REGULARITY OF THE INVERSION PROBLEM FOR THE STURM-LIOUVILLE DIFFERENCE EQUATION III. A CRITERION FOR REGULARITY OF THE INVERSION PROBLEM

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Abstract. We consider a difference equation

$$-h^{-2}\Delta^{(2)}y_n + q_n(h)y_n = f_n(h),\ n \in Z = \{0, \pm1, \pm2, \dots\}, \tag{1}$$

where $h \in (0, h_0]$, h_0 is a fixed positive number,

$$\Delta^{(2)}y_n = y_{n+1} - 2y_n + y_{n-1},\ n \in Z;\ f = \{f_n(h)\}_{n\in Z} \in L_p(h), p \in [1, \infty),$$

$$L_p(h) = \{f : \|f\|_{L_p(h)} < \infty\},\ \|f\|^p_{L_p(h)} = \sum_{n\in Z} |f_n(h)|^p h,\ \text{and}$$

$$0 \le q_n(h) < \infty,\ \sum_{k=-\infty}^{n} q_k(h) > 0,\ \sum_{k=n}^{\infty} q_k(h) > 0,\ n \in Z.$$

We obtain necessary and sufficient conditions under which assertions I) - II) hold together:

I) for a given $p \in [1, \infty)$, for any $f \in L_p(h)$, (1) has a unique solution $y = \{y_n(h)\}_{n\in Z} \in L_p(h)$ (regardless of h), and $y = (Gf)(h) \overset{\text{def}}{=} \{(Gf)_n(h)\}_{n\in Z}$, $(Gf)_n(h) = \sum_{m\in Z} G_{n,m}(h) f_m(h) h$, $n \in Z$.

II) $\|y\|_{L_p(h)} \le c(p)\|f\|_{L_p(h)}$ for any $f \in L_p(h)$.
Here $c(p)$ is an absolute positive constant, $\{G_{n,m}(h)\}_{n,m\in Z}$ is the difference Green function corresponding to (1).

1. Introduction

In this paper, we consider a difference Sturm-Liouville equation (three-point difference scheme)

$$-h^{-2}\Delta^{(2)}y_n + q_n(h)y_n = f_n(h), \quad n \in Z = \{0, \pm1, \pm2, \dots\}, \tag{1.1}$$

where here and throughout the paper $h \in (0, h_0]$, h_0 is a fixed positive number,

$$\Delta^{(2)}y_n = y_{n+1} - 2y_n + y_{n-1}, \quad n \in Z, \quad f \overset{\text{def}}{=} \{f_n(h)\}_{n\in Z} \in L_p(h),\ p \in [1,\infty),$$
$$L_p(h) \overset{\text{def}}{=} \{f : \|f\|_{L_p(h)} < \infty\}, \quad \|f\|^p_{L_p(h)} = \sum_{n\in Z} |f_n(h)|^p h,$$
$$0 \le q_n(h) < \infty,\ n \in Z,\ h \in (0, h_0]. \tag{1.2}$$

Our goal is to study conditions under which certain properties of the inversion of (1.1) do not depend on h. To be more precise, our main result (Theorem 1.1 below) is a criterion for I) – III) to hold together:

I) for a given $p \in [1,\infty)$, for any sequence $f \in L_p(h)$, (1.1) has a unique solution $y \in L_p(h)$ (regardless of h),

II) for any h, for any sequence $f \in L_p(h)$, the solution $y \in L_p(h)$ of (1.1) satisfies the inequality

$$\|y\|_{L_p(h)} \le c(p)\|f\|_{L_p(h)} \tag{1.3}$$

where c_p is an absolute positive constant;

III) for any h, for any sequence $f \in L_p(h)$, the solution $y \in L_p(h)$ of (1.1) admits the following representation:

$$y \overset{\text{def}}{=} (Gf)(h) \overset{\text{def}}{=} \{(Gf)_n(h)\}_{n\in Z}, \quad (Gf)_n(h) = \sum_{m\in Z} G_{n,m}(h) f_m(h) h. \tag{1.4}$$

Here $\{G_{n,m}(h)\}_{n,m\in Z}$ is the difference Green function corresponding to (1.1):

$$G_{n,m}(h) = \begin{cases} u_n(h)v_m(h), & n \ge m \\ u_m(h)v_n(h), & n \le m \end{cases} \tag{1.5}$$

and $\{u_n(h), v_n(h)\}_{n\in Z}$ is a special fundamental system of solutions of the equation (see §2)

$$h^{-2}\Delta^{(2)}z_n = q_n(h)z_n, \quad n \in Z. \tag{1.6}$$

Definition 1.1. [6, 7] If I) – III) hold, we say that the inversion problem for (1.1) is regular in $L_p(h)$.

It is known that the inversion problem for (1.1) is regular in $L_p(h)$, $p \in [1,\infty)$ provided

$$q_n(h) \ge \varepsilon > 0 \quad \text{for} \quad n \in Z,\ h \in (0, h_0] \tag{1.7}$$

(see [6, 7], the case $p = \infty$ is studied in [9]).

Thus the question whether I) – III) hold remains open for those equations (1.1) whose potential $q = \{q_n(h)\}_{n\in Z}$ is not separated from zero in the sense of (1.7) (for example, if q is an oscillating sequence). We study this problem in the present paper.

Let us now formulate our main result, Theorem 1.1. In Theorem 1.1 we assume that the sequence $q=\{q_n(h)\}_{n\in Z}$ satisfies, in addition to (1.2), another condition (1.8)

$$\sum_{k=-\infty}^{n} q_k(h)>0,\ \sum_{k=n}^{\infty} q_k(h)>0,\quad \text{for any}\quad n\in Z. \tag{1.8}$$

Note that the combined requirement (1.2) and (1.8) is satisfied, for example, for all non-negative oscillating sequences $\{q_n(h)\}_{n\in Z}$ which are not identically zero at $\pm\infty$. In addition to condition (1.8), we also need auxiliary sequences $\ell_n(h)_{n\in Z}$ and $\{d_n(h)\}_{n\in Z}$ which are well-defined provided (1.2) and (1.8) hold:

$$\ell_n(h)=\begin{cases}0 & \text{if}\quad q_n(h)h^2\ge 1\\ \min\limits_{j\ge 0}\left\{j: j\sum\limits_{s=n-j}^{n+j} q_s(h)h^2\ge 1\right\}, & \text{if}\quad q_n(h)h^2<1\end{cases} \tag{1.9}$$

$$d_n(h)=\begin{cases}h(1+q_n(h)h^2)^{-1} & \text{if}\quad \ell_n(h)=0\\ \ell_n(h)h, & \text{if}\quad \ell_n(h)\ne 0\end{cases} \tag{1.10}$$

The sequences $\ell_n(h)_{n\in Z}$ and $\{d_n(h)\}_{n\in Z}$ were first used in [1].

Theorem 1.1. *Let $p\in[1,\infty)$. Under conditions (1.2) and (1.8), the inversion problem of (1.1) in $L_p(h)$ is regular if and only if*

$$A(q)<\infty,\qquad A(q)\overset{def}{=}\sup_{h\in(0,h_0]}\sup_{n\in Z} d_n(h). \tag{1.11}$$

Corollary 1.1.1. *Suppose that the inversion problem for (1.1) is regular in $L_p(h)$. Then the solution $y=\{y_n(h)\}_{n\in Z}$ of (1.1) (see (1.4)) satisfies the inequality*

$$\|q^{1/p}y\|_{L_p(h)}=\left(\sum_{m\in Z} q_m(h)|y_m(h)|^p h\right)^{1/p}\le c(p)A(q)^{1/p'}\|f\|_{L_p(h)}. \tag{1.12}$$

Here $c(p)$ is an absolute positive constant. In particular, for $p=1$ the solution of (1.1) satisfies the following estimate:

$$\|h^{-2}\Delta^{(2)}y\|_{L_1(h)}+\|qy\|_{L_1(h)}\le 3\|f\|_{L_1(h)},\ f\in L_1(h),\ \Delta^{(2)}y\overset{def}{=}\{\Delta^{(2)}y_n\}_{n\in Z}. \tag{1.13}$$

Remark. For $q_n(h)\ge\varepsilon>0$, $n\in Z$, $h\in(0,h_0]$, estimate (1.13) was obtained in [12].

Condition (1.11) does not contain $q=\{q_n(n)\}_{n\in Z}$ explicitly and therefore some explanation is necessary. It is convenient to begin with possible applications of Theorem 1.1. It is known that a difference scheme (1.1) satisfying only I) – II) (and not I) – III) as in Definition 1.1) is said to be stable in $L_p(h)$. To study conditions for stability is one of the main problems of a priori analysis of concrete difference schemes (see, for example, [11]).

Note that if the inversion problem of (1.1) in $L_p(h)$ $p \in [1,\infty)$ is regular, then the difference scheme (1.1) is automatically stable. Hence Theorem 1.1 can be used as a sufficient condition ensuring stability of a given difference scheme (see [7] for such examples). However, the main application of Theorem 1.1 is that it can be used for obtaining various forms of stability criteria of the difference scheme (1.1). In particular, this method is used for proving equivalent theorems, Theorems 1.2 and 1.3 (the proof will be given in our forthcoming paper [2], see [9] for the case $p=\infty$). In these theorems, condition (1.2) is assumed to hold without special mention.

Theorem 1.2. *For any $p \in [1,\infty]$, the difference scheme* (1.1) *is stable in $L_p(h)$ if and only if condition* (1.8) *and inequality $A(q)<\infty$ hold (see* (1.11)*). In particular, one of the two following assertions holds:*

α) *the difference scheme* (1.1) *is stable in $L_p(h)$ for all $p\in[1,\infty]$;*

β) *for all $p\in[1,\infty]$ the difference scheme* (1.1) *is non-stable in $L_p(h)$.*

Moreover, in the case α) for any sequence $f\in L_p(h)$, $p\in[1,\infty]$ the solution $y\in L_p(h)$ of equation (1.1) *admits representation* (1.4).

Thus, we see that Theorem 1.1 is an assertion close to the unconditional criterion of Theorem 1.2. In particular, the "sufficiency part" of Theorem 1.2 is an immediate consequence of Theorem 1.1. In addition, Theorem 1.1 allows one to overcome all the technical difficulties arising in the proof of the "necessary" part of Theorem 1.2 (see [2]). Note that the conditions of Theorem 1.2 are more suited for studying conditions of schemes (1.1) of general form rather than for checking stability of concrete difference schemes ([7]). For applications, it is more convenient to use Theorem 1.3 where the meaning of condition (1.11) is finally clarified.

Theorem 1.3. *For any $p\in[1,\infty]$, the difference scheme* (1.1) *is stable in $L_p(h)$ if and only if there is an absolute positive constant c_0 such that $c_0h_0^{-1}\ge 1$ and $B>0$. Here*

$$B=\inf_{h\in(0,h_0]}\inf_{n\in z}\sum_{k=n-k_0(h)}^{n+k_0(h)} q_k(h)h, \qquad k_0(h)\overset{def}{=}\left[\frac{c_0}{h}\right]. \tag{1.14}$$

Thus, under condition (1.8), the complicated requirement (1.11) is equivalent to condition (1.14) which is simple and convenient for applications. For more details on (1.11), see [2].

Additional remarks. This work is based on statements from [6, 7]. It can be viewed as a natural continuation of [1, 3, 4, 5, 8, 12]. Here we strengthen and repeatedly use methods and tools from these works.

2. Preliminaries

Throughout the sequel we denote by c absolute positive constants whose values are not essential for exposition and which may differ even within a single chain of calculations. We also always assume that conditions (1.2) and (1.8) are satisfied.

Theorem 2.1. [6] *There is a FSS $\{u_n(h), v_n(h)\}_{n\in Z}$ of equation* (1.6) *such that for $n \in Z$, $h \in (0, h_0]$:*

$$0 < u_{n+1}(h) \le u_n(h), \qquad v_{n+1}(h) \ge v_n(h) > 0,$$
$$v_{n+1}(h)u_n(h) - u_{n+1}(h)v_n(h) = h, \qquad u_n(h) = v_n(h)\sum_{k=n}^{\infty}\frac{h}{v_k(h)v_{k+1}(h)}. \tag{2.1}$$

Moreover,

$$\lim_{n\to-\infty}\frac{v_n(h)}{u_n(h)} = 0, \quad \lim_{n\to\infty}\frac{u_n(h)}{v_n(h)} = 0.$$

Theorem 2.2. [6] *For $p \in [1,\infty)$, the inversion problem for* (1.1) *is regular in $L_p(h)$ if*

$$H < \infty, \quad H \overset{def}{=} \sup_{h\in(0,h_0]}\sup_{n\in Z}\sum_{m\in Z} G_{n,m}(h)h. \tag{2.2}$$

Theorem 2.3. [6] *Let $\rho_n(h) = u_n(h)v_n(h)$, $n \in Z$. For $n, m \in Z$ and $n \ne m$, the Green function $G_{n,m}(h)$ (see* (1.5)*) admits a representation of Davies-Harrell type (see* [10]*):*

$$G_{n,m}(h) = \begin{cases} \sqrt{\rho_n(h)\rho_m(h)}\prod_{k=n}^{m-1}\left[1+\frac{u_k(h)}{u_{k+1}(h)}\ \frac{h}{\rho_k(h)}\right]^{-1/2}, & n < m \\ \sqrt{\rho_n(h)\rho_m(h)}\prod_{k=m}^{n-1}\left[1+\frac{u_k(h)}{u_{k+1}(h)}\ \frac{h}{\rho_k(h)}\right]^{-1/2}, & n > m. \end{cases} \tag{2.3}$$

Theorem 2.4. [7] *For $n \in Z$, one has the following inequalities (see* (1.9) – (1.10)*):*

$$8^{-1}d_n(h) \le \rho_n(h) \le 16 d_n(h), \quad h \in (0, h_0]. \tag{2.4}$$

Note that technical assertions from [7] which are required for the proofs are given where needed.

3. Necessary conditions for regularity of the inversion problem

In this section, we prove the "necessary" part of Theorem 1.1. We need some auxiliary assertions.

Lemma 3.1. [1] *For every* $n \in Z,\ h \in (0, h_0]$, *the following inequalities hold (see* (1.9)*):*

$$\ell_n(h) - 1 \le \ell_{n+1}(h) \le \ell_n(h) + 1, \quad \ell_n(h) - 1 \le \ell_{n-1}(h) \le \ell_n(h) + 1. \tag{3.1}$$

Remark. Lemma 3.1 (as well as some other auxiliary assertions of this paper) was obtained in [1] under the assumption $q_n(h) \ge \varepsilon > 0$ for all $n \in Z,\ h \in (0, h_0]$. The paper [1] was never published in its detailed form, and Lemma 3.1 was not included in the text. In the sequel, references to [1] indicate that the corresponding assertions were used for the proof of results of [1].

Proof. The inequalities for $\ell_{n+1}(h)$ and $\ell_{n-1}(h)$ from (3.1) are verified in the same way; therefore, we only prove the estimates for $\ell_{n+1}(h)$. To prove the upper estimate from (3.1), we consider two separate cases: 1) $\ell_{n+1}(h) = 0$; 2) $\ell_{n+1}(h) \ne 0$.

1) If $\ell_{n+1}(h) = 0 \Rightarrow \ell_{n+1}(h) = 0 < 1 \le 1 + \ell_n(h)$ since $\ell_n(h) \ge 0$ for $n \in Z$.

2) Let $\ell_{n+1}(h) \ne 0$. Denote

$$P_j(n) = j \sum_{k=n-j}^{n+j} q_k(h)h^2, \quad j \ge 1,\ h \in (0, h_0]. \tag{3.2}$$

In (3.2), $j := j_0 = \ell_n(h) + 1$. Consider two separate cases: 2a) $\ell_n(h) = 0$; 2b) $\ell_n(h) \ne 0$.

2a) If $\ell_n(h) = 0$, then by (3.2) and (1.9) for $j = j_0 = 1$, we obtain

$$P_{j_0}(n+1) = \sum_{k=n+1-j_0}^{n+1+j_0} q_k(h)h^2 = \sum_{k=n}^{n+2} q_k(h)h^2 \ge q_n(h)h^2 \ge 1.$$

Therefore, $\ell_{n+1}(h) \le j_0 = \ell_n(h) + 1$.

2b) If $\ell_n(h) \ne 0$, then

$$P_{j_0}(n+1) = (\ell_n(h) + 1) \sum_{k=n-\ell_n(h)}^{n+\ell_n(h)+2} q_k(h)h^2 \ge \ell_n(h) \sum_{k=n-\ell_n(h)}^{n+\ell_n(h)} q_k(h)h^2 \ge 1,$$

and then $\ell_{n+1}(h) \le j_0 = \ell_n(h) + 1$ in view of (1.9), as required.

Let us now check the lower estimate from (3.1). If $\ell_n(h) \in \{0, 1\}$, then $\ell_n(h) - 1 \le 0 \le \ell_{n+1}(h)$ since $\ell_{n+1}(h) \ge 0$ for $n \in Z$, $h \in (0, h_0]$. For $\ell_n(h) \ge 2$, consider two separate cases: α) $\ell_n(h) > 2$; β) $\ell_n(h) = 2$.

α) If $\ell_n(h) > 2$, then for $j = j_0 = 2$, by (1.9) we obtain

$$P_{j_0}(n) = 2 \sum_{k=n-2}^{n+2} q_k(h)h^2 < 1 \Rightarrow q_{n+1}(h)h^2 < 1.$$

Hence $\ell_{n+1}(h)$ is defined by the lower row of (1.9). Let $j = j_0 = \ell_n(h) - 2$. Here $j_0 \geq 1$ and

$$P_{j_0}(n+1) = (\ell_n(h) - 2) \sum_{k=n-\ell_n(h)+3}^{n+\ell_n(h)-1} q_k(h)h^2 \leq (\ell_n(h) - 1) \sum_{k=n-\ell_n(h)+1}^{n+\ell_n(h)-1} q_k(h)h^2 < 1.$$

Hence $\ell_{n+1}(h) > \ell_n(h) - 2$ in view of (1.9), i.e., $\ell_{n+1}(h) \geq \ell_n(h) - 1$.

β) If $\ell_n(h) = 2$, from (1.9) it follows that

$$P_1(n) = \sum_{k=n-1}^{n+1} q_k(h)h^2 < 1 \Rightarrow q_{n+1}(h)h^2 < 1.$$

Hence, as above, $\ell_{n+1}(h) \geq 1$, i.e., $\ell_{n+1}(h) \geq 1 = 2 - 1 = \ell_n(h) - 1$. □

Lemma 3.2. [1] *For every $n \in Z$, the following inequalities hold:*

$$2^{-1}d_n(h) \leq d_m(h) \leq 3 \cdot 2^{-1}d_n(h) \quad \textit{if} \quad |m-n|h \leq 2^{-1}d_n(h), \; m \in Z. \tag{3.3}$$

Proof. Consider two separate cases: 1) $\ell_n(h) \geq 2$; 2) $\ell_n(h) < 2$.

1) If $\ell_n(h) \geq 2 \Rightarrow d_n(h) = \ell_n(h)h$, and the requirement on m in (3.3) is of the form

$$|m-n|h \leq 2^{-1}\ell_n(h)h \Rightarrow n - 2^{-1}\ell_n(h) \leq m \leq n + 2^{-1}\ell_n(h), \; m \in Z. \tag{3.4}$$

Let, for example, $m = n + k$, $0 \leq k \leq [2^{-1}\ell_n(h)]$. We use (3.1). Let us add inequalities (3.5) and (3.6) for $s = \overline{1,k}$:

$$\ell_{n+s}(h) \leq \ell_{n+s-1}(h) + 1, \quad s = 1, 2, \ldots, k \tag{3.5}$$

$$\ell_{n+s}(h) \geq \ell_{n+s-1}(h) - 1, \quad s = 1, 2, \ldots, k. \tag{3.6}$$

We obtain (3.7) and (3.8):

$$\ell_{n+k}(h) \leq \ell_n(h) + k \leq \ell_n(h) + [2^{-1}\ell_n(h)], \; k = 1, 2, \ldots, [2^{-1}\ell_n(h)], \tag{3.7}$$

$$\ell_{n+k}(h) \geq \ell_n(h) - k \geq \ell_n(h) - [2^{-1}\ell_n(h)], \; k = 1, 2, \ldots, [2^{-1}\ell_n(h)], \tag{3.8}$$

and, finally, (3.9) and (3.10):

$$\ell_n(h) - [2^{-1}\ell_n(h)] \leq \ell_m(h) \leq \ell_n(h) + [2^{-1}\ell_n(h)], \tag{3.9}$$
$$\text{for} \quad |m-n| \leq \frac{1}{2}\ell_n(h), \; m \in Z,$$

$$2^{-1}\ell_n(h) \leq \ell_m(h) \leq 3 \cdot 2^{-1}\ell_n(h), \; \text{for} \quad |m-n| \leq \frac{1}{2}\ell_n(h), \; m \in Z. \tag{3.10}$$

Since $\ell_n(h) \geq 2$, one has $\ell_m(h) \geq 1$, in view of (3.10) and $d_m(h) = \ell_m(h)h$ (see (1.9)). Therefore, (3.10) implies (3.3):

$$2^{-1}\ell_n(h)h = 2^{-1}d_n(h) \leq \ell_m(h)h = d_m(h) \leq 3. \cdot 2^{-1}\ell_n(h)h = 3 \cdot 2^{-1}d_n(h).$$

The case $m = n - k$, $k = \overline{0, [2^{-1}\ell_n(h)]}$ can be considered in a similar way.

2) Let $\ell_n(h) < 2$. Consider two separate cases: 2a) $\ell_n(h) = 1$; 2b) $\ell_n(h) = 0$.

2a) If $\ell_n(h) = 1$, then $d_n(h) = \ell_n(h)h = h$ (see (1.9)) $\Rightarrow |n - m|h \le 2^{-1}d_n(h) = 2^{-1}h \Rightarrow m = n$ and the assertion is obvious.

2b) If $\ell_n(h) = 0 \Rightarrow d_n(h) = h[1 + q_n(h)h^2]^{-1} \le \frac{h}{2}$ (see (1.9)). Hence, as in 2a), $m = n$; and the assertion is obvious.

□

Lemma 3.3. [7] *For $n \in Z$, $j \ge 1$, a FSS $\{u_n(h), v_n(h)\}_{n\in Z}$ of equation* (1.6) *satisfies the following equalities:*

$$j(v_{n+1}(h) - v_n(h)) = v_n(h) - v_{n-j}(h) + \sum_{s=0}^{j-1}\sum_{k=0}^{s} q_{n-k}(h)v_{n-k}(h)h^2, \tag{3.11}$$

$$j(u_{n-1}(h) - u_n(h)) = u_n(h) - u_{n+j}(h) + \sum_{s=0}^{j-1}\sum_{k=0}^{s} q_{n+k}(h)u_{n+k}(h)h^2. \tag{3.12}$$

Lemma 3.4. *A FSS $\{u_n(h), v_n(h)\}_{n\in Z}$ of* (1.6) *satisfies the following inequalities:*

$$j(v_{n+1}(h) - v_n(h)) \le v_n(h)[1 + j\sum_{k=n-j+1}^{n+j-1} q_k(h)h^2], \quad j \ge 1, \tag{3.13}$$

$$j(u_{n-1}(h) - u_n(h)) \le u_n(h)[1 + j\sum_{k=n-j+1}^{n+j-1} q_k(h)h^2], \quad j \ge 1. \tag{3.14}$$

Proof. From (2.1) and (3.11), it follows that

$$\begin{aligned} j(v_{n+1}(h) - v_n(h)) &\le v_n(h) + v_n(h)\sum_{s=0}^{j-1}\sum_{k=0}^{s} q_{n-k}(h)h^2 \\ &= v_n(h)[1 + jq_n(h)h^2 + (j-1)q_{n-1}(h)h^2 + \cdots + q_{n-j+1}(h)h^2] \\ &\le v_n(h)\left[1 + j\sum_{k=n-j+1}^{n+j-1} q_k(h)h^2\right]. \end{aligned}$$

The case (3.14) can be considered in a similar way.

□

Lemma 3.5. *Let $n \in Z$ and $\ell_n(h) \ge 1$. Then*

$$v_{n+1}(h) \le (1 + 4\ell_n(h)^{-1})v_n(h), \tag{3.15}$$

$$u_{n-1}(h) \le (1 + 4\ell_n(h)^{-1})u_n(h), \tag{3.16}$$

Proof. Both estimates are proved in the same way. Let us verify, say, (3.15). Consider two separate cases: 1) $\ell_n(h) \geq 2$; 2) $\ell_n(h) = 1$.

1) In (3.13), $j := \ell_n(h) - 1$. Then by (1.9) we get

$$\begin{aligned}(\ell_n(h)-1)[v_{n+1}(h) - v_n(h)] &\leq v_n(h)\left[1 + (\ell_n(h)-1)\sum_{k=n-\ell_n(h)+2}^{n+\ell_n(h)-2} q_k(h)h^2\right]\\ &\leq v_n(h)\left[1 + (\ell_n(h)-1)\sum_{k=n-\ell_n(h)+1}^{n+\ell_n(h)-1} q_k(h)h^2\right] \leq 2v_n(h).\end{aligned}$$

Since $\ell_n(h) - 1 \geq 2^{-1}\ell_n(h)$, this implies

$$2^{-1}\ell_n(h)[v_{n+1}(h) - v_n(h)] \leq 2v_n(h) \Rightarrow v_{n+1}(h) \leq (1 + 4\ell_n(h)^{-1})v_n(h).$$

2) Since $\ell_n(h) = 1$, one has $q_n(h)h^2 < 1$ (see (1.9)). Then from (2.1)) it follows that

$$v_{n+1}(h) - v_n(h) = v_n(h) - v_{n-1}(h) + q_n(h)h^2 v_n(h) \leq (1 + q_n(h)h^2)v_n(h) \leq 2v_n(h).$$

Therefore,

$$v_{n+1}(h) \leq 3v_n(h) = (1 + 2\ell_n(h)^{-1})v_n(h) \leq (1 + 4\ell_n(h)^{-1})v_n(h).$$

□

Lemma 3.6. *Let $n \in Z$. If $\ell_n(h) \geq 2$, then*

$$v_{n+[2^{-1}\ell_n(h)]+1}(h) \leq \exp(8)v_n(h), \tag{3.17}$$

$$u_{n-[2^{-1}\ell_n(h)]-1}(h) \leq \exp(8)u_n(h), \tag{3.18}$$

$$v_n(h) \leq \exp(8)v_{n-[2^{-1}\ell_n(h)]}(h), \tag{3.19}$$

$$u_n(h) \leq \exp(8)u_{n+[2^{-1}\ell_n(h)]}(h). \tag{3.20}$$

Proof. Inequalities (3.17) – (3.20) are checked in the same way. Let us prove, say, (3.17). Note that $\ell_k(h) \geq 1$ if

$$k = n, n+1, \ldots, n + [2^{-1}\ell_n(h)], \ \ \ell_n(h) \geq 2. \tag{3.21}$$

Indeed, Lemma 3.1 gives for such k

$$\begin{aligned}\ell_k(h) &\geq \min\{\ell_p(h) : p \in [n, n+1, \ldots, n + [2^{-1}\ell_n(h)]\} \geq \ell_n(h) - [2^{-1}\ell_n(h)]\\ &\geq 2[2^{-1}\ell_n(h)] - [2^{-1}\ell_n(h)] = [2^{-1}\ell_n(h)] \geq 1.\end{aligned}$$

In addition, $d_n(h) = \ell_n(h)h$ because $\ell_n(h) \ge 2$, and therefore from (3.3) it follows that

$$2^{-1}\ell_n(h) \le \ell_k(h) \le 3\cdot 2^{-1}\ell_n(h), \quad k = \overline{n-[2^{-1}\ell_n(h)], n+[2^{-1}\ell_n(h)]}. \tag{3.22}$$

For the same k as in (3.22), we obtain using (3.21), (3.15) and (3.22) that

$$v_{k+1}(h) \le (1+4\ell_k^{-1}(h))v_k(h) \le (1+8\ell_n^{-1}(h))v_k(h). \tag{3.23}$$

Let us multiply inequalities (3.22) over $k = n, n+1, \dots, n+[2^{-1}\ell_n(h)]$:

$$\begin{aligned} v_{n+[2^{-1}\ell_n(h)]+1}(h) &\le (1+8\ell_n(h)^{-1})^{[2^{-1}\ell_n(h)]+1}v_n(h) \\ &\le (1+8\ell_n(h)^{-1})^{2^{-1}\ell_n(h)+1}v_n(h) \le (1+8\ell_n(h)^{-1})^{\ell_n(h)}v_n(h) \\ &= \left[(1+8\ell_n(h)^{-1})^{8^{-1}\ell_n(h)}\right]^8 v_n(h) \le \exp(8)v_n(h). \end{aligned}$$

□

Corollary 3.6.1. *Let $n \in Z$. If $\ell_n(h) \ge 2$, then*

$$\alpha^{-1}v_n(h) \le v_k(h) \le \alpha v_n(h),\ k = \overline{n-[2^{-1}\ell_n(h)], n+[2^{-1}\ell_n(h)]},\ \alpha = \exp(8), \tag{3.24}$$

$$\alpha^{-1}u_n(h) \le u_k(h) \le \alpha u_n(h),\ k = \overline{n-[2^{-1}\ell_n(h)], n+[2^{-1}\ell_n(h)]}. \tag{3.25}$$

Proof. Estimates (3.24) – (3.25) follow from Lemma 3.6 and (2.1). □

Proof of Theorem 1.1. Necessity. We have to show that if I) – III) hold, then $A(q) < \infty$ (see §1). Let

$$\begin{aligned} A_0(h) &= \sup_{n\in Z}\{d_n(h) : \ell_n(h) = 0\}, \quad h \in (0, h_0] \\ A_1(h) &= \sup_{n\in Z}\{d_n(h) : \ell_n(h) = 1\}, \quad h \in (0, h_0] \\ A_2(h) &= \sup_{n\in Z}\{d_n(h) : \ell_n(h) \ge 2\}, \quad h \in (0, h_0]. \end{aligned}$$

Since for $\ell_n(h) = 0$ and $\ell_n(h) = 1$ we have, respectively,

$$d_n(h) = h[1+q_n(h)h^2]^{-1} \le h \le h_0; \quad d_n(h) = h \le h_0,$$

it follows that $\max\{A_0(h), A_1(h)\} \le h_0$ for $h \in (0, h_0]$ and $A(q) < \infty$ if $\tilde A(q) < \infty$, $\tilde A(q) \overset{\text{def}}{=} \sup_{h\in(0,h_0]} A_2(h)$.

Let us show that $\tilde A(q) < \infty$, provided I) – III) hold. Fix $h \in (0, h_0]$, and let $Z_2(h) = \{n \in Z : \ell_n(h) \ge 2\}$. For $n \in Z_2(h)$ set $f^{(n)} = \{f_i^{(n)}\}_{i\in Z}$

$$f_i^{(n)} = \begin{cases} 1, & \text{if} \quad |i-n| \le [2^{-1}\ell_n(h)] \\ 0, & \text{if} \quad |i-n| > [2^{-1}\ell_n(h)]. \end{cases} \tag{3.26}$$

Since I) – III) hold, equation (1.1) with (3.26) on the right-hand side has a unique solution $y^{(n)} = \{y_i^{(n)}(h)\}_{i\in Z}$ in $L_p(h)$, and it is determined by formula (1.4), i.e.,

$$y_i^{(n)}(h) = u_i(h)\sum_{m=-\infty}^{i} v_m(h) f_m^{(n)}(h)h + v_i(h)\sum_{i=m+1}^{\infty} u_m(h) f_m^{(n)}(h)h,\ i\in Z. \tag{3.27}$$

Let $i \in \overline{n-[2^{-1}\ell_n(h)], n+[2^{-1}\ell_n(h)]}$. In the following estimate we use (3.27), (3.26), (3.24)-(3.25), (2.4) and (1.10):

$$\begin{aligned} y_i^{(n)}(h) &\ge u_i(h)\sum_{m=n-[2^{-1}\ell_n(h)]}^{i} v_m(h) f_m^{(n)}(h)h + v_i(h)\sum_{m=i+1}^{n+[2^{-1}\ell_n(h)]} u_m(h) f_m^{(n)}(h)h \\ &= u_i(h)\sum_{m=n-[2^{-1}\ell_n(h)]}^{i} v_m(h)h + v_i(h)\sum_{m=i+1}^{n+[2^{-1}\ell_n(n)]} u_m(h)h \\ &\ge c^{-1}u_n(h)v_n(h)\sum_{m=n-[2^{-1}\ell_n(h)]}^{n+[2^{-1}\ell_n(h)]} h = c^{-1}\rho_n(h)(2[2^{-1}\ell_n(h)]+1)h \\ &\ge c^{-1}d_n(h)(\ell_n(h)h) = c^{-1}d_n(h)^2. \end{aligned} \tag{3.28}$$

Below we use (1.3), (3.28) and (3.26):

$$\begin{aligned} \infty &> \sup_{h\in(0,h_0]} \|G\|^p_{L_p(h)\to L_p(h)} = \sup_{h\in(0,h_0]}\sup_{0\neq f\in L_p(h)} \frac{\|Gf\|^p_{L_p(h)}}{\|f\|^p_{L_p(h)}} \\ &\ge \sup_{h\in(0,h_0]}\sup_{n\in Z_2(h)} \frac{\|y^{(n)}\|^p_{L_p(h)}}{\|f^{(n)}\|^p_{L_p(h)}} = \sup_{h\in(0,h_0]}\sup_{n\in Z_2(h)} \frac{\sum_{i=-\infty}^{\infty}|y_i^{(n)}(h)|^p\cdot h}{(2[2^{-1}\ell_n(h)]+1)h} \\ &\ge \sup_{h\in(0,h_0]}\sup_{n\in Z_2(h)} \frac{1}{2[2^{-1}\ell_n(h)]+1}\sum_{i=n-[2^{-1}\ell_n(h)]}^{n+[2^{-1}\ell_n(n)]} |y_i^{(n)}(h)|^p \\ &\ge c^{-1}\sup_{h\in(0,h_0]}\sup_{n\in Z_2(h)} \frac{d_n(h)^{2p}}{2[2^{-1}\ell_n(h)]+1}\sum_{i=n-[2^{-1}\ell_n(h)]}^{n+[2^{-1}\ell_n(h)]} 1 = c^{-1}\sup_{h\in(0,h_0]}\sup_{n\in Z_2(h)} d_n^{2p}(h). \end{aligned}$$

□

4. Sufficient conditions for regularity of the inversion problem

In this section, we prove the "sufficient" part of Theorem 1.1. We need some auxiliary assertions.

Lemma 4.1. *Consider the function*

$$\psi_n(h) = n - [2^{-1}\ell_n(h)], \quad n\in Z,\ h\in(0,h_0]. \tag{4.1}$$

For $n \in Z$, $h \in (0, h_0]$ one has

$$\psi_n(h) \le \psi_{n+1}(h) \le \psi_n(h) + 2, \tag{4.2}$$

$$\psi_n(h) - 2 \le \psi_{n-1}(h) \le \psi_n(h). \tag{4.3}$$

Proof. From (3.1) for $n \in Z$, $h \in (0, h_0]$, it follows that

$$[2^{-1}(\ell_n(h) - 1)] \le [2^{-1}\ell_{n+1}(h)] \le [2^{-1}(\ell_n(h) + 1)], \tag{4.4}$$

$$[2^{-1}(\ell_n(h) - 1)] \le [2^{-1}\ell_{n-1}(h)] \le [2^{-1}(\ell_n(h) + 1)]. \tag{4.5}$$

For $k = 0, 1, 2, \dots,$ we consider two separate cases: 1) $\ell_n(h) = 2k$; 2) $\ell_n(h) = 2k + 1$.

1) If $\ell_n(h) = 2k$ then $\psi_n(h) = n - k$, $[2^{-1}(\ell_n(h) - 1)] = k - 1$, $[2^{-1}(\ell_n(h) + 1)] = k$. Together with (4.4) – (4.5), this implies (4.6), and (4.6) implies (4.7):

$$[2^{-1}\ell_n(h)] - 1 = k - 1 = [2^{-1}(\ell_n(h) - 1)] \le [2^{-1}\ell_{n\pm1}(h)] \le [2^{-1}(\ell_n(h) + 1)] = k, \tag{4.6}$$

$$[2^{-1}\ell_n(h)] - 1 \le [2^{-1}\ell_{n\pm1}(h)] \le [2^{-1}\ell_n(h)]. \tag{4.7}$$

2) In this case $\psi_n(h) = n - k$, $[2^{-1}(\ell_n(h) - 1)] = k$, $[2^{-1}(\ell_n(h) + 1)] = k + 1$. Together with (4.4) – (4.5), this implies (4.8):

$$\begin{aligned} [2^{-1}\ell_n(h)] - 1 = k - 1 \le k = [2^{-1}(\ell_n(h) - 1)] \le [2^{-1}\ell_{n\pm1}(h)] \\ \le [2^{-1}(\ell_n(h) + 1)] = k + 1 = [2^{-1}\ell_n(h)] + 1. \end{aligned} \tag{4.8}$$

We thus have obtained the common conclusion of 1) – 2):

$$[2^{-1}\ell_n(h)] - 1 \le [2^{-1}\ell_{n\pm1}(h)] \le [2^{-1}\ell_n(h)] + 1, \ n \in Z, \ h \in (0, h_0]. \tag{4.9}$$

From (4.9), for $n \in Z$, $h \in (0, h_0]$, we derive (4.2) and (4.3):

$$\begin{aligned} \psi_n(h) + 2 = n - [2^{-1}\ell_n(h)] + 2 = (n+1) - ([2^{-1}\ell_n(h)] - 1) \ge n + 1 - [2^{-1}\ell_{n+1}(h)] \\ = \psi_{n+1}(h) \ge n + 1 - ([2^{-1}\ell_n(h)] + 1) = n - [2^{-1}\ell_n(h)] = \psi_n(h), \end{aligned}$$

$$\begin{aligned} \psi_n(h) = n - [2^{-1}\ell_n(h)] = (n-1) - ([2^{-1}\ell_n(h)] - 1) \ge n - 1 - [2^{-1}\ell_{n-1}(h)] \\ = \psi_{n-1}(h) \ge n - 1 - ([2^{-1}\ell_n(h)] + 1) = n - [2^{-1}\ell_n(h)] - 2 = \psi_n(h) - 2. \end{aligned}$$

□

Lemma 4.2. *For every $j \in Z$ there is $n \in Z$ such that*

$$j - 1 \le \psi_n(h) \le j. \tag{4.10}$$

Proof. Consider the function $m_j(h)$:

$$m_j(h) = \inf_{n \geq j} \{n : n \in Z,\ \psi_n(h) \geq j\}, \quad h \in (0, h_0]. \tag{4.11}$$

Let us verify that $m_j(h)$ is well defined, i.e., for every $j \in Z$ there exists $n \in Z$ such that $n \geq j$ and $\psi_n(h) \geq j$. Assume the contrary. Then there is $j_0 \in Z$ such that $\psi_n(h) < j_0$ for any $n \geq j_0$. If for $n \geq j_0$ the value $\ell_n(h)$ is bounded, this is impossible because in this case $\psi_n(h) \to \infty$ as $n \to \infty$. Let now $\ell_n(h) \to \infty$ as $n \to \infty$. Then for $n \gg 1$, (1.9) implies

$$(\ell_n(h) - 1) \sum_{k=n-\ell_n(h)+1}^{n+\ell_n(h)-1} q_k(h) h^2 < 1. \tag{4.12}$$

But $n - \ell_n(h) + 1 \leq n - [2^{-1}\ell_n(h)] \leq j_0$ and therefore from (4.12), it follows that

$$(\ell_n(h) - 1) \sum_{k=j_0}^{n+\ell_n(h)-1} q_k(h) h^2 < 1. \tag{4.13}$$

Then the condition $\ell_n(h) \to \infty$ as $n \to \infty$, and requirement (1.2) and (1.8) contradicts (4.13), as required. Let us now return to (4.10). Consider three separate cases: 1) $\psi_{m_j(h)}(h) = j$; 2) $\psi_{m_j(h)}(h) = j + 1$, 3) $\psi_{m_j(h)}(h) \geq j + 2$. Clearly, in the case 1) we have (4.10), and the lemma is proved. Let us verify that case 3) cannot occur. Indeed, if $\psi_{m_j(h)}(h) \geq j + 2$ then by Lemma 4.1 we get

$$\psi_{m_j(h)-1}(h) \geq \psi_{m_j(h)}(h) - 2 \geq j + 2 - 2 = j.$$

On the other hand, by definition (4.11) we have $\psi_{m_j(h)-1}(h) \leq j - 1$, contradiction. We now turn to case 2): $\psi_{m_j(h)}(h) = j + 1$. By Lemma 4.1, $\psi_{m_j(h)-1}(h) \geq \psi_{m_j(h)}(h) - 2 = j + 1 - 2 = j - 1$. Using definition (4.11) once again, we get $\psi_{m_j(h)-1}(h) \leq j - 1$. Hence $\psi_{m_j(h)-1} = j - 1$, and the lemma is proved. □

Throughout the sequel we denote $Z' = Z \setminus 0 = \{\pm 1, \pm 2, \dots\}$, $[m, p] = \{m, m+1, \dots, p\}$ for $m < p$; $[m, p] = \{m\}$ for $m = p$, and $m, p \in Z$. The sets $[m, p]$, $m \leq p$ will be called *segments*.

Definition 4.1. [1] Let $n \in Z$ be given. A system of segments $\Delta_s = [\Delta_s^-, \Delta_s^+]$, $\Delta_s^- \leq \Delta_s^+, \Delta_s^-$ and $\Delta_s^+ \in Z$, $s \in Z'$ is called a $Z(n)$-covering of Z if the following conditions hold:

1) $\Delta_s \cap \Delta_{s'} = \emptyset$ for $s \neq s'$,
2) $\bigcup_{s=-\infty}^{-1} \Delta_s = (\dots, n-2, n-1]$, $\quad \bigcup_{s=1}^{\infty} \Delta_s = [n+1, n+2, \dots)$.

Remark. The segments of a $Z(n)$-covering do not contain the point n.

Lemma 4.3. *For every $n \in Z$ there exists a sequence $\{k_s\}_{s\in Z'}$ such that one can form a $Z(n)$-covering of Z from the segments $\{\tilde\Delta_s, \tilde\Delta'_s\}_{s\in Z'}$ where*

1) $\tilde\Delta_s = \tilde\Delta'_s = [k_s, k_s] = \{k_s\}$ *if* $\ell_{k_s}(h) \in \{0,1\}$,
2) $\tilde\Delta_s = [k_s - [2^{-1}\ell_{k_s}(h)] + 1,\ k_s + [2^{-1}\ell_{k_s}(h)]]$ *if* $\ell_{k_s}(h) \ge 2$
   $\tilde\Delta'_s = [k_s - [2^{-1}\ell_{k_s}(h)],\ k_s + [2^{-1}\ell_{k_s}(h)]]$ *if* $\ell_{k_s}(h) \ge 2$.

Proof. Let us contruct the segments $\Delta_1, \Delta_2, \dots$ of a $Z(n)$-covering of Z. (The segments $\Delta_{-1}, \Delta_{-2}, \dots$ can be constructed in a similar way). Let $n \in Z$ be given. Consider three separate cases: 1) $\ell_{n+1}(h) = 0$; 2) $\ell_{n+1}(h) = 1$; 3) $\ell_{n+1}(h) \ge 2$.

1) If $\ell_{n+1}(h) = 0$ then $k_1 := n + 1$, $\Delta_1 := k_1$.
2) If $\ell_{n+1}(h) = 1$ then $k_1 := n + 1$, $\Delta_1 := k_1$.
3) If $\ell_{n+1}(h) \ge 2$ then by Lemma 4.2, there is a point $k_1 \in Z$ such that $k_1 \ge n$ and $n \le \psi_{k_1}(h) = k_1 - [2^{-1}\ell_{k_1}(h)] \le n + 1$.

Then, if $\psi_{k_1}(h) = n$, we set $\Delta_{k_1} := k_1 - [2^{-1}\ell_{k_1}(h)] + 1,\ k_1 + [2^{-1}\ell_{k_1}(h)]]$. If $\psi_{k_1}(h) = n + 1$, we set $\Delta_{k_1} := [k_1 - [2^{-1}\ell_{k_1}(h)], k_1 + [2^{-1}\ell_{k_1}(h)]]$.

Thus we have constructed the first term of the sequence $\{k_s\}_{s=1}^\infty$ (the number k_1) and the first segment Δ_1 of a $Z(n)$-covering of Z. Suppose that k_m and Δ_m, $m \ge 1$ are constructed. In order to construct k_{m+1} and Δ_{m+1}, one then has to make the same operations with the number Δ_m^+ as we made with the number n when constructing k_1 and Δ_1, etc. □

Remark. Assertions similar to Lemmas 4.1 – 4.3 were used in [1].

Lemma 4.4. *Let the segments $\{\Delta_s\}_{s\in Z'}$ form a $Z(n)$-covering of Z. Then for every $s \in Z'$ one has*

$$T_s(h) \overset{def}{=} \prod_{k\in\Delta_s}\left(1 + \frac{u_k(h)}{u_{k+1}(h)}\quad\frac{h}{\rho_k(h)}\right) \ge \gamma^{-1},\ \gamma^{-1} = \frac{50}{49}. \tag{4.14}$$

Proof. According to Lemma 4.3, we consider three separate cases: 1) $\ell_{k_s}(h) = 0$; 2) $\ell_{k_s}(h) = 1$; 3) $\ell_{k_s}(h) \ge 2$.

1) If $\ell_{k_s}(h) = 0$, then by definition (4.14), (2.1), (1.10) and (2.4), we get

$$T_s(h) = 1 + \frac{u_{k_s}(h)}{u_{k_s+1}(h)}\ \frac{h}{\rho_{k_s}(h)} \ge 1 + \frac{h}{\rho_{k_s}(h)} \ge 1 + \frac{h}{16 d_{k_s}(h)} = 1 + \frac{1 + q_{k_s}(h)h^2}{16} \ge \frac{9}{8} \ge \frac{50}{49}.$$

2) If $\ell_{k_s}(h) = 1$, then we proceed as in 1) and obtain

$$T_s(h) = 1 + \frac{u_{k_s}(h)}{u_{k_s+1}(h)}\ \frac{h}{\rho_{k_s}(h)} \ge 1 + \frac{h}{16\ell_{k_s}(h)h} = \frac{17}{16} > \frac{50}{49}.$$

3) If $\ell_{k_s}(h) \geq 2$, then we proceed as in 1) and use, in addition, Lemma 3.2. We get

$$\begin{aligned} T_s(h) &= \prod_{k\in\Delta_s}\left(1+\frac{u_k(h)}{u_{k+1}(h)}\,\frac{h}{\rho_k(h)}\right) \geq \prod_{k\in\Delta_s}\left(1+\frac{h}{\rho_k(h)}\right) \geq \prod_{k\in\Delta_s}\left(1+\frac{h}{16d_k(h)}\right) \\ &\geq \prod_{k\in\Delta_s}\left(1+\frac{h}{16}\cdot\frac{2}{3}\cdot\frac{1}{d_{k_s}(h)}\right) \geq \left(1+\frac{h}{24d_{k_s}(h)}\right)^{2[2^{-1}\ell_{k_s}(h)]} \\ &\geq \left(1+\frac{1}{24\ell_{k_s}(h)}\right)^{\ell_{k_s}(h)-1} \geq \frac{48}{49}\left(1+\frac{1}{24\ell_{k_s}(h)}\right)^{\ell_{k_s}(h)} \\ &\geq \frac{48}{49}\left(1+\frac{\ell_{k_s}(h)}{24\ell_{k_s}(h)}\right) = \frac{50}{49}. \end{aligned}$$

□

Corollary 4.4.1. *Let $A(q) < \infty$ (see (1.11), $\gamma = \frac{49}{50}$, and suppose that for some $n \in Z$ the segments $\{\Delta_s\}_{s\in Z}$ form a $Z(n)$-covering of Z. Then (see (1.5) and (2.3))*

$$G_{n,\Delta_s^-}(h) \leq 16A(q)\gamma^{\frac{s-1}{2}}, \quad s = 1, 2, \ldots \tag{4.15}$$

$$G_{n,\Delta_s^+}(h) \leq 16A(q)\gamma^{\frac{|s|-1}{2}}, \quad s = -1, -2, \ldots \tag{4.16}$$

Proof. Both inequalities are proved in the same way. Let us establish, say, (4.15). Consider two cases: 1) $s = 1$; 2) $s \geq 2$.

1) For $s = 1$ we get

$$G_{n,\Delta_1^-}(h) = v_n(h)u_{n+1}(h) \leq v_n(h)u_n(h) = \rho_n(h) \leq 16A(q).$$

2) If $s \geq 2$, then $\Delta_s^- \geq n+1$, and by (2.3) and (4.14) we get

$$\begin{aligned} G_{n,\Delta_s^-}(h) &= \sqrt{\rho_n(h)\rho_{\Delta_s^-}(h)}\prod_{k=n}^{\Delta_s^- -1}\left[1+\frac{u_k(h)}{u_{k+1}(h)}\,\frac{h}{\rho_k(h)}\right]^{-1/2} \\ &\leq 16A(q)\prod_{i=1}^{s-1}\prod_{k\in\Delta_i}\left[1+\frac{u_k(h)}{u_{k+1}(h)}\,\frac{h}{\rho_k(h)}\right]^{-1/2} = 16A(q)\prod_{i=1}^{s-1}T_i(h)^{-1/2} \\ &\leq 16A(q)\gamma^{\frac{s-1}{2}}. \end{aligned}$$

□

Proof of Theorem 1.1. Sufficiency. It suffices to show that (1.11)) implies (2.2). For fixed $n \in Z$ and $h \in (0, h_0]$, we consider the sum

$$H_n(h) = \sum_{m=-\infty}^{\infty} G_{n,m}(h)h = \sum_{m=-\infty}^{n-1} G_{n,m}(h)h + G_{n,n}(h)h + \sum_{m=n+1}^{\infty} G_{n,m}(h)h. \tag{4.17}$$

By (2.4) and (1.11), we get

$$G_{n,n}(h) = \rho_n(h)h \le 16d_n(h)h \le 16A(q)h_0,$$

and therefore it remains to estimate the sums in (4.17). The estimates for both sums are proved in the same way, so we only consider the second sum. Below we use a $Z(n)$-covering of Z, (4.15), (2.3), (2.1) and (1.10):

$$\sum_{m=n+1}^{\infty} G_{n,m}(h)h = \sum_{s=1}^{\infty}\sum_{k\in\Delta_s} G_{n,k}(h)h \le \sum_{s=1}^{\infty} G_{n,\Delta_s^-}(h)\left(\sum_{k\in\Delta_s} 1\right)h$$
$$\le cA(q)\sum_{s=1}^{\infty}(\ell_s(h)+1)h\gamma^{\frac{s-1}{2}} \le cA(q)[A(q)+h_0].$$

□

5. Consequences of the main result

In this section, we prove Corollary 1.1.1.

Proof of Corollary 1.1.1. We need the following lemma.

Lemma 5.1. [6] *One has*

$$\sum_{m\in Z} q_m(h)G_{n,m}(h)h \le 1, \quad n\in Z. \tag{5.1}$$

Suppose that assertions I) – III) from §1 hold. Let $f\in L_p(h)$, $p\in[1,\infty)$, and let y be the solution of (1.1) (see (1.4)). In the proof of (1.10) presented below, we use Hölder's inequality, (5.1), symmetry of $G_{n,m}(h)$ and a well-known theorem on changing the order of summation for multiple series [13, Ch.I, §6.2]:

$$\begin{aligned}\|q^{1/p}y\|_{L_p(h)} &= \left(\sum_{n\in Z} q_n(h)|y_n(h)|^p h\right)^{1/p}\\ &= \left[\sum_{n\in Z} q_n(h)\left|\sum_{m\in Z} G_{n,m}(h)f_m(h)h\right|^p h\right]^{1/p}\\ &\le \left\{\sum_{n\in Z} q_n(h)\left(\sum_{m\in Z} G_{n,m}(h)h\right)^{p/p'}\left(\sum_{m\in Z} G_{n,m}(h)|f_m(h)|^p h\right)h\right\}^{1/p}\\ &\le cA(q)^{1/p'}\left[\sum_{m\in Z}|f_m(h)|^p\left(\sum_{n\in Z} q_n(h)G_{n,m}(h)h\right)h\right]^{1/p}\\ &\le cA(q)^{1/p'}\|f\|_{L_p(h)}.\end{aligned}$$

Estimate (1.13) then follows from the triangle inequality. □

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