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Loop groups and discrete KdV equations

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Abstract
A study of fully discretized lattice equations associated with the KdV hierarchy is presented. Loop group methods give a systematic way of constructing discretizations of the equations in the hierarchy. The lattice KdV system of Nijhoff *et al* arises from the lowest order discretization of the trivial, lowest order equation in the hierarchy, $b_t = b_x$. Two new discretizations are also given, the lowest order discretization of the first nontrivial equation in the hierarchy, and a 'second order' discretization of $b_t = b_x$. The former, which is given the name *full lattice KdV*, has the (potential) KdV equation as a standard continuum limit. For each discretization a Bäcklund transformation is given and the soliton content is analysed. The full lattice KdV system has, like KdV itself, solitons of all speeds, whereas both other discretizations studied have a limited range of speeds (being discretizations of an equation with solutions only of a fixed speed).

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1. Introduction

Despite the fact that numerical simulations of PDEs of KdV type can be done quickly and accurately these days using standard spectral methods, it is still of interest to look at discretizations of such PDEs, and see how 'integrability properties' (elastic soliton scattering, existence of conserved quantities, etc) are affected by discretization, and in particular to see if there are 'integrable discretizations', that exhibit all the special properties of the underlying PDE. One can consider both 'partial' and 'full' discretizations; in the former, only the spatial coordinate is discretized, in the latter time is discretized too. This paper focuses on full discretizations.

The difference equation usually known as *discrete KdV* was first studied by Hirota [1]. Using a slightly different notation from that of [1], discrete KdV is the equation

$$\frac{1}{1+u_{n+1,m+1}} - \frac{1}{1+u_{n,m}} = c(u_{n+1,m} - u_{n,m+1}) \qquad (c \text{ constant}). \tag{1}$$

This is a discretization of KdV, but in a rather unusual sense. The main justification for the name 'discrete KdV' is that (1) has a bilinear formulation and a family of soliton solutions very similar to those of KdV (see also [2] for rational solutions). The study of discrete KdV was taken further by Nijhoff and collaborators (see [3] for a review and references). The work of Nijhoff *et al* focuses on the equation

$$\left(1+\frac{b_{n,m+1}-b_{n+1,m}}{p-q}\right)\left(1+\frac{b_{n,m}-b_{n+1,m+1}}{p+q}\right) = 1 \qquad (p, q \text{ constant}), \tag{2}$$

which they call *lattice KdV*. In fact, this equation is a 'potential form' of discrete KdV, in the sense that if $b_{n,m}$ satisfies (2), then it is easy to check that

$$u_{n,m} = \frac{b_{n-1,m-1} - b_{n,m}}{p+q} \tag{3}$$

satisfies (1) with $c = (p+q)/(p-q)$. (This is supposed to be an analogue of the fact that if $b(x,t)$ satisfies the 'potential KdV' equation $b_t = \frac{1}{4}b_{xxx} + \frac{3}{2}b_x^2 + \delta(t)$ for some function $\delta(t)$, then $u = b_x$ satisfies the KdV equation $u_t = \frac{1}{4}u_{xxx} + 3uu_x$.)

Nijhoff *et al*'s lattice KdV equation has an advantage over Hirota's discrete KdV in that it is easier to see its continuum limit (in the usual sense, to be explained shortly) as well as at least one nonstandard continuum limit in which it reduces to the potential KdV equation. On substituting $p = 1/h$ and $q = 1/k$, (2) becomes

$$\begin{aligned} -\frac{b_{n+1,m+1} - b_{n,m+1} + b_{n+1,m} - b_{n,m}}{h} &+ \frac{b_{n+1,m+1} - b_{n+1,m} + b_{n,m+1} - b_{n,m}}{k} \\ &+ (b_{n,m} - b_{n+1,m+1})(b_{n,m+1} - b_{n+1,m}) = 0. \end{aligned} \tag{4}$$

Taking the standard continuum limit will be taken to mean replacing $b_{n,m}$ by $b(x,t)$, $b_{n+1,m}$ by $b(x+h,t)$, $b_{n,m+1}$ by $b(x,t+k)$, $b_{n+1,m+1}$ by $b(x+h,t+k)$, expanding in powers of h and k and ignoring all but leading order terms. It is clear that in this limit the first term in (4) gives $-2b_x$, the second $2b_t$ and the third 0. Thus, *in the standard continuum limit, lattice KdV is simply a discretization of* $b_t = b_x$. A nonstandard continuum limit of (4) that gives the potential KdV equation is as follows: make the same replacements as before, expand in powers of h and k, but keep not only the leading order terms but also all terms of order h and h^2. This gives

$$-2\left(b_x + \frac{h}{2}b_{xx} + \frac{h^2}{6}b_{xxx}\right) + b_t + \left(b_t + hb_{tx} + \frac{h^2}{2}b_{txx}\right) + h^2 b_x^2 = 0. \tag{5}$$

Now write $b = \tilde{b} - (h/2)\tilde{b}_x$. Ignoring terms of order h^3 and above, the last equation can be written as

$$\tilde{b}_t = \tilde{b}_x - \frac{h^2}{3}\left(\frac{1}{4}\tilde{b}_{xxx} + \frac{3}{2}\tilde{b}_x^2\right). \tag{6}$$

This is a 'linear combination' of the flow obtained in the standard continuum limit with the potential KdV flow.

The foregoing discussions raise a variety of questions. The relationship of KdV/potential KdV and discrete KdV/lattice KdV as it stands is rather cryptic and requires some clarification. It would also be good to have another integrable lattice equation from which KdV/potential KdV can be obtained by taking a standard continuum limit. If this is possible, then it would be good to know just what freedoms there are in constructing integrable discretizations. Finally, though this is a question that will not be addressed in the current paper, given an integrable lattice equation, just how much freedom is there in taking the continuum limit?

This paper discusses the subject of discretizations of KdV using loop group methods. The basic fact behind the loop group approach to KdV is that the KdV equation (or, more precisely, the Lax pair for the KdV equation) is simply a 'disguised' version of the Frobenius-integrable pair of linear first-order constant-coefficient ODEs:

$$\partial_x U = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} U, \qquad \partial_t U = \lambda \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} U \tag{7}$$

(here U is a 2×2 matrix function of x, t, λ). The relation of the above system with KdV will be explained fully in section 2. In greater generality, the Nth flow ($N = 1, 3, 5, \ldots$) in the KdV hierarchy is associated with the system

$$\partial_x U = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} U, \qquad \partial_{t_n} U = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}^N U, \tag{8}$$

which reduces to the standard system (7) when $N = 3$. The approach proposed in this paper for constructing integrable discretizations of KdV is simply to discretize the system (7) or (8) (any explicit scheme for numerical integration of ODEs can be used) and then to apply the necessary 'disguise' to translate this system into a discrete KdV. Section 3 is devoted to the simplest discretization of (8) with $N = 1$, namely

$$U_{n+1,m} = \left[I + h \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} \right] U_{n,m}, \qquad U_{n,m+1} = \left[I + k \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} \right] U_{n,m}. \tag{9}$$

This is just a first-order Euler scheme with different step sizes in the x and t directions. This scheme gives rise to the lattice KdV equation, which, as shown above, is a first-order discretization of the $N = 1$ flow in the potential KdV hierarchy, $b_t = b_x$. As an application of the loop group formulation, a Bäcklund transformation for (2) is given, and soliton solutions are derived (cf [1]). A brief analysis of the soliton solutions is given, which helps clarify the rather schizophrenic nature of the lattice KdV equation, which on the one hand is a (nonlinear) discretization of $b_t = b_x$, and on the other displays features of potential KdV.

Section 4 is devoted to the simplest discretization of (7), namely

$$U_{n+1,m} = \left[I + h \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} \right] U_{n,m}, \qquad U_{n,m+1} = \left[I + k\lambda \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} \right] U_{n,m}. \tag{10}$$

As expected, this gives rise to a system which is, in a natural way, a first-order discretization of the potential KdV equation. The system is a little complicated, involving two auxiliary fields (reminiscent of the discretization of the sinh-Gordon equation given in [4]), but it seems this is the price that has to be paid to have an integrable lattice equation that has potential KdV as a natural continuum limit. The Bäcklund transformation and soliton solutions are derived for this system too.

Section 5 considers another discretization of (8) for $N = 1$, namely

$$U_{n+1,m} = \left[I + h \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} + \frac{h^2}{2} \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}^2 \right] U_{n,m} = \begin{pmatrix} 1 + \dfrac{h^2\lambda}{2} & h \\ h\lambda & 1 + \dfrac{h^2\lambda}{2} \end{pmatrix} U_{n,m}, \tag{11}$$

$$U_{n,m+1} = \left[I + k \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} \right] U_{n,m}. \tag{12}$$

This example is worked out mainly to illustrate that the method can be extended to arbitrary order discretizations of (7) and (8), establishing that there is quite a lot of freedom in constructing integrable discretizations. Section 6 contains some concluding remarks.

I conclude the introduction with a brief mention of some relevant literature. The approach to discretization taken in this paper is closely related to the approach of discretizing the scattering problem, which was first proposed by Ablowitz and Ladik [5], and recently has been revisited by Boiti *et al* [6]. Several potentially interesting applications of discretizations of equations of KdV type have emerged recently. Nijhoff *et al* [3] were the first to notice the link between lattice KdV and the *discrete conformal map* equation:

$$\frac{(z_{n,m}-z_{n+1,m})(z_{n,m+1}-z_{n+1,m+1})}{(z_{n,m}-z_{n,m+1})(z_{n+1,m}-z_{n+1,m+1})}=s \qquad (s \text{ constant}), \tag{13}$$

which in the case $s=-1$ is a natural discretization of the Cauchy–Riemann conditions. Techniques related to those of this paper have been applied to (13) in [7]. Equation (13) may well play a significant role in the field of numerical conformal mapping. Discretizations of KdV and related equations have also been shown to have a role in a variety of other numerical algorithms [8].

2. KdV as a linear constant-coefficient flow

This section contains a summary of results from [9], relating the (potential) KdV equation with the linear constant coefficient flow (7). A rather more mathematical description can be found in [10].

The general solution of (7) is

$$U(x,t,\lambda)=\exp\left(x\begin{pmatrix}0&1\\ \lambda&0\end{pmatrix}+t\lambda\begin{pmatrix}0&1\\ \lambda&0\end{pmatrix}\right)U(0,0,\lambda). \tag{14}$$

Assume that the function $U(0,0,\lambda)$ is defined for $|\lambda|=1$, and has nonzero determinant; in other words it is an element of the *loop group* $LGL_2(\mathbb{C})$ [11]. Then, evidently so is $U(x,t,\lambda)$. Now, a typical element $g(\lambda)$ of the loop group $LGL_2(\mathbb{C})$ can be written as a product $S^{-1}(\lambda)Y(\lambda)$ where $Y(\lambda)$ is holomorphic for $|\lambda|<1$ and $S(\lambda)$ is holomorphic for $|\lambda|>1$ with $S(\infty)=I$. This is the so-called *Birkhoff factorization theorem*, see [11], chapter 8. So let us write

$$U(x,t,\lambda)=S^{-1}(x,t,\lambda)Y(x,t,\lambda), \tag{15}$$

(with Y holomorphic in $|\lambda|<1$, S holomorphic in $|\lambda|>1$ and $S(x,t,\infty)=I$) and let us try to find differential equations satisfied by the two 'components' Y and S of U. Substituting (15) into (7), multiplying on the left by S and on the right by Y^{-1} gives

$$-S_xS^{-1}+Y_xY^{-1}=S\begin{pmatrix}0&1\\ \lambda&0\end{pmatrix}S^{-1}, \qquad -S_tS^{-1}+Y_tY^{-1}=\lambda S\begin{pmatrix}0&1\\ \lambda&0\end{pmatrix}S^{-1}. \tag{16}$$

Now, if

$$S=I+\frac{1}{\lambda}\begin{pmatrix}a_1(x,t)&b_1(x,t)\\ c_1(x,t)&d_1(x,t)\end{pmatrix}+\frac{1}{\lambda^2}\begin{pmatrix}a_2(x,t)&b_2(x,t)\\ c_2(x,t)&d_2(x,t)\end{pmatrix}+\cdots \tag{17}$$

then, a brief calculation shows

$$S\begin{pmatrix}0&1\\ \lambda&0\end{pmatrix}S^{-1} = \begin{pmatrix}-b&1\\ \lambda-v&b\end{pmatrix}+\mathrm{O}(\lambda^{-1}), \tag{18}$$

$$\lambda S\begin{pmatrix}0&1\\ \lambda&0\end{pmatrix}S^{-1} = \begin{pmatrix}-b\lambda-B&\lambda+v-b^2\\ \lambda^2-v\lambda-V&b\lambda+B\end{pmatrix}+\mathrm{O}(\lambda^{-1}), \tag{19}$$

where $b = -b_1$, $v = a_1 - d_1$, $B = c_1 - b_2 + a_1 b_1$, $V = a_2 - d_2 + a_1 d_1 - b_1 c_1 - a_1^2$. Substitute these results in (16). On the left-hand side of the equations in (16), since Y is holomorphic in $|\lambda| < 1$, $Y_x Y^{-1}$ and $Y_t Y^{-1}$ can be written as power series in λ, and since S is holomorphic in $|\lambda| > 1$ with $S(x, t, \infty) = I$, $S_x S^{-1}$ and $S_t S^{-1}$ can be written as power series in $1/\lambda$ with no constant term. Thus, from the non-negative powers of λ in (16), after substituting (18) and (19), it follows that

$$Y_x Y^{-1} = \begin{pmatrix} -b & 1 \\ \lambda - v & b \end{pmatrix}, \tag{20}$$

$$Y_t Y^{-1} = \begin{pmatrix} -b\lambda - B & \lambda + v - b^2 \\ \lambda^2 - v\lambda - V & b\lambda + B \end{pmatrix}. \tag{21}$$

If $X = Y_x Y^{-1}$, $T = Y_t Y^{-1}$ then X, T must satisfy the zero-curvature equation

$$X_t - T_x + [X, T] = 0. \tag{22}$$

Substituting the forms (20) and (21) into the zero-curvature equation, required to be true for all λ, gives the following system of equations:

$$v = b_x + b^2, \tag{23}$$

$$B = \tfrac{1}{2} b_{xx} + b b_x, \tag{24}$$

$$V_x = \left(\tfrac{1}{4} b_{xxx} + \tfrac{1}{2} b_x^2 + b b_{xx} + b^2 b_x\right)_x, \tag{25}$$

$$b_t = \tfrac{1}{2} b_{xxx} + 2 b_x^2 + b b_{xx} + b^2 b_x - V. \tag{26}$$

The third equation can be integrated to give $V = \frac{1}{4} b_{xxx} + b b_{xx} + \frac{1}{2} b_x^2 + b^2 b_x - \delta(t)$, where δ is an arbitrary function of t alone. Using this in the last equation gives

$$b_t = \tfrac{1}{4} b_{xxx} + \tfrac{3}{2} b_x^2 + \delta(t). \tag{27}$$

All this can be summarized in the following result.

Proposition 2.1. *Let $U(0, 0, \lambda)$ be an element of $LGL_2(\mathbb{C})$; let $S^{-1}Y$ be the Birkhoff decomposition of*

$$U(x, t, \lambda) = \exp\left(x \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} + t\lambda \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} \right) U(0, 0, \lambda)$$

and let $b(x, t)$ be (-1) times the component of $1/\lambda$ in the $1, 2$-entry of S. Then, $b(x, t)$ is a solution, possibly with singularities, of the potential KdV equation (27) for some function δ.

The reason for the phrase 'possibly with singularities' here is because for some values of x and t, $U(x, t, \lambda)$ might leave the dense open set of $LGL_2(\mathbb{C})$ where Birkhoff decomposition is possible (it can be proved that these values are isolated). It is important for the purposes of this paper to note that although the above proposition makes no mention of the linear constant-coefficient flow (7), the heart of its proof is that this flow induces, via Birkhoff decomposition, the matrix Lax pair (20) and (21) for the potential KdV equation. Note also that the first equation of the Lax pair (20) gives the usual relation of KdV with the Schrödinger equation. Writing either column of Y as $\binom{\psi}{\phi}$, (20) gives $\psi_{xx} = (\lambda - 2b_x)\psi$.

There are many applications of the above result, of which only one will be discussed here, the construction of the standard Bäcklund transformation for potential KdV. The idea behind this Bäcklund transformation is as follows: suppose the element $U(0, 0, \lambda)$ of the loop group gives a solution $U(x, t, \lambda)$ of the linear system (7) with Birkhoff decomposition $S^{-1}Y$

and corresponding potential KdV solution $b(x,t)$. Let us now try to find the potential KdV solution corresponding to the element

$$\sqrt{\frac{\lambda-\theta}{\lambda}}\begin{pmatrix}0 & 1\\ \lambda & 0\end{pmatrix}U(0,0,\lambda)\begin{pmatrix}0 & \dfrac{1}{\lambda-\theta}\\ 1 & 0\end{pmatrix} \tag{28}$$

with $0<\theta<1$. The new solution of the linear system (7) is

$$\begin{aligned}\sqrt{\frac{\lambda-\theta}{\lambda}}&\begin{pmatrix}0 & 1\\ \lambda & 0\end{pmatrix}U(x,t,\lambda)\begin{pmatrix}0 & \dfrac{1}{\lambda-\theta}\\ 1 & 0\end{pmatrix}\\ &=\sqrt{\frac{\lambda-\theta}{\lambda}}\begin{pmatrix}0 & 1\\ \lambda & 0\end{pmatrix}S^{-1}(x,t,\lambda)Y(x,t,\lambda)\begin{pmatrix}0 & \dfrac{1}{\lambda-\theta}\\ 1 & 0\end{pmatrix}.\end{aligned} \tag{29}$$

To perform the new Birkhoff decomposition, a certain matrix and its inverse must be inserted as follows:

$$\begin{aligned}\sqrt{\frac{\lambda-\theta}{\lambda}}&\begin{pmatrix}0 & 1\\ \lambda & 0\end{pmatrix}S^{-1}(x,t,\lambda)\begin{pmatrix}\alpha & 1\\ \lambda-\theta+\alpha\beta & \beta\end{pmatrix}^{-1}\\ &\times\begin{pmatrix}\alpha & 1\\ \lambda-\theta+\alpha\beta & \beta\end{pmatrix}Y(x,t,\lambda)\begin{pmatrix}0 & \dfrac{1}{\lambda-\theta}\\ 1 & 0\end{pmatrix}.\end{aligned} \tag{30}$$

The aim is to choose α and β so that this is written in Birkhoff factorized form, i.e. so that

$$\tilde{S}(x,t,\lambda)=\sqrt{\frac{\lambda}{\lambda-\theta}}\begin{pmatrix}\alpha & 1\\ \lambda-\theta+\alpha\beta & \beta\end{pmatrix}S(x,t,\lambda)\begin{pmatrix}0 & \dfrac{1}{\lambda}\\ 1 & 0\end{pmatrix} \tag{31}$$

is holomorphic in $|\lambda|>1$ and satisfies $\tilde{S}(x,t,\infty)=I$, and

$$\tilde{Y}(x,t,\lambda)=\begin{pmatrix}\alpha & 1\\ \lambda-\theta+\alpha\beta & \beta\end{pmatrix}Y(x,t,\lambda)\begin{pmatrix}0 & \dfrac{1}{\lambda-\theta}\\ 1 & 0\end{pmatrix} \tag{32}$$

is holomorphic in $|\lambda|<1$. Inserting the expansion (17) in (31), the former condition requires $\beta=b$. For the latter condition it is just necessary to check $\tilde{Y}$ does not have a pole at $\lambda=\theta$, and this requires $\alpha=-Y_{21}(x,t,\theta)/Y_{11}(x,t,\theta)$. Finally, it is necessary to compute the new solution of the potential KdV equation, i.e. the component of $1/\lambda$ in the 1, 2-entry of $\tilde{S}$. A brief calculation shows this is simply $-\alpha$. Utilizing the Lax pair (20) and (21) it is straightforward to determine properties of α leading to the following result.

Proposition 2.2. *If b is a solution of the potential KdV equation (27) and ψ satisfies*

$$\psi_{xx}=(\theta-2b_x)\psi,\qquad \psi_t=-\tfrac{1}{2}b_{xx}\psi+(\theta+b_x)\psi_x, \tag{33}$$

then $\tilde{b}=b+\psi_x/\psi$ is also a solution of potential KdV, for the same function δ.

Equations (33) comprise the standard scalar Lax pair for the KdV equation. Applying the Bäcklund transformation to the x-independent solution $b(t)=\int\delta(t)\,\mathrm{d}t$ gives the 1-soliton solutions

$$b(x,t)=\int\delta(t)\,\mathrm{d}t+\sqrt{\theta}\tanh(\sqrt{\theta}(x+\theta t)+C) \tag{34}$$

and the singular solutions

$$b(x,t) = \int \delta(t)\,\mathrm{d}t + \sqrt{\theta}\coth(\sqrt{\theta}(x+\theta t) + C), \tag{35}$$

where in both formulae C is a constant. The easiest way to apply the Bäcklund transformation again to these solutions is to use the *Bianchi permutability theorem* that states that the two-parameter family of solutions obtained by applying first the Bäcklund transformation with parameter θ_1 and then the Bäcklund transformation with parameter θ_2 is the same as the two-parameter family of solutions obtained by applying the two Bäcklund transformations in the reverse order. See [9] for a detailed discussion of this. The Bianchi permutability theorem can be used to derive an algebraic expression for the solutions obtained by applying two Bäcklund transformations (see [12] section 5.4.3).

Proposition 2.3. *If b is a solution of the potential KdV equation (27), and b_1 and b_2 are solutions obtained by applying Bäcklund transformations with parameters θ_1 and θ_2, respectively, to b, then*

$$B = b + \frac{\theta_1 - \theta_2}{b_1 - b_2} \tag{36}$$

is a solution obtained by applying the two Bäcklund transformations successively to b, in either order.

Applying this result using a 1-soliton solution for b_1, a singular solution for b_2 and $\theta_2 > \theta_1$ gives the 2-soliton solution

$$\begin{aligned} b(x,t) &= \int \delta(t)\,\mathrm{d}t + \frac{\theta_1 - \theta_2}{\sqrt{\theta_1}\tanh\alpha_1 - \sqrt{\theta_2}\coth\alpha_2} \qquad \begin{cases}\alpha_1 = \sqrt{\theta_1}(x+\theta_1 t) + C_1 \\ \alpha_2 = \sqrt{\theta_2}(x+\theta_2 t) + C_2\end{cases} \\ &= \int \delta(t)\,\mathrm{d}t + \sqrt{\theta_1}\tanh\alpha_1 + \sqrt{\theta_2}\tanh\alpha_2 \\ &\quad - \frac{\theta_1\tanh\alpha_2\,\mathrm{sech}^2\alpha_1 + \sqrt{\theta_1\theta_2}\tanh\alpha_1\,\mathrm{sech}^2\alpha_2}{\sqrt{\theta_2} - \sqrt{\theta_1}\tanh\alpha_1\tanh\alpha_2}. \end{aligned} \tag{37}$$

This concludes our presentation of the basic theory of the KdV equation and its relation with the linear system (7) which will be imitated for discrete systems in later sections.

3. Discretizations I: lattice KdV

The aim in this section is to follow the procedures of the last section as closely as possible, but replacing the solution $U(x,t,\lambda)$ of (7) by the solution $U_{nm}(\lambda)$ of the lattice equation (9), which has the general solution

$$\begin{aligned} U_{n,m}(\lambda) &= \left[I + h\begin{pmatrix}0 & 1\\ \lambda & 0\end{pmatrix}\right]^n \left[I + k\begin{pmatrix}0 & 1\\ \lambda & 0\end{pmatrix}\right]^m U_{0,0}(\lambda) \\ &= \frac{1}{4}\begin{pmatrix} (1+h\sqrt{\lambda})^n + (1-h\sqrt{\lambda})^n & \frac{1}{\sqrt{\lambda}}((1+h\sqrt{\lambda})^n - (1-h\sqrt{\lambda})^n) \\ \sqrt{\lambda}((1+h\sqrt{\lambda})^n - (1-h\sqrt{\lambda})^n) & (1+h\sqrt{\lambda})^n + (1-h\sqrt{\lambda})^n \end{pmatrix} \\ &\quad\times \begin{pmatrix} (1+k\sqrt{\lambda})^m + (1-k\sqrt{\lambda})^m & \frac{1}{\sqrt{\lambda}}((1+k\sqrt{\lambda})^m - (1-k\sqrt{\lambda})^m) \\ \sqrt{\lambda}((1+k\sqrt{\lambda})^m - (1-k\sqrt{\lambda})^m) & (1+k\sqrt{\lambda})^m + (1-k\sqrt{\lambda})^m \end{pmatrix} U_{0,0}(\lambda). \end{aligned} \tag{38}$$

Suppose $U_{n,m}(\lambda)$ has a Birkhoff factorization $S_{n,m}^{-1}(\lambda)Y_{n,m}(\lambda)$. Substituting in (9) and rearranging gives

$$Y_{n+1,m}Y_{n,m}^{-1} = S_{n+1,m}\left[I + h\begin{pmatrix}0 & 1\\ \lambda & 0\end{pmatrix}\right]S_{n,m}^{-1}, \tag{39}$$

$$Y_{n,m+1}Y_{n,m}^{-1} = S_{n,m+1}\left[I + k\begin{pmatrix}0 & 1\\ \lambda & 0\end{pmatrix}\right]S_{n,m}^{-1}. \tag{40}$$

Writing

$$S_{n,m} = I + \frac{1}{\lambda}\begin{pmatrix}a_{n,m} & -b_{n,m}\\ c_{n,m} & d_{n,m}\end{pmatrix} + \cdots \tag{41}$$

and comparing non-negative powers of λ on both sides of (39) and (40) gives

$$Y_{n+1,m} = \begin{pmatrix}1 - hb_{n+1,m} & h\\ h\lambda + h(d_{n+1,m} - a_{n,m}) & 1 + hb_{n,m}\end{pmatrix}Y_{n,m}, \tag{42}$$

$$Y_{n,m+1} = \begin{pmatrix}1 - kb_{n,m+1} & k\\ k\lambda + k(d_{n,m+1} - a_{n,m}) & 1 + kb_{n,m}\end{pmatrix}Y_{n,m}. \tag{43}$$

There is one further simplification that can be made in these equations. Equation (39) (and similarly (40)) can be written in the form

$$S_{n,m}S_{n+1,m}^{-1}Y_{n+1,m}Y_{n,m}^{-1} = S_{n,m}\left[I + h\begin{pmatrix}0 & 1\\ \lambda & 0\end{pmatrix}\right]S_{n,m}^{-1}. \tag{44}$$

Taking the determinant gives

$$\det\left(S_{n,m}S_{n+1,m}^{-1}\right)\det\left(Y_{n+1,m}Y_{n,m}^{-1}\right) = 1 - h^2\lambda. \tag{45}$$

The Birkhoff factorization theorem applies for scalars (elements of $LGL_1(\mathbb{C})$) too, so from this it can be deduced that $\det(Y_{n+1,m}Y_{n,m}^{-1}) = 1 - h^2\lambda$ (and $\det(S_{n,m}S_{n+1,m}^{-1}) = 1$). Applying this to (42) (and the corresponding result $\det(Y_{n,m+1}Y_{n,m}^{-1}) = 1 - k^2\lambda$ to (43)) gives

$$Y_{n+1,m} = \begin{pmatrix}1 - hb_{n+1,m} & h\\ h\lambda + b_{n,m} - b_{n+1,m} - hb_{n,m}b_{n+1,m} & 1 + hb_{n,m}\end{pmatrix}Y_{n,m}, \tag{46}$$

$$Y_{n,m+1} = \begin{pmatrix}1 - kb_{n,m+1} & k\\ k\lambda + b_{n,m} - b_{n,m+1} - kb_{n,m}b_{n,m+1} & 1 + kb_{n,m}\end{pmatrix}Y_{n,m}. \tag{47}$$

Up to a rescaling this is precisely Nijhoff *et al*'s scalar Lax pair for the lattice KdV equation [3]. Writing

$$L_{n,m} = \begin{pmatrix}1 - hb_{n+1,m} & h\\ h\lambda + b_{n,m} - b_{n+1,m} - hb_{n,m}b_{n+1,m} & 1 + hb_{n,m}\end{pmatrix}, \tag{48}$$

$$M_{n,m} = \begin{pmatrix}1 - kb_{n,m+1} & k\\ k\lambda + b_{n,m} - b_{n,m+1} - kb_{n,m}b_{n,m+1} & 1 + kb_{n,m}\end{pmatrix}, \tag{49}$$

equations (46) and (47) are just

$$Y_{n+1,m} = L_{n,m}Y_{n,m}, \qquad Y_{n,m+1} = M_{n,m}Y_{n,m}, \tag{50}$$

and for consistency $L_{n,m+1}M_{n,m} = M_{n+1,m}L_{n,m}$. (This last equation plays the role of the zero-curvature equation in the continuous case.) Substituting the forms found for $L_{n,m}$, $M_{n,m}$ in the consistency condition gives lattice KdV (4). Thus, the analogue of proposition 2.1 is obtained.

Proposition 3.1. *Let $U_{0,0}(\lambda)$ be an element of $LGL_2(\mathbb{C})$; let $S_{n,m}^{-1}(\lambda)Y_{n,m}(\lambda)$ be the Birkhoff decomposition of*

$$U_{n,m}(\lambda) = \left[I + h\begin{pmatrix}0 & 1\\ \lambda & 0\end{pmatrix}\right]^n \left[I + k\begin{pmatrix}0 & 1\\ \lambda & 0\end{pmatrix}\right]^m U_{0,0}(\lambda) \tag{51}$$

and let $b_{n,m}$ be (-1) times the component of $1/\lambda$ in the $1,2$-entry of $S_{n,m}$. Then, $b_{n,m}$ is a solution, possibly with singularities, of the lattice KdV equation (4).

In fact, there is no reason why $U_{0,0}(\lambda)$ should not, in this case, be dependent on h and k. So, in principle, the class of solutions of lattice KdV occurring this way is much larger than the corresponding class of solutions of (potential) KdV.

Let us attempt to find a Bäcklund transformation and soliton solutions for lattice KdV, proceeding as in section 2. Making the replacement

$$U_{0,0}(\lambda) \to \sqrt{\frac{\lambda-\theta}{\lambda}}\begin{pmatrix}0 & 1\\ \lambda & 0\end{pmatrix} U_{0,0}(\lambda)\begin{pmatrix}0 & \dfrac{1}{\lambda-\theta}\\ 1 & 0\end{pmatrix} \tag{52}$$

gives

$$U_{n,m}(\lambda) \to \sqrt{\frac{\lambda-\theta}{\lambda}}\begin{pmatrix}0 & 1\\ \lambda & 0\end{pmatrix} U_{n,m}(\lambda)\begin{pmatrix}0 & \dfrac{1}{\lambda-\theta}\\ 1 & 0\end{pmatrix} \tag{53}$$

and

$$S_{n,m}(\lambda) \to \sqrt{\frac{\lambda}{\lambda-\theta}}\begin{pmatrix}\alpha_{n,m} & 1\\ \lambda-\theta+\alpha_{n,m}b_{n,m} & b_{n,m}\end{pmatrix} S_{n,m}(\lambda)\begin{pmatrix}0 & \dfrac{1}{\lambda}\\ 1 & 0\end{pmatrix}, \tag{54}$$

$$Y_{n,m}(\lambda) \to \begin{pmatrix}\alpha_{n,m} & 1\\ \lambda-\theta+\alpha_{n,m}b_{n,m} & b_{n,m}\end{pmatrix} Y_{n,m}(\lambda)\begin{pmatrix}0 & \dfrac{1}{\lambda-\theta}\\ 1 & 0\end{pmatrix}, \tag{55}$$

where $\alpha_{n,m} = -(Y_{n,m})_{21}(\theta)/(Y_{n,m})_{11}(\theta)$. The new solution of lattice KdV is simply $-\alpha_{n,m}$. Using (46) and (47) to find properties of $\alpha_{n,m}$ gives the Bäcklund transformation.

Proposition 3.2. *If $b_{n,m}$ is a solution of the lattice KdV equation (4) and $\psi_{n,m}$ satisfies*

$$\frac{\psi_{n+2,m} - 2\psi_{n+1,m} + \psi_{n,m}}{h^2} = \theta\psi_{n,m} - \left(\frac{b_{n+2,m} - b_{n,m}}{h}\right)\psi_{n+1,m}, \tag{56}$$

$$\frac{\psi_{n,m+1} - \psi_{n,m}}{k} = \frac{\psi_{n+1,m} - \psi_{n,m}}{h} + (b_{n+1,m} - b_{n,m+1})\psi_{n,m}, \tag{57}$$

then, $\tilde{b}_{n,m} = b_{n+1,m} + (\psi_{n+1,m} - \psi_{n,m})/(h\psi_{n,m})$ is also a solution of lattice KdV.

The first equation here is a natural discretization of the first equation in (33), and is the discretization of the Schrödinger equation studied in [6]. The second equation is, however, completely unrelated to that in (33). To get 1-soliton and singular solutions, the Bäcklund transformation can be applied to the trivial solution $b_{n,m} = 0$. This gives solutions of the form

$$b_{n,m} = \sqrt{\theta}\,\frac{A(1+h\sqrt{\theta})^n(1+k\sqrt{\theta})^m - B(1-h\sqrt{\theta})^n(1-k\sqrt{\theta})^m}{A(1+h\sqrt{\theta})^n(1+k\sqrt{\theta})^m + B(1-h\sqrt{\theta})^n(1-k\sqrt{\theta})^m} \qquad (A, B \text{ constants}). \tag{58}$$

If $A : B$ is positive and $h, k < 1/\sqrt{\theta}$ this gives 1-soliton solutions:

$$b_{n,m} = \sqrt{\theta}\tanh(n\tanh^{-1}(h\sqrt{\theta}) + m\tanh^{-1}(k\sqrt{\theta}) + C) \qquad (C \text{ constant}). \tag{59}$$

If $A : B$ is negative and $h, k < 1/\sqrt{\theta}$, (58) gives singular solutions:

$$b_{n,m} = \sqrt{\theta}\coth(n\tanh^{-1}(h\sqrt{\theta}) + m\tanh^{-1}(k\sqrt{\theta}) + C). \tag{60}$$

The Bianchi permutability theorem applies equally here in the discrete case, and this can be used to give the discrete version of proposition 2.3.

Proposition 3.3. *If $b_{n,m}$ is a solution of the lattice KdV equation (4), and $b^{(1)}_{n,m}$ and $b^{(2)}_{n,m}$ are solutions obtained by applying Bäcklund transformations with parameters θ_1 and θ_2, respectively, to $b_{n,m}$, then,*

$$B_{n,m} = b_{n,m} + \frac{\theta_1 - \theta_2}{b^{(1)}_{n,m} - b^{(2)}_{n,m}} \tag{61}$$

is a solution obtained by applying the two Bäcklund transformations successively to $b_{n,m}$, in either order.

Proof. Writing $q_{n,m} = (\psi_{n+1,m} - \psi_{n,m})/(h\psi_{n,m})$, the Bäcklund transformation can be written as $b_{n,m} \to b_{n+1,m} + q_{n,m}$, where $q_{n,m}$ satisfies the discrete Riccati equation [13]

$$q_{n+1,m} = \frac{q_{n,m}(1 - hb_{n+2,m} + hb_{n,m}) + (h\theta - b_{n+2,m} + b_{n,m})}{1 + hq_{n,m}} \tag{62}$$

(for the sake of brevity I only look at the first equation in (56) and (57)). Alternatively, after a little algebra, the transformation can be written as $b_{n,m} \to \tilde{b}_{n,m}$, where $b_{n,m}, \tilde{b}_{n,m}$ are related by

$$\frac{\tilde{b}_{n+1,m} - \tilde{b}_{n,m} + b_{n+1,m} - b_{n,m}}{h} = \theta + (b_{n,m} - \tilde{b}_{n+1,m})(\tilde{b}_{n,m} - b_{n+1,m}). \tag{63}$$

Using the Bianchi permutability theorem and the premises of the theorem, it is known that applying the BT with parameter θ_1 to $b_{n,m}$ gives $b^{(1)}_{n,m}$, applying the BT with parameter θ_2 to $b_{n,m}$ gives $b^{(2)}_{n,m}$, and applying *either* the BT with parameter θ_2 to $b^{(1)}_{n,m}$ *or* the BT with parameter θ_1 to $b^{(2)}_{n,m}$ gives the same solution $B_{n,m}$. This implies four equations:

$$\frac{b^{(1)}_{n+1,m} - b^{(1)}_{n,m} + b_{n+1,m} - b_{n,m}}{h} = \theta_1 + (b_{n,m} - b^{(1)}_{n+1,m})(b^{(1)}_{n,m} - b_{n+1,m}), \tag{64}$$

$$\frac{b^{(2)}_{n+1,m} - b^{(2)}_{n,m} + b_{n+1,m} - b_{n,m}}{h} = \theta_2 + (b_{n,m} - b^{(2)}_{n+1,m})(b^{(2)}_{n,m} - b_{n+1,m}), \tag{65}$$

$$\frac{B_{n+1,m} - B_{n,m} + b^{(1)}_{n+1,m} - b^{(1)}_{n,m}}{h} = \theta_2 + (b^{(1)}_{n,m} - B_{n+1,m})(B_{n,m} - b^{(1)}_{n+1,m}), \tag{66}$$

$$\frac{B_{n+1,m} - B_{n,m} + b^{(2)}_{n+1,m} - b^{(2)}_{n,m}}{h} = \theta_1 + (b^{(2)}_{n,m} - B_{n+1,m})(B_{n,m} - b^{(2)}_{n+1,m}). \tag{67}$$

Adding the first and last of these equations and subtracting the other two gives

$$2(\theta_1 - \theta_2) = (B_{n,m} - b_{n,m})(b^{(1)}_{n,m} - b^{(2)}_{n,m}) + (B_{n+1,m} - b_{n+1,m})(b^{(1)}_{n+1,m} - b^{(2)}_{n+1,m}). \tag{68}$$

The general solution of this is clearly

$$(B_{n,m} - b_{n,m})(b^{(1)}_{n,m} - b^{(2)}_{n,m}) = (\theta_1 - \theta_2) + (-1)^n F(m), \tag{69}$$

where F is an arbitrary function of m. Using the second equation in (56) and (57) it is possible to show that $F(m) = 0$. □

All that remains to be done in this section is to briefly discuss the nature of soliton solutions of lattice KdV, and in particular how they compare to those of continuum KdV. From (59) the speed of the soliton with parameter θ is

$$c = \frac{h \tanh^{-1}(k\sqrt{\theta})}{k \tanh^{-1}(h\sqrt{\theta})}. \tag{70}$$

(The formal definition of the 'speed' is the number c such that the solution depends on m, n only through the combination $(nh + cmk)$.) Recall that the parameter θ is limited by the requirements $h\sqrt{\theta}, k\sqrt{\theta} < 1$. Thus, we have the following.

Proposition 3.4. *For $h = k$ the soliton solutions of lattice KdV (4) all have speed 1. For $h < k$ there are solitons with all speeds greater than 1. For $h > k$ there are solitons with all speeds between 0 and 1.*

Proof. The result for $h = k$ is obvious. Switching h and k switches c and $1/c$ so it is just necessary to check the result for, say, $h < k$. As θ tends to 0, c tends to 1, and as θ tends to $1/k^2$ (which is less than $1/h^2$) c tends to ∞. So the result will be proved if we can establish that c is a monotonic increasing function of θ for $0 < \theta < 1/k^2$. Writing $z = k\sqrt{\theta}$ and $\alpha = h/k < 1$,

$$c = \frac{\alpha \tanh^{-1} z}{\tanh^{-1} \alpha z} \tag{71}$$

and it is necessary to check this is a monotonic function of z on $0 < z < 1$ for α fixed between 0 and 1. Differentiating gives

$$\frac{\mathrm{d}c}{\mathrm{d}z} = \frac{\alpha}{[\tanh^{-1}(\alpha z)]^2(1-z^2)(1-\alpha^2 z^2)}[(1-\alpha^2 z^2)\tanh^{-1}(\alpha z) - \alpha(1-z^2)\tanh^{-1}(z)]. \tag{72}$$

All the terms except the last are evidently positive. The last term can be written $g(\alpha z) - \alpha g(z)$ where $g(z) = (1 - z^2)\tanh^{-1}(z)$. Thus, it is necessary to show $g(\alpha z) > \alpha g(z)$. But this follows immediately from the convexity of g, which is trivial as

$$\frac{\mathrm{d}^2 g}{\mathrm{d}z^2} = -2\tanh^{-1} z - \frac{2z}{1-z^2} < 0 \qquad \text{for } 0 < z < 1. \tag{73}$$

□

Proposition 3.4 does a lot to clarify the relationship of lattice KdV (4) with its standard continuum limit $b_t = b_x$ on the one hand, and potential KdV (27) on the other. The linear equation $b_t = b_x$ admits solitons of speed 1, but, since it is linear, the solitons can be of arbitrary amplitude. The indirect method of discretization used has given rise to a *nonlinear* discretization, except when $k = h$ (when (4) can be written as a product of linear factors). The family of speed 1 solitons with arbitrary amplitude is perturbed, after discretization, into a family of solitons with a nontrivial speed–amplitude relation. For small h, k the low amplitude solitons (those with $\sqrt{\theta} \ll 1/h, 1/k$) must have speed close to 1, and indeed this is the case. For larger amplitudes the speeds can change substantially, giving a range of speeds ranging from 1 to either 0 or ∞. Since, now, there are solitons of different speeds, and the necessary algebraic structure has been preserved, the phenomena associated with KdV will emerge, in particular elastic soliton scattering. Thus, from a phenomenological viewpoint, lattice KdV is closer to potential KdV than the linear equation $b_t = b_x$. There are, however, several fundamental differences: first, the range of soliton speeds in lattice KdV is limited to speeds either less than or greater than 1. Second, there are many solutions of lattice KdV that do not have natural continuum limits; for example, solutions (58) in the case where θ exceeds $1/h$ or $1/k$ (or both).

4. Discretizations II: the simplest natural discretization

This section is devoted to the simple discretization (10) of (7), which, as explained in the introduction, should give an integrable lattice equation, which has potential KdV as its *standard* continuum limit. The general solution of (10) is given by (38) on replacing k with $k\lambda$.

Once again, suppose $U_{n,m}(\lambda)$ has a Birkhoff factorization $S_{n,m}^{-1}(\lambda)Y_{n,m}(\lambda)$, and substitute in (10) to get

$$Y_{n+1,m}Y_{n,m}^{-1} = S_{n+1,m}\left[I + h\begin{pmatrix}0 & 1\\ \lambda & 0\end{pmatrix}\right]S_{n,m}^{-1}, \tag{74}$$

$$Y_{n,m+1}Y_{n,m}^{-1} = S_{n,m+1}\left[I + k\lambda\begin{pmatrix}0 & 1\\ \lambda & 0\end{pmatrix}\right]S_{n,m}^{-1}. \tag{75}$$

Writing

$$S_{n,m} = I + \frac{1}{\lambda}\begin{pmatrix}a_{n,m} & -b_{n,m}\\ c_{n,m} & d_{n,m}\end{pmatrix} + \frac{1}{\lambda^2}\begin{pmatrix}\tilde{a}_{n,m} & \tilde{b}_{n,m}\\ \tilde{c}_{n,m} & \tilde{d}_{n,m}\end{pmatrix} + \cdots \tag{76}$$

and employing the relations $\det(Y_{n+1,m}Y_{n,m}^{-1}) = 1 - h^2\lambda$, $\det(Y_{n,m+1}Y_{n,m}^{-1}) = 1 - k^2\lambda^3$ (obtained by left-multiplying (74) and (75), respectively, by $S_{n,m}S_{n+1,m}^{-1}$ and $S_{n,m}S_{n,m+1}^{-1}$, taking the determinant and factorizing) gives

$$Y_{n+1,m} = L_{n,m}Y_{n,m}, \qquad Y_{n,m+1} = M_{n,m}Y_{n,m}, \tag{77}$$

where

$$L_{n,m} = \begin{pmatrix}1 - hb_{n+1,m} & h\\ h\lambda + b_{n,m} - b_{n+1,m} - hb_{n,m}b_{n+1,m} & 1 + hb_{n,m}\end{pmatrix}, \tag{78}$$

$$M_{n,m} = k\begin{pmatrix}-\lambda b_{n,m+1} + \Sigma_{n,m} + \Delta_{n,m} - \beta_{n,m}b_{n,m+1} & \lambda + \beta_{n,m}\\ \lambda^2 - \lambda(\beta_{n,m} + b_{n,m}b_{n,m+1}) - b_{n,m}b_{n,m+1}\beta_{n,m} & \lambda b_{n,m} + \Sigma_{n,m}\\ +\beta_{n,m}^2 + \Delta_{n,m}(b_{n,m} + b_{n,m+1}) + \Sigma_{n,m}(b_{n,m} - b_{n,m+1}) & -\Delta_{n,m} + \beta_{n,m}b_{n,m}\end{pmatrix}. \tag{79}$$

The matrix M depends on three lattice fields β, Δ, Σ in addition to the basic lattice field b, but Σ is determined via the relation

$$\Sigma_{n,m} = \sqrt{\frac{1}{k^2} + \Delta_{n,m}^2 + \beta_{n,m}^3}. \tag{80}$$

Substituting these ansätze into the consistency equation (50) gives the following three equations for the three fundamental fields b, β, Δ:

$$\beta_{n+1,m} + \beta_{n,m} = \frac{b_{n+1,m} + b_{n+1,m+1} - b_{n,m} - b_{n,m+1}}{h} + (b_{n+1,m+1} - b_{nm})(b_{n,m+1} - b_{n+1,m}), \tag{81}$$

$$\Delta_{n+1,m} + \Delta_{n,m} = \left(\frac{\beta_{n+1,m} - \beta_{n,m}}{h}\right)\left(-1 + \frac{h}{2}(b_{n+1,m+1} + b_{n+1,m} - b_{n,m+1} - b_{n,m})\right), \tag{82}$$

$$\left(\frac{\beta_{n+1,m} - \beta_{n,m}}{h}\right)\left(\frac{b_{n,m+1} + b_{n+1,m+1} - b_{n,m} - b_{n+1,m}}{k}\right) = \frac{\sqrt{1 + k^2(\Delta_{n+1,m}^2 + \beta_{n+1,m}^3)} - \sqrt{1 + k^2(\Delta_{n,m}^2 + \beta_{n,m}^3)}}{\frac{1}{2}hk^2}. \tag{83}$$

Note the equations involve b at four points ($b_{n,m}$, $b_{n+1,m}$, $b_{n,m+1}$, $b_{n+1,m+1}$) but β and Δ at only two ($\beta_{n,m}$, $\beta_{n+1,m}$, $\Delta_{n,m}$, $\Delta_{n+1,m}$). The system (81)–(83) will be given the title *full lattice*

KdV; as will shortly be shown, unlike standard lattice KdV, full lattice KdV displays, for certain choices of h and k, solitons with the full range of speeds. Full lattice KdV also has, as expected, potential KdV as a standard continuum limit: replacing $b_{n,m}$ by $b(x,t)$, $b_{n+1,m}$ by $b(x+h,t)$, $b_{n,m+1}$ by $b(x,t+k)$, $b_{n+1,m+1}$ by $b(x+h,t+k)$, and similarly for β and Δ, and then taking the limit $h,k\to 0$, the equations (81)–(83) become

$$2\beta = 2b_x, \qquad 2\Delta = -\beta_x, \qquad 2\beta_x b_t = (\Delta^2+\beta^3)_x. \tag{84}$$

Eliminating β and Δ from these yields potential KdV $b_t = \frac{1}{4}b_{xxx} + \frac{3}{2}b_x^2$.

There are analogues for full lattice KdV of all the results of the previous sections.

Proposition 4.1. *Let $U_{0,0}(\lambda)$ be an element of $LGL_2(\mathbb{C})$; let $S_{n,m}^{-1}(\lambda)Y_{n,m}(\lambda)$ be the Birkhoff decomposition of*

$$U_{n,m}(\lambda) = \left[I + h\begin{pmatrix}0 & 1\\ \lambda & 0\end{pmatrix}\right]^n \left[I + k\lambda\begin{pmatrix}0 & 1\\ \lambda & 0\end{pmatrix}\right]^m U_{0,0}(\lambda), \tag{85}$$

and let $b_{n,m}$ be (-1) times the component of $1/\lambda$ in the $1,2$-entry of $S_{n,m}$. Then, $b_{n,m}$ is a solution, possibly with singularities, of the full lattice KdV system (81)–(83).

By '$b_{n,m}$ is a solution of full lattice KdV,' I mean that there exist fields β,Δ for which equations (81)–(83) hold. In practice, once b is known, the easiest way to determine β,Δ will be directly from equations (81) and (82). In the previous proposition, the other fields can actually be determined from S if this is known in full: if the expansion of S in powers of $1/\lambda$ is as in (76), then

$$\beta_{n,m} = a_{n,m+1} - d_{n,m} - b_{n,m}b_{n,m+1}, \tag{86}$$

$$\begin{aligned}\Delta_{n,m} = \tfrac{1}{2}\big(&\tilde{b}_{n,m+1} + \tilde{b}_{n,m} - c_{n,m+1} - c_{n,m} - b_{n,m}d_{n,m+1} - b_{n,m+1}d_{n,m}\\ &+ (b_{n,m+1}+b_{n,m})(a_{n,m}+a_{n,m+1}-b_{n,m}b_{n,m+1})\big),\end{aligned} \tag{87}$$

$$\begin{aligned}\Sigma_{n,m} = \frac{1}{k} + \frac{1}{2}\big(&\tilde{b}_{n,m+1} - \tilde{b}_{n,m} + c_{n,m+1} - c_{n,m} + b_{n,m}d_{n,m+1} - b_{n,m+1}d_{n,m}\\ &+ (b_{n,m+1}-b_{n,m})(a_{n,m}+a_{n,m+1}-b_{n,m}b_{n,m+1})\big).\end{aligned} \tag{88}$$

The Bäcklund transformation takes the following form.

Proposition 4.2. *If $b_{n,m},\beta_{n,m},\Delta_{n,m}$ is a solution of the full lattice KdV equations (81)–(83) and $\psi_{n,m}$ satisfies*

$$\frac{\psi_{n+2,m} - 2\psi_{n+1,m} + \psi_{n,m}}{h^2} = \theta\psi_{n,m} - \left(\frac{b_{n+2,m}-b_{n,m}}{h}\right)\psi_{n+1,m}, \tag{89}$$

$$\begin{aligned}\frac{\psi_{n,m+1}-\psi_{n,m}}{k} = (\theta+\beta_{n,m})&\frac{\psi_{n+1,m}-\psi_{n,m}}{h}\\ &+\left((b_{n+1,m}-b_{n,m+1})(\theta+\beta_{n,m}) + \Delta_{n,m} + \Sigma_{n,m} - \frac{1}{k}\right)\psi_{n,m},\end{aligned} \tag{90}$$

then, $b^{\text{new}}_{n,m} = b_{n+1,m} + (\psi_{n+1,m}-\psi_{n,m})/(h\psi_{n,m})$ is also a solution of full lattice KdV. The fields β,Δ are replaced by β^{new}, Δ^{new}, respectively, which are given by the following algebraic equations:

$$\beta^{\text{new}}_{n,m} + \beta_{n,m} = \theta + (b_{n,m} - b^{\text{new}}_{n,m+1})(b^{\text{new}}_{n,m} - b_{n,m+1}), \tag{91}$$

$$\Delta^{\text{new}}_{n,m} + \Delta_{n,m} = \tfrac{1}{2}\left(\beta^{\text{new}}_{n,m} - \beta_{n,m}\right)\left(b^{\text{new}}_{n,m+1} + b^{\text{new}}_{n,m} - b_{n,m+1} - b_{n,m}\right). \tag{92}$$

The Bäcklund transformation for continuum KdV (proposition 2.2) is recovered in the limit $h,k\to 0$, since $\beta_{n,m}\to b_x$, $\Delta_{n,m}\to -\frac{1}{2}b_{xx}$, $\Sigma_{n,m} - 1/k \to 0$. Note the difference between the second equation in (89) and (90) and the discrete evolution proposed in [6]. The solutions obtained using the Bäcklund transformation on the vacuum solution $b_{n,m} = \beta_{n,m} = \Delta_{n,m} = 0$ are given by (58) with k replaced by $k\theta$. In particular, writing $t(n, m)$ in place of $\tanh(n \tanh^{-1}(h\sqrt{\theta}) + m \tanh^{-1}(k\theta\sqrt{\theta}) + C)$, it is straightforward to verify that the soliton solution is given by

$$b_{n,m} = \sqrt{\theta}t(n, m), \tag{93}$$

$$\beta_{n,m} = \frac{t(n, m+1) - t(n, m)}{k\sqrt{\theta}}, \tag{94}$$

$$\Delta_{n,m} = \frac{t(n, m+1)^2 - t(n, m)^2}{2k}, \tag{95}$$

$$\Sigma_{n,m} = \frac{1}{k} + \frac{(t(n, m+1) - t(n, m))^2}{2k}. \tag{96}$$

Before exploring the phenomenology of these solitons, note that since the proof of proposition 3.3 is based almost entirely on the first equation of the scalar Lax pair (56) and (57), it is no surprise that it goes through verbatim to full lattice KdV, as given by the following proposition.

Proposition 4.3. *If $b_{n,m}$ is a solution of the full lattice KdV system (81)–(83), and $b^{(1)}_{n,m}$ and $b^{(2)}_{n,m}$ are solutions obtained by applying Bäcklund transformations with parameters θ_1 and θ_2, respectively, to $b_{n,m}$, then,*

$$B_{n,m} = b_{n,m} + \frac{\theta_1 - \theta_2}{b^{(1)}_{n,m} - b^{(2)}_{n,m}} \tag{97}$$

is a solution obtained by applying the two Bäcklund transformations successively to $b_{n,m}$, in either order.

Since the formulae (91) and (92) for applying the Bäcklund transformation to the fields β, Δ are already pure algebraic, there is no need to consider them in proposition 4.3.

It just remains to investigate the speed–amplitude relation of the soliton solutions. The soliton speed is

$$c = \frac{h \tanh^{-1}(k\theta\sqrt{\theta})}{k \tanh^{-1}(h\sqrt{\theta})}, \tag{98}$$

where the range of the parameter θ is limited by the requirements $k\theta\sqrt{\theta}, h\sqrt{\theta} < 1$. Writing $\alpha = hk^{-1/3}$ and $z = \sqrt{\theta}k^{1/3}$ gives

$$c = k^{-2/3}\frac{\alpha \tanh^{-1}(z^3)}{\tanh^{-1}(\alpha z)}. \tag{99}$$

See figure 1. For $\alpha < 1 \Leftrightarrow h^3 < k$ the speed is a monotonic increasing function of z (or θ), going from 0 as $z \to 0$ to ∞ as $z \to 1$. Thus, for this range of parameters the soliton content exactly mirrors that of continuum KdV. For $\alpha = 1 \Leftrightarrow h^3 = k$ the speed is a monotonic increasing function of z (or θ), going from 0 as $z \to 0$ to 1 as $z \to 1$. For $\alpha > 1 \Leftrightarrow h^3 > k$ there is an interesting effect that c increases from 0, reaches a maximum value, and then decreases again to 0 as z approaches $1/\alpha$. Thus, for these choices of h, k there is a limited set of speeds, but for all but the fastest there are solitons of two different amplitudes; furthermore, these can be superposed to give other types of soliton solutions. Note that if our interest in discretizations of KdV were for the purposes of numerical simulation, we would presumably want both h and k small and of the same order of magnitude, and thus be in the $h^3 < k$ regime, where the soliton phenomenology is correct.

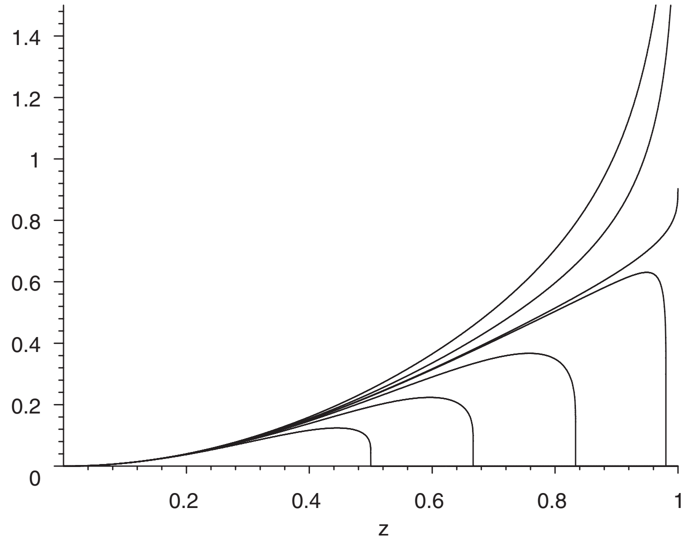

Figure 1. The function $\alpha \tanh^{-1}(z^3)/\tanh^{-1}(\alpha z)$ for $\alpha = 0.1, 0.8, 1, 1.02, 1.2, 1.5, 2.0$ (from top to bottom).

5. Discretizations III: a second-order discretization

In this section our method is applied to the discretization (11) and (12) of (8) with $N = 1$. The resulting system is of limited intrinsic interest, the main point here is to illustrate that our methods can, in principle, be extended to give a whole range of integrable discretizations of equations in the KdV hierarchy. One interesting point that emerges is the form of the related discretization of the Schrödinger equation.

Following the usual procedure, assuming $U_{n,m}(\lambda)$ has a Birkhoff decomposition $S_{n,m}^{-1}(\lambda)Y_{n,m}(\lambda)$ and writing

$$S_{n,m} = I + \frac{1}{\lambda}\begin{pmatrix} a_{n,m} & -b_{n,m} \\ c_{n,m} & d_{n,m} \end{pmatrix} + \cdots \tag{100}$$

and so on, gives the system

$$Y_{n+1,m} = L_{n,m}Y_{n,m}, \qquad Y_{n,m+1} = M_{n,m}Y_{n,m}, \tag{101}$$

where

$$L_{n,m} = \begin{pmatrix} 1 + \frac{1}{2}h^2\lambda - hb_{n+1,m} + \frac{1}{2}h^2(a_{n+1,m} - a_{n,m}) & h + \frac{1}{2}h^2(b_{n,m} - b_{n+1,m}) \\ h\lambda + b_{n,m} - b_{n+1,m} - h\Delta & 1 + \frac{1}{2}h^2\lambda + hb_{n,m} + \frac{1}{2}h^2(a_{n,m} - a_{n+1,m}) \end{pmatrix}, \tag{102}$$

$$M_{n,m} = \begin{pmatrix} 1 - kb_{n,m+1} & k \\ k\lambda + b_{n,m} - b_{n,m+1} - kb_{n,m}b_{n,m+1} & 1 + kb_{n,m} \end{pmatrix} \tag{103}$$

and

$$\Delta = \frac{\frac{1}{2}(b_{n,m}^2 + b_{n+1,m}^2) - \frac{1}{2}h(a_{n+1,m} - a_{n,m})(b_{n+1,m} + b_{n,m}) + \frac{1}{4}h^2(a_{n,m} - a_{n+1,m})^2}{1 + \frac{1}{2}h(b_{n,m} - b_{n+1,m})}. \tag{104}$$

The consistency condition $L_{n,m+1}M_{n,m} = M_{n+1,m}L_{n,m}$ unravels into two equations for the fields a, b. Introducing the combinations

$$\Sigma_{n,m} = \tfrac{1}{2}(a_{n+1,m+1} - a_{n,m+1} + a_{n+1,m} - a_{n,m}), \tag{105}$$

$$\Delta_{n,m} = \tfrac{1}{2}(a_{n+1,m+1} - a_{n,m+1} - a_{n+1,m} + a_{n,m}), \tag{106}$$

the equations can be written as

$$\Sigma_{n,m} = \frac{1}{2}\left(\frac{b_{n+1,m+1} - b_{n,m+1} - b_{n+1,m} + b_{n,m}}{k} + b_{n+1,m+1}b_{n+1,m} - b_{n,m+1}b_{n,m}\right), \tag{107}$$

$$\begin{aligned}
0 = &\left(\frac{h\Delta_{n,m}}{2} + \frac{1}{k}\right)(b_{n+1,m+1} + b_{n,m+1} - b_{n+1,m} - b_{n,m}) \\
&+ \left(\frac{h^2\Delta_{n,m}}{4k} - \frac{1}{h}\right)(b_{n+1,m+1} - b_{n,m+1} + b_{n+1,m} - b_{n,m}) \\
&+ \frac{h^2}{4}\left(\Delta_{n,m}(b_{n+1,m+1}b_{n,m} - b_{n+1,m}b_{n,m+1}) - \Delta_{n,m}^2 - \Sigma_{n,m}^2\right) - \frac{h\Delta_{n,m}}{k} \\
&+ \frac{h}{4k}(b_{n,m+1}^2 + b_{n+1,m}^2 - b_{n,m}^2 - b_{n+1,m+1}^2 + 4b_{n+1,m+1}b_{n,m} - 4b_{n+1,m}b_{n,m+1}) \\
&+ \frac{h^2}{8k}\begin{pmatrix} b_{n+1,m+1}^2 b_{n+1,m} - b_{n,m}^2 b_{n,m+1} + b_{n,m+1}^2 b_{n,m} - b_{n+1,m+1}^2 b_{n,m} \\ -b_{n+1,m}^2 b_{n+1,m+1} + b_{n+1,m}^2 b_{n,m+1} - b_{n,m+1}^2 b_{n+1,m} + b_{n+1,m+1} b_{n,m}^2 \end{pmatrix} \\
&+ \frac{1}{2}\begin{pmatrix} 3b_{n+1,m}b_{n+1,m+1} + 3b_{n,m}b_{n,m+1} - b_{n+1,m+1}b_{n,m} \\ -2b_{n,m}b_{n+1,m} - b_{n+1,m}b_{n,m+1} - 2b_{n+1,m+1}b_{n,m+1} \end{pmatrix} \\
&+ \frac{h}{4}\begin{pmatrix} 3b_{n+1,m+1}b_{n+1,m}b_{n,m+1} + 3b_{n+1,m}b_{n+1,m+1}b_{n,m} \\ -3b_{n,m+1}b_{n,m}b_{n+1,m+1} - 3b_{n+1,m}b_{n,m}b_{n,m+1} \\ -b_{n+1,m+1}^2 b_{n+1,m} - b_{n+1,m}^2 b_{n+1,m+1} + b_{n,m}^2 b_{n,m+1} + b_{n,m+1}^2 b_{n,m} \end{pmatrix} \\
&+ \frac{h^2}{8}\begin{pmatrix} b_{n,m+1}^2 b_{n,m}^2 + b_{n+1,m}^2 b_{n+1,m+1}^2 + 2b_{n+1,m}b_{n+1,m+1}b_{n,m}b_{n,m+1} \\ -b_{n+1,m}^2 b_{n+1,m+1}b_{n,m+1} - b_{n+1,m+1}b_{n,m}^2 b_{n,m+1} \\ -b_{n,m+1}^2 b_{n,m}b_{n+1,m} - b_{n+1,m}b_{n+1,m+1}^2 b_{n,m} \end{pmatrix}.
\end{aligned} \tag{108}$$

Since the field a only appears in the equations through the combinations Δ and Σ, which only depend on a through differences, solutions of this system are only defined up to addition of a constant to a. The analogue of propositions 2.1, 3.1 and 4.1 is

Proposition 5.1. *Let $U_{0,0}(\lambda)$ be an element of $LGL_2(\mathbb{C})$, let $S_{n,m}^{-1}(\lambda)Y_{n,m}(\lambda)$ be the Birkhoff decomposition of*

$$U_{n,m}(\lambda) = \begin{pmatrix} 1 + \frac{h^2\lambda}{2} & h \\ h\lambda & 1 + \frac{h^2\lambda}{2} \end{pmatrix}^n \begin{pmatrix} 1 & k \\ k\lambda & 1 \end{pmatrix}^m U_{0,0}(\lambda) \tag{109}$$

and let $a_{n,m}$ and $b_{n,m}$ be, respectively, the $1,1$-entry and (-1) times the $1,2$-entry in the $1/\lambda$ component of $S_{n,m}$. Then, $a_{n,m}, b_{n,m}$ is a solution, possibly with singularities, of the system (107) and (108).

The system (107) and (108), despite its algebraic complexity, is an integrable discretization of the equation $b_t = b_x$ in every sense that lattice KdV is. The soliton solutions are given as follows.

Proposition 5.2. *The system (107) and (108) has soliton solutions*

$$b_{n,m} = \sqrt{\theta}\tanh\left(n\tanh^{-1}\left(\frac{h\sqrt{\theta}}{1+\frac{1}{2}h^2\theta}\right) + m\tanh^{-1}(k\sqrt{\theta}) + C\right), \tag{110}$$

$$a_{n,m} = \text{constant}, \tag{111}$$

with speed

$$c = \frac{h\tanh^{-1}(k\sqrt{\theta})}{k\tanh^{-1}(h\sqrt{\theta}/(1+\frac{1}{2}h^2\theta))}. \tag{112}$$

In greater generality, it can be shown that if instead of equation (11) a pth order approximation

$$U_{n+1,m} = \left[\sum_{i=0}^{p}\frac{h^i}{i!}\begin{pmatrix}0 & 1\\ \lambda & 0\end{pmatrix}^i\right]U_{n,m} \tag{113}$$

is used, then the speed of the soliton solution is

$$c = \frac{h\tanh^{-1}(k\sqrt{\theta})}{k\tanh^{-1}(s_p(h\sqrt{\theta})/c_p(h\sqrt{\theta}))}, \tag{114}$$

where $c_p(x)$ and $s_p(x)$ are, respectively, the order p truncations of the Taylor series for $\cosh(x)$ and $\sinh(x)$ (ignoring terms of order x^{p+1} and higher). It is straightforward to verify that for small x

$$\frac{1}{x}\tanh^{-1}\left(\frac{s_p(x)}{c_p(x)}\right) = \begin{cases}1+\mathrm{O}(x^p) & x \text{ even},\\ 1+\mathrm{O}(x^{p+1}) & x \text{ odd}.\end{cases} \tag{115}$$

Thus, for small h, the dependence of the soliton speed on h becomes weaker as p increases. Likewise, the order of accuracy in k can be increased. (The distinction between the even and odd cases in (115), that for odd p there is a 'free' extra order of magnitude accuracy, means that (107) and (108), for which $p = 2$, is actually no more accurate in this regard than standard lattice KdV, with $p = 1$. The equations obtained from $p = 3$ can be written down, but due to their length I have restricted the discussion to the $p = 2$ case.)

Returning to the formula (112), note that if $v = k\sqrt{\theta}$, $w = h\sqrt{\theta}$, then,

$$c = \frac{(1/v)\tanh^{-1}v}{(1/w)\tanh^{-1}(w/(1+\frac{1}{2}w^2))}. \tag{116}$$

The function in the numerator increases monotonically from 1 to ∞ as v goes from 0 to 1. The function in the denominator decreases monotonically from 1 to 0 as v goes from 0 to ∞. Thus, for the current discretization c can only take values greater than 1.

The soliton solutions just presented can be found using the Bäcklund transformation, which is obtained as in previous sections.

Proposition 5.3. *If $b_{n,m}$, $a_{n,m}$ is a solution of (107) and (108) and $\psi_{n,m}$ satisfies*

$$\begin{aligned}\frac{\psi_{n+2,m} - 2\psi_{n+1,m} + \psi_{n,m}}{h^2} &= \theta\psi_{n+1,m}\left(1 + \frac{h}{4}(b_{n,m} - b_{n+2,m})\right)\\
&+ \frac{(b_{n+1,m} - b_{n,m})\psi_{n+2,m} - 3(b_{n+2,m} - b_{n,m})\psi_{n+1,m} + (b_{n+2,m} - b_{n+1,m})\psi_{n,m}}{2h}\\
&- \frac{h^2\theta^2}{4}\left(1 + \frac{h(b_{n+1,m} - b_{n+2,m})}{2}\right)\psi_{n,m} + \frac{a_{n+2,m} - 2a_{n+1,m} + a_{n,m}}{2}\psi_{n+1,m}\\
&+ \frac{(b_{n+2,m} + b_{n,m})b_{n+1,m} - 2b_{n,m}b_{n+2,m}}{2}\psi_{n+1,m}\\
&+ \frac{h}{4}\begin{pmatrix}a_{n,m}b_{n+1,m} + a_{n+1,m}b_{n+2,m} + a_{n+2,m}b_{n,m}\\ -a_{n,m}b_{n+2,m} - a_{n+1,m}b_{n,m} - a_{n+2,m}b_{n+1,m}\end{pmatrix}\psi_{n+1,m},\end{aligned} \tag{117}$$

$$\frac{\psi_{n,m+1}-\psi_{n,m}}{k} = \frac{1}{1+\frac{1}{2}h(b_{n,m}-b_{n+1,m})}\left(\frac{\psi_{n+1,m}-\psi_{n,m}}{h} + (b_{n+1,m}-b_{n,m+1})\psi_{n,m}\right.$$
$$\left. - \frac{h}{2}(\theta + a_{n+1,m} - a_{n,m} + b_{n,m+1}(b_{n,m}-b_{n+1,m}))\psi_{n,m}\right), \qquad (118)$$

then,

$$b^{\mathrm{new}}_{n,m} = \frac{b_{n+1,m} + (\psi_{n+1,m}-\psi_{n,m})/(h\psi_{n,m}) + \frac{1}{2}h^2(a_{n,m}-a_{n+1,m}-\theta)}{1+\frac{1}{2}h(b_{n,m}-b_{n+1,m})}, \qquad (119)$$

$$a^{\mathrm{new}}_{n,m} = b_{n,m}b^{\mathrm{new}}_{n,m} - a_{n,m} + \text{constant} \qquad (120)$$

is also a solution of (107) and (108).

All formulae in the previous proposition have been written in a manner that hopefully makes it clear in what sense they are modifications of the corresponding formulae in proposition 3.2. The surprising feature of the discretization of the Schrödinger equation in proposition 3.2, equation (56), is that in it the parameter θ multiplies $\psi_{n,m}$, not $\psi_{n+1,m}$, which would seem more natural. The new discretization just presented, equation (107), has θ multiplying $\psi_{n+1,m}$. But the cost of this is the introduction of many new terms, including a term proportional to θ^2, multiplying $\psi_{n,m}$. It can be checked that the new discretization (107) is a second-order approximation to the Schrödinger equation, while (56) is only first order. This is the justification for the title of this section.

6. Concluding remarks

In this paper, I have presented a systematic approach towards integrable discretizations, based on the loop group approach to integrable systems. Three integrable discretizations have been examined in detail, one known, the lattice KdV system of Nijhoff *et al*, and two new, one of which I have called *full lattice KdV*, as it would seem to be the first discrete integrable system with (potential) KdV as its standard continuum limit. For each integrable discretization, a Bäcklund transformation has been given and soliton solutions have been analysed. Unlike the lattice KdV system of Nijhoff *et al*, which only displays solitons with speeds below, above or equal to 1, full lattice KdV has the full range of soliton speeds (for suitable choices of h and k).

Full lattice KdV would seem to merit further attention. Our plans include conducting numerical studies, and trying to work out a suitable inverse scattering formalism. Another issue that has not been touched upon in this paper is the subject of tau functions for discretizations. The linear flows on a loop group that underlie KdV can be extended to the central extension of the group, and one would expect the same to be true for the discretizations looked at in this paper.

The formalism developed here can also be extended to look at integrable discretizations of KdV on non-rectangular lattices, see [14].

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