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A Combinatoric Proof and Generalization of Ferguson's Formula for k-generalized Fibonacci Numbers

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Various generalizations of the Fibonacci numbers have been proposed, studied and applied over the years (see [5] for a brief list). Probably the best known are the k-generalized Fibonacci numbers $F_n^{(k)}$ (also known as the k-fold Fibonacci, k-th order Fibonacci, k-Fibonacci or polynacci numbers), satisfying

$$
\begin{aligned}
F_n^{(k)} &= F_{n-1}^{(k)} + F_{n-2}^{(k)} + \ldots + F_{n-k}^{(k)} , \qquad n \geq k , \\
F_n^{(k)} &= 0 , \qquad 0 \leq n \leq k-2 , \\
F_{k-1}^{(k)} &= 1 .
\end{aligned}
$$

An exhaustive bibliography of papers on the k-generalized Fibonacci numbers would cover pages, so we just give a few references. The paper of Miles [9] seems to be the oldest well-known paper on the subject, though Knuth [6] (section 5.4.2) cites a work of Schlegel [13] dating from 1894. Numerous interesting results can be found in the pages of the Fibonacci Quarterly, see for example [2, 3, 7]. There are significant applications in computer science [6] and probability theory [11, 4, 8], the latter of which will be important for our purposes. Also much is known about "weighted" k-generalized Fibonacci numbers, with different coefficients in the recursion relation, see for example [1].

In this paper we look at a particular case of weighted k-generalized Fibonacci numbers, which we call the (k,p)-generalized Fibonacci numbers. We define these by

$$F_n^{(k,p)} = \frac{1-p}{p}\left(F_{n-1}^{(k,p)} + F_{n-2}^{(k,p)} + \ldots + F_{n-k}^{(k,p)}\right) , \qquad n \geq k , \tag{1}$$

$$F_n^{(k,p)} = 0 , \qquad 0 \leq n \leq k-2 , \tag{2}$$

$$F_{k-1}^{(k,p)} = \frac{p}{1-p} . \tag{3}$$

Here $k \geq 2$ is an integer and p is a real with $0 < p < 1$. For $p = \frac{1}{2}$ we recover the k-generalized Fibonacci numbers. The reader will have no trouble checking that

$$F_{k+n}^{(k,p)} = p^{-n} , \qquad n = 0, 1, \ldots k-1 . \tag{4}$$

For fixed k, p we have a representation

$$F_n^{(k,p)} = \sum_{i=1}^{k} C_i \lambda_i^n , \tag{5}$$

where $\lambda_1, \ldots, \lambda_k$ are the roots (presumed distinct) of

$$\lambda^k = \frac{1-p}{p}(\lambda^{k-1} + \lambda^{k-2} + \ldots + \lambda + 1) ,$$

and $C_1, \ldots, C_k$ are chosen to satisfy the initial conditions (2)-(3). The behavior of the λ_i for general p and sufficiently large k is similar to the behavior for $p = \frac{1}{2}$ described in [9]: there is a real root, which we will call λ_1, between $\lambda = 1$ and $\lambda = \frac{1}{p}$, and the remaining roots are all inside the circle $|\lambda| = 1$. The contribution from λ_1 dominates the sum (5), and in particular $\lim_{n\to\infty} \left(F_{n+1}^{(k,p)} / F_n^{(k,p)} \right) = \lambda_1$. As k increases, λ_1 converges rapidly to $\frac{1}{p}$.

The focus of this paper will be on an interesting formula for the $F_n^{(k,p)}$, very different from the Binet-type representation (5). In the case of the standard k-generalized Fibonacci numbers ($p = \frac{1}{2}$), and in particular in the case of the standard Fibonacci numbers ($p = \frac{1}{2}, k = 2$), our formula reduces to an expression found by Ferguson [2]. Ferguson proved his result using generating functions; we offer a combinatoric proof, which highlights the importance of the the (k, p)-generalized Fibonacci numbers in success run problems. The combinatoric proof nicely explains why $F_n^{(k,p)}$ depends on the greatest integer not exceeding $(n-k+1)/(k+1)$. This integer is simply the maximum possible number of distinct runs of k successes in a sequence of $n-k$ trials.

Notation. Let $H(x)$ denote the Heaviside function defined by

$$H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Theorem. For $n \geq k$

$$F_n^{(k,p)} = \sum_{r=0}^{\lfloor \frac{n-k+1}{k+1} \rfloor} (-1)^r p^{k(r+1)-n} (1-p)^{r-1} \left[(1-p) \binom{n-k(r+1)}{r} + \binom{n-k(r+1)}{r-1} \right] . \tag{6}$$

Here we understand that for any non-negative integer N, $\binom{N}{-1} = 0$, and for any real x, $\lfloor x \rfloor$ denotes the largest integer not exceeding x.

Proof. Let $P_n^{(k,p)}$ denote the probability that there is at least one run of at least k successes in a sequence of n identical Bernoulli trials, each having probability p of success. The possible outcomes of the trials can be partitioned as follows: either the first trial results in failure, or the first trial results in success and the second in failure, or the first two in success and the third in failure, ..., or the first $k-1$ result in success and the kth in failure, or the first k all result in success. Thus, using the law of total probability we have

$$P_n^{(k,p)} = q(P_{n-1}^{(k,p)} + pP_{n-2}^{(k,p)} + p^2 P_{n-3}^{(k,p)} + \ldots + p^{k-1} P_{n-k}^{(k,p)}) + p^k \ , \qquad n \geq k \ , \tag{7}$$

where we have written $q = 1 - p$. The initial conditions for this recursion are $P_n^{(k,p)} = 0$, $0 \leq n \leq k-1$. If we write $P_n^{(k,p)} = 1 - p^n R_n^{(k,p)}$ we find

$$R_n^{(k,p)} = \frac{q}{p}(R_{n-1}^{(k,p)} + R_{n-2}^{(k,p)} + \ldots + R_{n-k}^{(k,p)}) \ , \qquad n \geq k \ ,$$

with initial conditions $R_n^{(k,p)} = p^{-n}$, $0 \leq n \leq k-1$. Comparing with (1) and (4) we deduce that $R_n^{(k,p)} = F_{k+n}^{(k,p)}$, or

$$F_n^{(k,p)} = p^{k-n}(1 - P_{n-k}^{(k,p)}) \ , \qquad n \geq k \ . \tag{8}$$

We now consider computing $P_n^{(k,p)}$ using the inclusion–exclusion principle. Let A_i, $i = 1, 2, \ldots, n-k+1$, denote the event that there is a *minimal* run of k successes starting on the ith trial. By "minimal" we mean that there is a run of precisely k successes starting on the ith trial. Thus for $i = 1, 2, \ldots, n-k$, being in A_i means that trials $i, i+1, \ldots, i+k-1$ result in success, but trial $i+k$ results in failure (so in particular, the event that the first $k+1$ trials result in success but trial $k+2$ results in failure is contained in A_2 but not A_1). For $i = n-k+1$, being in A_i means just that the last k trials (trials $n-k+1, n-k+2, \ldots, n$) result in success. With this definition of the events A_i, $P_n^{(k,p)}$ is simply the probability of their union. So applying the inclusion–exclusion principle we have

$$\begin{aligned} P_n^{(k,p)} &= \sum_i \mathbf{P}(A_i) - \sum_{i<j} \mathbf{P}(A_i \cap A_j) + \sum_{i<j<l} \mathbf{P}(A_i \cap A_j \cap A_l) \\ &\quad + \ldots + (-1)^{r-1} \sum_{i_1<i_2<\ldots<i_r} \mathbf{P}(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_r}) + \ldots \ . \end{aligned}$$

Computing the first term is straightforward. We have

$$\mathbf{P}(A_i) = \begin{cases} p^k q & 1 \leq i \leq n-k \\ p^k & i = n-k+1 \end{cases} \ ,$$

and so

$$\sum_i \mathbf{P}(A_i) = \left(p^k q(n-k) + p^k\right) H(n-k) \ . \tag{9}$$

The factor $H(n-k)$ here reflects the fact that unless $n \geq k$ we cannot have a success run. Moving to the second term, we can only have a pair of success runs if $n \geq 2k+1$. Because of our choice of A_i as the event that there is a minimal success run starting at the ith trial, A_i and A_j are mutually exclusive if $|j-i| \leq k$, and in full generality we have

$$\mathbf{P}(A_i \cap A_j) = \begin{cases} 0 & |j-i| \leq n \\ (p^k q)^2 & 1 \leq i,\ j > i+k,\ j < n-k+1 \\ p^{2k} q & 1 \leq i,\ j > i+k,\ j = n-k+1 \end{cases} .$$

Thus

$$\begin{aligned} \sum_{i<j} \mathbf{P}(A_i \cap A_j) &= \sum_{i=1}^{n-2k-1} \sum_{j=i+k+1}^{n-k} p^{2k} q^2 + \sum_{i=1}^{n-2k} p^{2k} q \\ &= \left(\frac{1}{2} p^{2k} q^2 (n-2k)(n-2k-1) + p^{2k} q (n-2k) \right) H(n-2k-1) . \end{aligned}$$

The third term in the inclusion–exclusion principle is

$$\begin{aligned} \sum_{i<j<l} \mathbf{P}(A_i \cap A_j \cap A_l) &= \sum_{i=1}^{n-3k-2} \sum_{j=i+k+1}^{n-2k-1} \sum_{l=j+k+1}^{n-k} p^{3k} q^3 + \sum_{i=1}^{n-3k-1} \sum_{j=i+k+1}^{n-2k} p^{3k} q^2 \\ &= \left(\frac{1}{6} p^{3k} q^3 (n-3k)(n-3k-1)(n-3k-2) \right. \\ &\quad \left. + \frac{1}{2} p^{3k} q^2 (n-3k)(n-3k-1) \right) H(n-3k-2) . \end{aligned}$$

A pattern is clearly emerging. For arbitrary $r \geq 1$ we have

$$\begin{aligned} \sum_{i_1<i_2<\ldots<i_r} \mathbf{P}(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_r}) &= \sum_{i_1=1}^{n+1-r(k+1)} \sum_{i_2=i_1+(k+1)}^{n+1-(r-1)(k+1)} \cdots \sum_{i_r=i_{r-1}+(k+1)}^{(n+1)-(k+1)} p^{rk} q^r \\ &+ \sum_{i_1=1}^{n+2-r(k+1)} \sum_{i_2=i_1+(k+1)}^{n+2-(r-1)(k+1)} \cdots \sum_{i_{r-1}=i_{r-2}+(k+1)}^{n+2-2(k+1)} p^{rk} q^{r-1} . \end{aligned}$$

Lemma. For positive integers N, r, K with $N > rK$

$$\sum_{i_1=1}^{N-rK} \sum_{i_2=i_1+K}^{N-(r-1)K} \cdots \sum_{i_r=i_{r-1}+K}^{N-K} 1 = \binom{N-rK+r-1}{r} . \tag{10}$$

Proof. We consider arrangements of $N-1$ objects with the constraints that there are K objects of type 1 that appear in succession in a prescribed order, followed (not necessarily immediately) by K objects of type 2 that appear in succession in a prescribed order, followed in turn by K objects of type 3 that appear in succession in a prescribed order, and so on, up to K objects of type r (that appear in succession in a prescribed order). Furthermore, the remaining $N-1-Kr$ objects are identical. The left hand side of (10) counts the number of such arrangements by counting the ways to place the first object of type 1, the first object of type 2 etc. A more sensible way to count, however, is to notice that we can treat each of the blocks of objects of types $1, 2, \ldots, r$ as metaobjects, and then we are counting arrangements of just $N-1-r(K-1)$ objects in which r have prescribed order and the remaining $N-1-Kr$ are identical. The number of such arrangements is clearly just the number of ways to chose r from $N-rK+r-1$. •

Returning now to the proof of the main theorem, using the lemma we can write down all the terms in the inclusion–exclusion principle, and we have

$$P_n^{(k,p)} = \sum_{r=1}^{\infty}(-1)^{r-1}\left[\binom{n-rk}{r}p^{rk}q^r + \binom{n-rk}{r-1}p^{rk}q^{r-1}\right]H\left((n+1)-r(k+1)\right) \ .$$

The Heaviside functions restrict the sum to be over a finite range, and thus we reach the final result

$$P_n^{(k,p)} = \sum_{r=1}^{\left\lfloor\frac{n+1}{k+1}\right\rfloor}(-1)^{r-1}p^{rk}(1-p)^{r-1}\left[(1-p)\binom{n-rk}{r} + \binom{n-rk}{r-1}\right] \ . \tag{11}$$

Using this in (8) we obtain the result in the theorem. •

Corollary 1.[2] For $n \geq k$

$$F_n^{(k)} = \sum_{r=0}^{\left\lfloor\frac{n-k+1}{k+1}\right\rfloor}(-1)^r 2^{n+1-kr-k-r}\left[\frac{1}{2}\binom{n-k(r+1)}{r} + \binom{n-k(r+1)}{r-1}\right] \ . \tag{12}$$

Corollary 2.[2] For $n \geq 2$

$$F_n = \sum_{r=0}^{\left\lfloor\frac{n-1}{3}\right\rfloor}(-1)^r 2^{n-3r-1}\left[\frac{1}{2}\binom{n-2r-2}{r} + \binom{n-2r-2}{r-1}\right] \ . \tag{13}$$

Corollary 3. Define

$$\Sigma_N^{(k,p)} = \sum_{r=0}^{\left\lfloor\frac{N}{k+1}\right\rfloor}\binom{N-kr}{r}\left(\frac{1}{p}\right)^{N-kr-r}\left(\frac{p-1}{p}\right)^r \ .$$

(This is the sum of terms in a diagonal in a generalization of Pascal's triangle, see [12].) Then

$$F_n^{(k,p)} = p\left(\Sigma_{n-k+1}^{(k,p)} - \Sigma_{n-2k}^{(k,p)}\right) \ . \tag{14}$$

(The proof of this is an elementary manipulation of binomial coefficients.)

Comment 1. It is well-known that

$$F_n = \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-r}{r} \ ,$$

i.e. that the standard Fibonacci numbers can be expressed as a diagonal sum of Pascal's triangle. In (14) we see that for any k, the k-generalized Fibonacci numbers can be expressed as the difference of two diagonal sums of a generalized Pascal's triangle. In [12] it is shown that the $\Sigma_n^{(k,p)}$ obey the recursion relation

$$\Sigma_n^{(k,p)} = \frac{1}{p}\Sigma_{n-1}^{(k,p)} + \frac{p-1}{p}\Sigma_{n-k-1}^{(k,p)} \ .$$

The roots of the characteristic polynomial of this recursion are exactly those of the recursion (1), plus the root 1. Thus we should be able to write the $F_n^{(k,p)}$ as a linear combination of the $\Sigma_n^{(k,p)}$. The simplicity of (14), however, is remarkable.

Comment 2. There is an interesting way to generate the result (11) from the recursion relation (7). We write (7) in the form

$$P_n^{(k,p)} = q\sum_{i=1}^{k} p^{i-1}P_{n-i}^{(k,p)} + p^k \ , \quad n \geq k \ , \qquad P_n^{(k,p)} = 0 \ , \quad 0 \leq n < k \ . \tag{15}$$

Consider the modified recursion

$$\overline{P_n^{(k,p)}} = q\sum_{i=1}^{\infty} p^{i-1}\overline{P_{n-i}^{(k,p)}} + p^k \ , \quad n \geq k \ , \qquad \overline{P_n^{(k,p)}} = 0 \ , \quad n < k \ , \tag{16}$$

which determines quantities $\overline{P_n^{(k,p)}}$, where now n runs over all the integers, even negative. We can regard the $\overline{P_n^{(k,p)}}$ as a "first approximation" to the $P_n^{(k,p)}$. Remarkably (16) has the exact solution

$$\overline{P_n^{(k,p)}} = p^k\left(1 + q(n-k)\right)H(n-k) \ ,$$

as can be verified by a tedious calculation. This is precisely the first term (9) in the inclusion–exclusion principle! We continue by writing a recursion for the differences $Q_n^{(k,p)} = P_n^{(k,p)} - \overline{P_n^{(k,p)}}$. Subtracting (16) from (15) gives

$$\begin{aligned} Q_n^{(k,p)} &= q\sum_{i=1}^{k} p^{i-1}Q_{n-i}^{(k,p)} - q\sum_{i=k+1}^{\infty} p^{i-1}\overline{P_{n-i}^{(k,p)}} \\ &= q\sum_{i=1}^{k} p^{i-1}Q_{n-i}^{(k,p)} - q\sum_{i=k+1}^{n-k} p^{i+k-1}\left(1 + q(n-i-k)\right) \\ &= q\sum_{i=1}^{k} p^{i-1}Q_{n-i}^{(k,p)} - q(n-2k)p^{2k}H(n-2k-1) \ , \quad n \geq k \ . \end{aligned} \tag{17}$$

This recursion should be solved with initial conditions $Q_n^{(k,p)} = 0$, $0 \le n < k$, and in fact we clearly have $Q_n^{(k,p)} = 0$ for $0 \le n < 2k+1$. Again, we "approximate" (17) with the recursion

$$\overline{Q_n^{(k,p)}} = q \sum_{i=1}^{\infty} p^{i-1} \overline{Q_{n-i}^{(k,p)}} - q(n-2k)p^{2k} \ , \quad n \ge 2k+1 \ , \tag{18}$$

with initial conditions $\overline{Q_n^{(k,p)}} = 0$, $n < 2k+1$. This recursion can also be solved exactly, to give

$$\overline{Q_n^{(k,p)}} = -p^{2k} q(n-2k) \left(1 + \frac{1}{2} q(n-2k-1)\right) H(n-2k-1) \ ,$$

which is the second term in the inclusion–exclusion formula (with the correct sign). This procedure can be continued. There may be some relation with ∞-generalized Fibonacci numbers [10].

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