity of I) – II) only depends on the properties of the sequence $q(h)$. Note that under condition (1.4):

$$0 < \varepsilon \le q_n(h) < \infty \quad \forall\, n \in Z,\ \forall\, h \in (0,h_0] \tag{1.4}$$

which is stronger than (1.2), the difference scheme (1.1) becomes stable in $L_p(h)$ for all $p \in [1,\infty]$ (see [4], [2]). Therefore the problem on validity of I) – II) only arises when the sequence $q(h)$ is not separated from zero for $n \in Z$, $h \in (0,h_0]$. In the latter case the study of the properties of the solution $y(h) \in L_p(h)$ of (1.1) is much more difficult (see [3], [2]). Perhaps this is the main reason why the problem of stablity of (1.1) in $L_p(h)$, $p \in [1,\infty)$ was studied under some additional assumptions, as, for example, in [3].

In this paper, we show, among other things, that additional assumptions in [3] are irrelevant and the problem of stability of (1.1) in $L_p(h)$, $p \in [1,\infty)$ can be reduced to a problem which has already been studied in [3] (see Section 3). We now briefly describe our main results. Our main statement (Theorem 3.1) contains a criterion for the validity of I) – II) and is expressed in terms of a certain auxiliary sequence $\{d_n(h)\}_{n\in Z}$. The sequence $\{d_n(h)\}_{n\in Z}$ is an average characteristic of the sequence $\{q_n(h)\}_{n\in Z}$, and for given $n \in Z$ and $h \in (0,h_0]$, it may not be directly expressed in terms of the values of $q_n(h)$. Such a form of a criterion of stability of (1.1) may be inconvenient for applications, and therefore we complete Theorem 3.1 by equivalent Theorems 3.2 and 3.3. These assertions can be formulated in terms of $\{q_n(n)\}_{n\in Z}$ which is more reasonable for the investigation of concrete difference schemes. Note that Theorems 3.2 and 3.3 are easy consequences of our main Theorem 3.1. (Recall that Theorem 3.1, in turn, easily follows from the main result of [3]) (see Theorem 2.3 in Section 2). Nevertheless, we state them as theorems because they contain possible new approaches to the study of the difference scheme (1.1) which differ from those suggested in Theorem 3.1. For example, in Section 7 we consider a basic variant of uses of the application of Theorem 3.2 (see [5], [2] for applications of Theorem 3.1 to concrete difference schemes). Namely, we consider the problem of numerical inversion of the equation

$$-y''(x)+q(x)y(x)=f(x),\quad x\in R \tag{1.5}$$

where $f(x)\in L_p(R)$, $p\in[1,\infty]$ and

$$0\le q(x)\in L_1^{\mathrm{loc}}(R),\quad x\in R. \tag{1.6}$$

(Equation (1.5), together with condition (1.6), is denoted below (1.5) – (1.6).) In connection with (1.5) – (1.6), in Section 7 we study a standard difference scheme (1.1) with

$$q_n(h)=\frac{1}{2h}\int_{x_n-h}^{x_n+h}q(t)dt,\quad x_n=nh,\quad n\in Z,\quad h\in(0,1]. \tag{1.7}$$

(Such a scheme is denoted (1.1) – (1.7).)

Using Theorem 3.2, we show (see Theorem 3.4) that the difference scheme (1.1) – (1.7) is stable in $L_p(h)$, $p\in[1,\infty]$ if and only if the initial equation (1.5) – (1.6) is correctly solvable in $L_p(R)$. Usually, it is not so hard to check the latter condition (see [6] and Theorem 2.4 in Section 2), and we thus get a definite answer to the question of stability of (1.1) – (1.7) by studying the properties of the coefficient $q(x)$ of the initial differential problem (1.5) – (1.6). Thus, from the point of "stability", the standard difference scheme (1.1) – (1.7) turned out to be an ideal model for (1.5) – (1.6) and certainly deserves special attention. Therefore, the study of (1.1) – (1.7) will be continued in our forthcoming paper. In particular, our primary goal is to study the problem on covergence of the solution of (1.1) – (1.7) in the nodes x_n, $n\in Z$ to the solution of (1.5) – (1.6) as $h\to 0$. We believe that the results of this paper will be used to solve the latter problem.

2. Preliminaries

Throughout the sequel the letter c stands for absolute positive constants which are not essential for exposition and may differ even within a single chain of calculations. We denote by h an arbitrary number from the segment $(0,h_0]$. We assume that condition (1.2) holds without special mentioning.

Lemma 2.1. [4] *Suppose that for every $n\in Z$, we have*

$$\sum_{k=-\infty}^{n} q_k(h)>0,\quad \sum_{k=n}^{\infty} q_k(h)>0. \tag{2.1}$$

Then there exists a fundametnal system of solutions (FSS) $\{u(h),v(h)\}\overset{def}{=}\{u_n(h),v_n(h)\}_{n\in Z}$ of equation (2.2)*:*

$$h^{-2}\Delta^{(2)}z_n=q_n(h)z_n,\quad n\in Z \tag{2.2}$$

such that the solutions $u(h),v(h)$ satisfy the relation

$$\begin{aligned}
&0<u_{n+1}(h)\le u_n(h),\quad v_{n+1}(h)\ge v_n(h)>0,\quad n\in Z\\
&v_{n+1}(h)u_n(h)-v_n(h)u_{n+1}(h)=h,\\
&u_n(h)=v_n(h)\sum_{k=n}^{\infty}\frac{h}{v_k(h)v_{k+1}(h)},\quad n\in Z\\
&\lim_{n\to-\infty}\frac{v_n(h)}{u_n(h)}=\lim_{n\to\infty}\frac{u_n(h)}{v_n(h)}=0\\
&\lim_{n\to-\infty}u_n(h)=\lim_{n\to\infty}v_n(h)=\infty.
\end{aligned} \tag{2.3}$$

Throughout this section we assume that (2.1) holds. Denote by $G_{n,m}(h)$ the difference Green function corresponding to equation (1.1):

$$G_{n,m}(h)=\begin{cases}u_n(h)v_m(h) & \text{if}\quad n\ge m\\ \\ u_m(h)v_n(h) & \text{if}\quad n\le m\end{cases},\qquad n,m\in Z \tag{2.4}$$

$$\rho(h)=G_{n,m}(h)\Big|_{n=m}=u_n(h)v_n(h),\quad n\in Z.$$

Theorem 2.1. [4] *Let $n,m\in Z$ and $n\ne m$. Then the Green function $G_{n,m}(h)$ admits a representation of the Davies-Harrell type (see* [7]*):*

$$G_{n,m}(h)=\sqrt{\rho_n(h)\rho_m(h)}\cdot\begin{cases}\prod\limits_{k=n}^{m-1}\left[1+\frac{u_k(h)}{u_{k+1}(h)}\ \frac{h}{\rho_k(h)}\right]^{-1/2}, & n<m\\ \prod\limits_{k=m}^{n-1}\left[1+\frac{u_k(h)}{u_{k+1}(h)}\ \frac{h}{\rho_k(h)}\right]^{-1/2}, & n>m\end{cases} \tag{2.5}$$

Consider auxiliary sequences $\{\ell_n(h)\}_{n\in Z}$ and $\{d_n(h)\}_{n\in Z}$;

$$\ell_n(h) = \begin{cases} 0, & \text{if} \quad q_n(h)h^2 \geq 1 \\ \min\limits_{j\geq 0}\left\{ j : j\cdot \sum\limits_{k=n-j}^{n+j} q_k(h)h^2 \geq 1 \right\}, & \text{if} \quad q_n(h)h^2 < 1 \end{cases} \tag{2.6}$$

$$d_n(h) = \begin{cases} \frac{h}{1+q_n(h)h^2}, & \text{if} \quad \ell_n(h) = 0 \\ \ell_n(h)h, & \text{if} \quad \ell_n(h) \neq 0 \end{cases} \tag{2.7}$$

These sequences were introduced in [1] and used in [5], [4], [3], [2]. Below we state various properties of the FSS $\{u(h), v(h)\}$ of equation (2.2) and of the Green function $G_{n,m}(h)$ in terms of $\ell_n(h)$ and $d_n(h)$.

Lemma 2.2. [3] *For every $n \in Z$, we have*

$$c^{-1} \leq \frac{v_k(h)}{v_n(h)},\ \frac{u_k(h)}{u_n(h)} \leq c \quad \textit{for} \quad |k-n| \leq \left[\frac{\ell_n(h)}{2}\right]. \tag{2.8}$$

Theorem 2.2. [5] *For every $n \in Z$ we have*

$$8^{-1}d_n(h) \leq \rho_n(h) = u_n(h)v_n(h) \leq 16 d_n(h). \tag{2.9}$$

Denote $Z' = Z \setminus 0 = \{\pm1, \pm2, \dots\}$, $[m,p] = \{m, m+1, \dots, p\}$ for $m < p$, $[m,p] = p$ for $m = p$ and $m, p \in Z$. We call the sets $[m,p]$, $m \leq p$ *segments.*

Definition 2.1. [3] Let $n \in Z$ be given. A system of segments $\Delta_s = [\Delta_s^-, \Delta_s^+]$, $\Delta_s^- \leq \Delta_s^+$ and $\Delta_s^-, \Delta_s^+ \in Z$, $s \in Z'$ is called a $Z(n)$-*covering of* Z if the following conditions hold:

1) $\Delta_s \cap \Delta_s' = \emptyset$ for $s \neq s'$;
2) $\bigcup\limits_{s=-\infty}^{-1} \Delta_s = (\dots, n-2, n-1]$, $\bigcup\limits_{s=1}^{\infty} = [n+1, n+2, \dots)$.

Remark 1. The segments of a $Z(n)$-covering of Z do not contain the point n.

Lemma 2.3. [3] *For every $n \in Z$ there is a sequence $\{k_s\}_{s\in Z'}$ such that one can form a $Z(n)$-covering of Z from the segments $\{\tilde\Delta_s, \tilde\Delta_s'\}_{s\in Z'}$. Here*

1) $\tilde\Delta_s = \tilde\Delta_s' = [k_s, k_s] = k_s \quad \textit{if} \quad \ell_{k_s}(h) \in \{0, 1\}$,
2) $\tilde\Delta_s = [k_s - [2^{-1}\ell_{k_s}(h)] + 1, k_s + [2^{-1}\ell_{k_s}(h)]] \quad \textit{if} \quad \ell_{k_s}(h) \geq 2$,
$\tilde\Delta_s' = [k_s - [2^{-1}\ell_{k_s}(h)], k_s + [2^{-1}\ell_{k_s}(h)]] \quad if \quad \ell_{k_s} \geq 2$.

Lemma 2.4. [3] *Suppose that segments $\{\Delta_s\}_{s\in Z'}$ form a $Z(n)$-covering from Lemma 2.3. Then for any $s \in Z'$ we have*

$$T_s(h) \overset{def}{=} \prod_{k\in\Delta_s}\left(1 + \frac{u_k(h)}{u_{k+1}(h)}\ \frac{h}{\rho_k(h)}\right) \geq \gamma^{-1}, \quad \gamma^{-1} = \frac{50}{49}. \tag{2.10}$$

Lemma 2.5. [4] *Let $f(h) = \{f_n(h)\}_{n\in Z}$ be a sequence such that the series*

$$y_n(h) \overset{def}{=} (Gf)_n(h) \overset{def}{=} \sum_{m\in Z} G_{n,m}(h)f_m(h)h, \quad n \in Z \tag{2.11}$$

absolutely converges for every $n \in Z$. Then the sequence $y(h) = \{y_n(h)\}_{n\in Z}$ is a solution to equation (1.1)*.*

Lemma 2.6. [3] *Denote*

$$H = \sup_{h\in(0,h_0]} \sup_{n\in Z} \sum_{m\in Z} G_{n,m}(h)h \tag{2.12}$$

$$A = \sup_{h\in(0,h_0]} \sup_{n\in Z} d_n(h). \tag{2.13}$$

Then

$$H \le cA(A + h_0). \tag{2.14}$$

Lemma 2.7. [2] *Suppose that inequalities* (2.1) *hold and $A < \infty$ (see* (2.13)*). Then for every $n \in Z$ we have the following estimates (see* (2.7)*):*

$$c^{-1}d_n(h) \le \sum_{m\in Z} G_{n,m}(h)h \le c\sqrt{d_n(h)}. \tag{2.15}$$

Definition 2.2. We say that the inverse problem for equation (1.1) is regular in the space $L_p(h)$, $p \in [1,\infty)$ if together with statements I) – II) (see §2), the following statement also holds:

III) for every $h \in (0, h_0]$ and for any $f(h) \in L_p(h)$, $p \in [1,\infty)$, the solution $y(h) \in L_p(h)$ of equation (1.1) admits representation (2.11).

Theorem 2.3. [3] *Suppose that condition* (2.1) *holds. For $p \in [1,\infty)$, the inversion problem for equation* (1.1) *is regular in the space $L_p(h)$ if and only if $A < \infty$ (see* (2.13)*).*

We also need one result from [6]. Consider equation (1.5) assuming that (1.6) holds. Below by a solution of (1.5) we mean any function $y(x)$ such that $y(x)$, $y'(x) \in AC^{\text{loc}}(R)$ and equality (1.5) holds almost everywhere in R. We say that equation (1.5) is correctly solvable in a given space $L_p(R)$, $p \in [1,\infty]$ if for any function $f(x) \in L_p(x)$, there is a unique solution $y(x) \in L_p(R)$ to (1.5) and the following inequality holds:

$$\|y(x)\|_p \le c(p)\|f(x)\|_p, \quad \forall\ f(x) \in L_p(R)$$

where $c(p) \in (0,\infty)$ is an absolute constant.

Theorem 2.4. [6] *Let $p \in [1, \infty]$ be given. Equation* (1.5) *is correctly solvable in $L_p(R)$ if and only there is $a \in (0, \infty)$ such that $q_0(a) > 0$. Here*

$$q_0(a) = \inf_{x \in R} \int_{x-a}^{x+a} q(t)dt. \tag{2.16}$$

3. Statement of Results

Below we present three equivalent statements: Theorems 3.1, 3.2 and 3.3. Each of these assertions contains a criterion for stability of the difference scheme (1.1) in $L_p(h)$, $p \in [1, \infty)$. Theorem 3.1 is intended for investigation of general properties of the solution of equation (1.1) (see, for example, the proof of Corollary 3.1.1 in Section 4), and Theorems 3.2 and 3.3 are more convenient as practical criteria for checking stability of concrete difference schemes (see, for example, the proof of Theorem 3.4 in Section 7). Note that for the sake of completeness, we also include the case $p = \infty$. The proofs for $p = \infty$ can be found in [2].

Theorem 3.1. *For every $p \in [1, \infty]$ the difference scheme* (1.1) *is stable in $L_p(h)$ if and only if condition* (2.1) *holds and $A < \infty$ (see* (2.13)*). In particular, one of the following assertions holds:*

A) *The difference scheme* (1.1) *is stable in $L_p(h)$ for all $p \in [1, \infty]$*

B) *For all $p \in [1, \infty]$ the difference scheme* (1.1) *is non-stable in $L_p(h)$.*

In addition, in case A), for any sequence $f(h) \in L_p(h)$, $p \in [1, \infty]$, the solution $y(h) \in L_p(h)$ of (1.1) *admits representation* (2.11).

Corollary 3.1.1. *Suppose that the difference scheme* (1.1) *is stable in $L_p(h)$, $p \in [1, \infty]$. Then for every right-hand side $f(h) \in L_p(h)$, the solution $y(h) \in L_p(h)$ of* (1.1) *satisfies the following inequalities:*

$$\|q(h)^{1/p} y(h)\|_{L_p(h)} \le c\|f(h)\|_{L_p(h)}, \tag{3.1}$$

$$\|r(h)y(h)\|_{L_p(h)} \le c\|f(h)\|_{L_p(h)}. \tag{3.2}$$

Here $r(h) = \{d_n(h)^{-1/2}\}_{n \in Z}$ (see (2.7)*). In addition, for $p = 1$ we have*

$$\|\Delta^{(2)} y(h)\|_{L_1(h)} + \|q(h)y(h)\|_{L_1(h)} \le 3\|f(h)\|_{L_1(h)}. \tag{3.3}$$

Here $\Delta^{(2)} y(h) = \{\Delta^{(2)} y_n(h)\}_{n \in Z}$.

Remarks. In case (1.4) estimate (3.3) was obtained in [11]. Inequality (3.3) means that for $f(h) \in L_1(h)$ a stable difference scheme (1.1) can be decomposed into summands which are uniformly bounded in $L_1(h)$. A problem on decomposability of difference equations appeared as an analogue of the corresponding problem for a differential operator which was first studied in [8], [9].

Theorem 3.2. *For any $p \in [1, \infty]$ the difference scheme (1.1) is stable in $L_p(h)$ if and only if there is an absolute positive constant c_0 such that $c_0 h_0^{-1} \geq 1$ and $B > 0$. Here*

$$B = \inf_{h \in (0,h_0]} \inf_{n \in Z} \sum_{k=n-k_0(h)}^{n+k_0(h)} q_k(h) h, \quad k_0(h) = [c_0 h^{-1}] \tag{3.4}$$

Theorem 3.3. *For any $p \in [1, \infty]$, the difference scheme (1.1) is stable in $L_p(h)$ if and only if there is an absolute positive constant c_0 such that $c_0 h_0^{-1} \geq 1$ and $S > 0$. Here*

$$S = \inf_{h \in (0,h_0]} \inf_{n \in Z} \frac{1}{2k_0(h)+1} \sum_{k=n-k_0(h)}^{n+k_0(h)} q_k(h), \quad k_0(h) = [c_0 h^{-1}]. \tag{3.5}$$

Corollary 3.3.1. *If for at least one $h \in (0, h_0]$ any of the following equalities*

$$\lim_{n \to -\infty} q_n(h) = 0, \quad \lim_{n \to \infty} q_n(h) = 0, \tag{3.6}$$

holds, then the difference scheme (1.1) is non-stable in $L_p(h)$ for any $p \in [1, \infty]$.

In particular, the difference scheme (1.1) is non-stable in $L_p(h)$ for any $p \in [1, \infty]$ if for at least one $h \in (0, h_0]$, any of the following inequalities holds:

$$\sum_{n=-\infty}^{0} q_n(h) < \infty, \quad \sum_{n=0}^{\infty} q_n(h) < \infty. \tag{3.7}$$

From Theorem 3.3 it follows that if (1.4) holds, then for every c_0 (such that $c_0 h_0^{-1} \geq 1$) the following inequalities hold:

$$S = \inf_{h \in [0,h_0]} \inf_{n \in Z} \frac{1}{2k_0(h)+1} \sum_{k=n-k_0(h)}^{n+k_0(h)} q_k(h) \geq \inf_{h \in (0,h_0]} \inf_{n \in Z} \frac{1}{2k_0(h)+1} \sum_{k=n-k_0(h)}^{n+k_0(h)} \varepsilon = \varepsilon > 0,$$

and therefore in this case the difference scheme (1.1) is stable in $L_p(h)$, $p \in [1, \infty]$. The proof of this assertion and short and transparent because using Theorem 3.3 shows its efficiency (compare with the proof of the same result in [4] independent of Theorem 3.3. The next theorem establishes a relationship between the difference scheme (1.1) – (1.7) and the equations (1.5) – (1.6).

Theorem 3.4. *For any $p \in [1, \infty]$ the difference scheme* (1.1) – (1.7) *is stable in $L_p(h)$ if and only if equation* (1.5) *with condition* (1.6) *is correctly solvable in $L_p(R)$.*

4. Proof the First Stability Criterion

In this section we prove Theorem 3.1.

Proof of Theorem 3.1. Necessity.

Suppose that for some $p \in [1, \infty)$ the difference scheme (1.1) is stable in $L_p(h)$. Let us show that inequalities (2.1) hold. Assume the contrary. For example, assume that there exist $n \in Z$ and $h_1 \in (0, h_0]$ such that

$$\sum_{n=n_0}^{\infty} q_k(h_1) = 0. \tag{4.1}$$

Then, in view of (1.2), we obtain

$$q_n(h_1) = 0 \quad \text{for} \quad n \geq n_0. \tag{4.2}$$

Let us show that equalities (4.2) contradict assertions I) – II) from Section 1. Without loss of generality, we may (and shall) assume that $n_0 = -3$. Let m_0 be a sufficiently large natural number. For any arbitrary natural number $m \geq m_0$, consider the sequence $y^{(m)} = \{y_n^{(m)}\}_{n \in Z}$ where

$$y_n^{(m)} = \begin{cases} h_1^2, & \text{if} \quad n = 0 \\ h_1^2 + (n-1)h_1^4, & \text{if} \quad n = 1, 2, \dots, m \\ 0, & \text{if} \quad n \notin \{0, 1, 2, \dots, m\} \end{cases} \tag{4.3}$$

From (4.2) and (4.3) it immediately follows that equalities (1.1) hold for $h = h_1$ and $f_n(h) = f_n^{(m)}$ where

$$f_n^{(m)} = \begin{cases} -1, & \text{if} \quad n = -1 \\ 1, & \text{if} \quad n = 0 \\ -h_1^2, & \text{if} \quad n = 1 \\ 1 + mh_1^2, & \text{if} \quad n = m \\ -1 - (m-1)h_1^2, & \text{if} \quad n = m+1 \\ 0, & \text{if} \quad n \notin \{-1, 0, 1, m, m+1\}, \ n \in Z \end{cases} \tag{4.4}$$

Let $f^{(m)} = \{f_n^{(m)}\}_{n \in Z}$. Since $m \geq m_0 \gg 1$, from (4.4) it follows that

$$\|f^{(m)}\|_{L_p(h_1)}^p = \{2 + h_1^{2p} + (1 + mh_1^2)^p + [1 + (m-1)h_1^2]^p\}h_1 \leq c(h_0)m^p h_1^{2p+1}. \tag{4.5}$$

By (4.5), we have $f^{(m)} \in L_p(h_1)$ and therefore, according to I) (see Section 1), for $h = h_1$ in the space $L_p(h_1)$, there exists a unique solution to (1.1) with $f(h_1) = f^{(m)}$. Since by (4.3) we have $y^{(m)} \in L_p(h_1)$, the sequence $y^{(m)}$ coincides with this solution. Note that for any $m \ge m_0 \gg 1$, we have an obvious estimate

$$\|y^{(m)}\|^p_{L_p(h_1)} = \left\{2h_1^p + \sum_{n=2}^{m}[h_1^2 + (n-1)h_1^4]^p\right\} h_1 \ge c(p)h_1^{4p+1}m^{p+1}. \tag{4.6}$$

Now, using (1.3), (4.6) and (4.5), we get

$$\begin{aligned} c(p)h_1^{4p+1}m^{p+1} \le \|y^{(m)}\|^p_{L_p(h_1)} \le c\|f^{(m)}\|^p_{L_p(h_1)} \le c(h_0)h_1^{2p+1}m^p \\ \Rightarrow \quad h_1^{2p} \le cm^{-1}, \quad \forall\, m \ge m_0 \gg 1 \quad \Rightarrow \quad h_1 = 0, \end{aligned}$$

a contradiction.

Hence inequalities (2.1) hold, as required. Let us now show that condition (2.1) and assertions I) – II) (see Section 1) imply assertion III) from Definition 2.2. Let $f(h) = \{f_n(h)\}_{n\in Z} \in L_p(h)$ and $y(h) = \{y_n(h)\}_{n\in Z}$ be the unique solution of (1.1) in $L_p(h)$, and let $\{u(h), v(h)\}$ be a FSS of (2.2) with properties (2.3). Let us multiply the equalities

$$-\Delta^{(2)}y_k(h) + q_k(h)y_k(h)h^2 = f_k(h)h^2, \quad k \in Z$$

by $v_k(h)$ and add up for $k \in [m,n]$, $m \le n$.

$$\begin{aligned} \sum_{k=m}^{n} f_k(h)v_k(h)h^2 &= -\sum_{k=m}^{n}(\Delta^{(2)}y_k)v_k(h) + \sum_{k=m}^{n} q_k(h)v_k(h)y_k(h)h^2 \\ &= -\sum_{k=m}^{n} y_k\Delta^{(2)}v_k(h) + \sum_{k=m}^{n} q_k(h)v_k(h)y_k(h)h^2 + y_nv_{n+1}(h) \\ &\quad - y_{n+1}v_n(h) + y_mv_{m-1}(h) - v_my_{m-1}(h) \\ &= \sum_{k=m}^{n}[-\Delta^{(2)}v_k(h) + q_k(h)v_k(h)h^2]y_k + y_nv_{n+1}(h) - y_{n+1}v_n(h) \\ &\quad + y_mv_{m-1}(h) - v_my_{m-1} \\ &\Rightarrow \quad y_nv_{n+1}(h) - y_{n+1}v_n(h) + y_mv_{m-1}(h) - v_my_{m-1} \\ &= \sum_{k=m}^{n} f_k(h)v_k(h)h^2, \quad m \le n. \end{aligned} \tag{4.7}$$

Since $y(h) \in L_p(h)$, we conclude that $y_m \to 0$ as $m \to -\infty$. Then from (2.3) it follows that the left-hand side of (4.7) tends to $y_nv_{n+1}(h) - y_{n+1}v_n(h)$ as $m \to -\infty$. Hence the sum in the right-hand side of (4.7) has a limit as $m \to -\infty$, and we obtain

$$y_nv_{n+1}(h) - y_{n+1}v_n(h) = \sum_{k=-\infty}^{n} v_k(h)f_k(h)h^2, \quad n \in Z. \tag{4.8}$$

Similarly, we verify equality (4.9):

$$y_{n+1}u_n(h) - u_{n+1}(h)y_n = \sum_{k=n+1}^{\infty} u_k(h)f_k(h)h^2, \quad n \in Z. \tag{4.9}$$

Let us now multiply (4.8) and (4.9) by $u_n(h)$ and $v_n(h)$, respectively, and add up the resulting equalities. Then (2.3) implies

$$\begin{aligned} u_n(h)[y_n v_{n+1}(h) - y_{n+1}v_n(h)] + v_n(h)[y_{n+1}(h)u_n(h) - u_{n+1}(h)y_n] \\ = y_n[v_{n+1}(h)u_n(h) - v_n(h)u_{n+1}(h)] = y_n h = \sum_{k\in Z} G_{n,k}(h)f_k(h)h^2. \end{aligned}$$

After cancelling $h \in (0, h_0]$ in the last equality, we obtain the needed representation:

$$y_n = y_n(h) = \sum_{m\in Z} G_{n,m}(h)f_m(h)h, \quad n \in Z. \tag{4.10}$$

Thus we conclude that conditions (2.1) hold, and the inversion problem for (1.1) is regular in $L_p(h)$ for some $p \in [1, \infty)$ (see Definition 2.2). Then by Theorem 2.3 we have $A < \infty$, as required.

Proof of Theorem 3.1. Sufficiency.

The main assertion of the theorem immediately follows from Theorem 2.3. Moreover, since the conditions obtained for the stability of the scheme (1.1) in $L_p(h)$ do not depend on $p \in [1, \infty)$ and are exact, we obtain the alternative given in A) and B) (see Theorem 3.1). Finally, in case A) we have the representation (4.10) (see above) which concludes the proof of all the assertions of Theorem 3.1. □

Proof of Corollary 3.1.1. If a difference scheme is stable, then the inequalities (2.1) and $A < \infty$ hold (see (2.13)). Then inequalities (3.1) and (3.3) follows from Theorem 2.3 (see [3]). Let us check inequality (3.2). We need the following lemma.

Lemma 4.1. *Suppose that inequalities* (2.1) *hold. Then*

$$c^{-1} \le \frac{d_k(h)}{d_n(h)} \le c \quad \text{for} \quad |k-n| \le \left[\frac{\ell_n(h)}{2}\right], \ n \in Z. \tag{4.11}$$

If, in addition, $A < \infty$ *(see* (2.13)*), then*

$$\sup_{h\in(0,h_0]} \sup_{n\in Z} \sum_{m\in Z} \frac{1}{\sqrt{d_m(h)}} G_{n,m}(h)h < \infty. \tag{4.12}$$

Proof. Estimates (4.11) were obtained in [3]. To prove the second inequality, we divide the sum in (4.12) into three summands:

$$\sum_{m\in Z}\frac{1}{\sqrt{d_m(h)}}G_{n,m}(h)h=\sum_{m=-\infty}^{n-1}\frac{1}{\sqrt{d_m(h)}}G_{n,m}(h)h$$
$$+\frac{G_{n,n}(h)}{\sqrt{d_n(h)}}+\sum_{m=n+1}^{\infty}\frac{1}{\sqrt{d_m(h)}}G_{n,m}(h)h:=S_1^{(n)}(h)+S_2^{(n)}(h)+S_3^{(n)}(h).$$

Let us estimate $S_k^{(n)}(h), k=\overline{1,3}$ separately. First, from (2.9), it follows that

$$S_2^{(n)}(h)=\frac{G_{n,n}(h)h}{\sqrt{d_n(h)}}=\frac{\rho_n(h)h}{\sqrt{d_n(h)}}\le c\sqrt{d_n(h)}h\le c\sqrt{A}h_0.$$

The sums $S_1^{(n)}$ and $S_2^{(n)}(h)$ can be estimated in the same way, so we only consider, say, $S_3^{(n)}(h)$. In the following, we use Lemmas 2.1, 2.3, 2.4, inequalities (2.9), (4.11), and formula (2.4):

$$\begin{aligned}
S_3^{(h)}(h)&=\sum_{m=n+1}^{\infty}\frac{G_{n,m}(h)h}{\sqrt{d_m(h)}}=\sum_{s=1}^{\infty}\sum_{k\in\Delta_s}\frac{G_{n,k}(h)h}{\sqrt{d_k(h)}}\\
&\le c\sum_{s=1}^{\infty}\frac{1}{\sqrt{d_{\Delta_s^-}(h)}}G_{n,\Delta_s^-}(h)\left(\sum_{k\in\Delta_s}1\right)h=c\frac{v_n(h)u_{n+1}(h)h}{\sqrt{d_{n+1}(h)}}\\
&\quad+c\sum_{s=2}^{\infty}\sqrt{\frac{\rho_n(h)\rho_{\Delta_s^-}(h)}{d_{\Delta_s^-}(h)}}\prod_{i=1}^{s-1}\prod_{k\in\Delta_i}\left[1+\frac{u_k(h)}{u_{k+1}h)}\ \frac{h}{\rho_k(h)}\right]^{-1/2}\left(2\left[\frac{\ell_{k_s}(h)}{2}\right]+1\right)h\\
&\le c\frac{\rho_{n+1}(h)h}{\sqrt{d_{n+1}(h)}}+c\sqrt{d_n(h)}\sum_{s=2}^{\infty}\prod_{i=1}^{s-1}T_s(h)^{-1/2}(\ell_{k_s}(h)h+h)\\
&\le c\sqrt{d_{n+1}(h)}h+c\sqrt{d_n(h)}\sum_{s=2}^{\infty}(d_{k_s}(h)+h)\gamma^{\frac{s-1}{2}}\le c\sqrt{A}h_0+c\sqrt{A}(A+h_0)\sum_{s=2}^{\infty}\gamma^{\frac{s-1}{2}}\\
&=c<\infty.
\end{aligned}$$

□

In the following, to prove inequalities (3.2), we use Hölder's inequality, estimates (4.12) and (2.15), the symmetry of the Green function $G_{n,m}(h)$, and the theorem on the change of summation order for multiple series with non-negative terms ([13, Ch.I,§6.2]):

$$\begin{aligned}
\left\| \frac{1}{r(h)} y(h) \right\|_{L_p(h)}^p &= \sum_{n \in Z} \frac{1}{r_n(h)^p} |y_n(h)|^p h \\
&= \sum_{n \in Z} \frac{1}{r_n(h)^p} \left| \sum_{m \in Z} G_{n,m}(h) f_m(h) h \right|^p h \\
&\le \sum_{n \in Z} \frac{1}{r_n(h)^p} \left[\sum_{m \in Z} G_{n,m}(h) h \right]^{p/p'} \left[\sum_{m \in Z} G_{n,m}(h) |f_m(h)|^p h \right] h \\
&\le c \sum_{n \in Z} \frac{1}{\sqrt{d_n(h)}} \left[\sum_{m \in Z} G_{n,m}(h) |f_m(h)|^p h \right] h \\
&= c \sum_{m \in Z} |f_m(h)|^p \left[\sum_{n \in Z} \frac{1}{\sqrt{d_n(h)}} G_{n,m}(h) h \right] h \\
&\le c \sum_{m \in Z} |f_m(h)|^p h = c \| f(h) \|_{L_p(h)}^p .
\end{aligned}$$

□

5. Proof of the Second Stability Criterion

In this section we prove Theorem 3.2.

Proof of Theorem 3.2. Necessity.

Suppose that the difference scheme (1.1) is stable in $L_p(h)$, $p \in [1, \infty]$. Then, by Theorem 3.1, condition (2.1) holds, and therefore the sequences $\{\ell_n(h)\}_{n \in Z}$ and $\{d_n(h)\}_{n \in Z}$ (see (2.6) – (2.7)) are well defined. For arbitrary $n \in Z$ and $h \in (0, h_0]$, we have two possibilities:

1) $\ell_n(h) \neq 0$

2) $\ell_n(h) = 0$.

First consider case 1). Since by Theorem 3.1 we have $A < \infty$ (see (2.13)), from (2.7) it follows that

$$d_n(h) = \ell_n(h) h \le A < \infty \quad \Rightarrow \quad \ell_n(h) \le c_0 h^{-1}. \tag{5.1}$$

We choose the constant in (5.1) so that $c_0 h_0^{-1} \ge 1$, and set $k_0(h) = [c_0 h^{-1}]$. Clearly, $k_0(h) \ge 1$ because $c_0 h^{-1} \ge c_0 h_0^{-1} \ge 1$. Therefore (5.1) implies

$$\ell_n(h) \le k_0(h) \le c_0 h^{-1}, \quad k_0(h) h \le c_0. \tag{5.2}$$

Now, using (5.1) and (2.6), we get

$$
\begin{aligned}
c_0 \sum_{k=n-k_0(h)}^{n+k_0(h)} q_k(h)h &= c_0h^{-1} \sum_{k=n-k_0(h)}^{n+k_0(h)} q_k(h)h^2 \\
&\ge \ell_n(h) \sum_{k=n-\ell_n(h)}^{n+\ell_n(h)} q_k(h)h^2 \ge 1 \quad \Rightarrow \quad \sum_{k=n-k_0(h)}^{n+k_0(h)} q_k(h)h \ge c_0^{-1}.
\end{aligned}
\tag{5.3}
$$

In the case 2), with the same choice of $k_0(h)$, we obtain (see (2.6)):

$$
\begin{aligned}
&\sum_{k=n-k_0(h)}^{n+k_0(h)} q_k(h)h^2 \ge q_n(h)h^2 \ge 1 \\
&\qquad\Rightarrow \quad \sum_{k=n-k_0(h)}^{n+k_0(h)} q_k(h)h \ge \frac{c_0}{h}\cdot\frac{1}{c_0} \ge \frac{c_0}{h_0}\cdot\frac{1}{c_0} \ge c_0^{-1}.
\end{aligned}
$$

This immediately implies that $B > 0$ (see (3.4)).

Proof of Theorem 3.2. Sufficiency.

Let c_0 be a constant such that $c_0h_0^{-1} \ge 1$ and $B > 0$ (see (3.4)). If necessary, let us take a larger c_0 in order to obtain the inequality $c_0B \ge 1$. Then, clearly, inequalities (2.1) hold, and for all $n \in Z$ and $h \in (0, h_0]$ the functions $\ell_n(h)$ and $d_n(h)$ (see (2.6) – (2.7)) are defined. Furthermore, for $k_0(h) = [c_0h^{-1}]$ we have

$$
\sum_{k=n-k_0(h)}^{n+k_0(h)} q_k(h)h \ge B \ge \frac{1}{c_0}, \quad n \in Z, \quad h \in (0, h_0]. \tag{5.4}
$$

Clearly, $k_0(h) \ge 1$ because $c_0h^{-1} \ge c_0h_0^{-1} \ge 1$. Therefore, $2k_0(h) \ge c_0h^{-1}$, and we get

$$
\begin{aligned}
2k_0(h) \sum_{k=n-2k_0(h)}^{n+2k_0(h)} q_k(h)h^2 &= 2\left[\frac{c_0}{h}\right] \sum_{k=n-2k_0(h)}^{n+2k_0(h)} q_k(h)h^2 \\
&\ge \frac{c_0}{h} \sum_{k=n-k_0(h)}^{n+k_0(h)} q_k(h)h^2 = c_0 \sum_{k=n-k_0(h)}^{n+k_0(h)} q_k(h) \ge c_0B \ge 1.
\end{aligned}
\tag{5.5}
$$

From (5.5), it follows that if $q_k(h)h^2 < 1$, then $\ell_n(h) \le 2k_0(h)$ (see (2.6)), and therefore $d_n(h) = \ell_n(h)h \le 2[c_0h^{-1}]h \le 2c_0$. Moreover, if $q_n(h)h^2 \ge 1$ then (see (2.7)):

$$
d_n(h) = \frac{h}{1 + q_n(h)h^2} \le \frac{h}{2} < h_0.
$$

Hence $A < \infty$ (see (2.13)), and it remains to refer to Theorem 3.1. $\square$

6. Proof of the Third Stability Criterion

In this section we prove Theorem 3.3.

Proof of Theorem 3.1. Necessity.

Suppose that the difference scheme (1.1) is stable in $L_p(h)$, $p \in [1, \infty]$. By Theorem 3.2, there is a constant c_0 such that $c_0 h_0^{-1} \geq 1$, and for all $n \in Z$ and $h \in (0, h_0]$ we have (see (3.4)):

$$B_n(h) \overset{\text{def}}{=} \sum_{k=n-k_0(h)}^{n+k_0(h)} q_k(h)h \geq B > 0 \quad \text{for} \quad k_0(h) = [c_0 h^{-1}]. \tag{6.1}$$

Since $c_0 h^{-1} \geq c_0 h_0^{-1} \geq 1$, we get $k_0(h) \geq 1$ and also $c_0 k_0(h)^{-1} \geq h$. Then from (6.1), for all $n \in Z$ and $h \in (0, h_0]$, it follows that

$$0 < B \leq B_n(h) \leq \frac{c_0}{k_0(h)} \sum_{k=n-k_0(h)}^{n+k_0(h)} q_k(h) \leq 3c_0 \left[\frac{1}{2k_0(h)+1} \sum_{k=n-k_0(h)}^{n+k_0(h)} q_n(h)\right].$$

Hence $S \geq (3c_0)^{-1} B > 0$, as required (see (3.5)).

Proof of Theorem 3.1. Sufficiency.

Suppose that there is a constant c_0 such that $c_0 h_0^{-1} \geq 1$ and for all $n \in Z$ and $h \in (0, h_0]$, the following inequality holds:

$$S_n(h) \overset{\text{def}}{=} \frac{1}{2k_0(h)+1} \sum_{k=n-k_0(h)}^{n+k_0(h)} q_k(h) \geq S > 0, \quad k_0(h) = [c_0 h^{-1}]. \tag{6.2}$$

Since $2k_0(h) \geq c_0 h^{-1}$, from (6.2) it follows (see (6.1)) that

$$\begin{aligned} 0 < S \leq S_n(h) &\leq \frac{1}{2k_0(h)} \sum_{k=n-k_0(h)}^{n+k_0(h)} q_k(h) \leq \frac{1}{c_0} \sum_{k=n-k_0(h)}^{n+k_0(h)} q_k(h)h \\ &= \frac{1}{c_0} B_n(h) \quad \Rightarrow \quad B_n(h) \geq c_0 S \quad \Rightarrow \quad B > 0 \quad \text{(see (3.4))}. \end{aligned}$$

It remains to refer to Theorem 3.2. □

Proof of Corollary 3.3.1. Suppose that for some $h_1 \in (0, h_0]$, say, the second equality of (3.6) holds. Then for a given $\varepsilon > 0$, there exists $n_0(\varepsilon)$ such that $q_n(h_1) \leq \varepsilon$ for any $n \geq h_0(\varepsilon)$. Fix some c_0 such that $c_0 h_0^{-1} \geq 1$, and set $k_0(h_1) = [c_0 h_1^{-1}]$. With such a choice of c_0 and $k_0(h_1)$, for any $n \ge n_0(\varepsilon) + k_0(h_1)$, we have (see (6.2)):

$$S_n(h_1) = \frac{1}{2k_0(h_1)+1} \sum_{k=n-k_0(h_1)}^{n+k_0(h_1)} q_k(h_1) \le \frac{\varepsilon}{2k_0(h_1)+1} \sum_{k=n-k_0(h_1)}^{n+k_0(h_1)} 1 = \varepsilon. \tag{6.3}$$

From (6.3), it follows that

$$\inf_{n\in Z} S_n(h_1) = 0 \quad \Rightarrow \quad S = \inf_{h\in(0,h_0]} \inf_{n\in Z} S_n(h) = 0.$$

It remains to refer to Theorem 3.3. □

7. Main Example

In this section we prove Theorem 3.4.

Proof of Theorem 3.1. Necessity.

Suppose that equations (1.5) – (1.6) is correctly solvable in $L_p(R)$, $p \in [1,\infty]$. Then by Theorem 2.4, there exist $a > 0$ and $\varepsilon > 0$ such that

$$\int_{x-a}^{x+a} q(t)dt \ge \varepsilon > 0, \quad \forall\, x \in R. \tag{7.1}$$

Without loss of generality, we may (and shall) assume $a \ge 1$. Consider the difference scheme (1.1) – (1.7) with $h_0 = 1$. Let $c_0 = 2a$, $k_0(h) = [2ah^{-1}]$. Then, clearly, $k_0(h) \ge ah^{-1}$, and therefore for any $n \in Z$, we get (see (6.1)):

$$\begin{aligned} 2B_n(h) &= 2\sum_{k=n-k_0(h)}^{n+k_0(h)} q_k(h)h = \sum_{k=n-k_0(h)}^{n+k_0(h)} \int_{x_k-h}^{x_k+h} q(t)dt \\ &\ge \int_{x_n-k_0(h)h}^{x_n+k_0(h)h} q(t)dt \ge \int_{x_n-a}^{x_n+a} q(t)dt \ge \varepsilon > 0. \end{aligned}$$

Hence $B > 0$ (see (3.4)), and by Theorem 3.2 we conclude that the difference scheme (1.1) – (1.7) is stable in $L_p(h)$, $p \in [1,\infty]$.

Proof of Theorem 3.4. Sufficiency.

Let $h_0 = 1$, and let the difference scheme (1.1) – (1.7) be stable in $L_p(h)$, $p \in [1,\infty]$. Then there exist $c_0 = c_0 h_0^{-1} \ge 1$ and $\varepsilon > 0$ such that

$$\sum_{k=n-k_0(h)}^{n+k_0(h)} q_k(h)h \ge \varepsilon \quad \text{for} \quad k_0(h) = [c_0h^{-1}],\ h \in (0,1]. \tag{7.2}$$

From (7.2), taking into account (1.7), it follows that

$$\varepsilon \leq \sum_{k=n-k_0(h)}^{n+k_0(h)} q_k(h)h = \frac{1}{2}\sum_{k=n-k_0(h)}^{n+k_0(h)} \int_{x_k-h}^{x_k+h} q(t)dt$$
$$\leq \int_{x_n-k_0(h)h}^{x_n+k_0(h)h} q(t)dt \leq \int_{x_n-c_0}^{x_n+c_0} q(t)dt, \quad n \in Z.$$

Thus for $x_n = nh$, $h \in (0,1]$ and for any $n \in Z$, we have

$$\int_{x_n-c_0}^{x_n+c_0} q(t)dt \geq \varepsilon > 0. \tag{7.3}$$

Then for any $x \in (x_n - h, x_n + h)$ and any $n \in Z$, $h \in (0,1]$, we get

$$\int_{x-2c_0}^{x+2c_0} q(t)dt \geq \int_{x_n-c_0}^{x_n+c_0} q(t)dt \geq \varepsilon > 0. \tag{7.4}$$

From (7.4) it follows that $q_0(2c_0) > 0$ (see (2.16)), and by Theorem 2.4, we conclude that equations (1.5) – (1.6) is correctly solvable in $L_p(R)$, $p \in [1,\infty]$. $\square$

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