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The Camassa Holm Equation: Conserved Quantities and the Initial Value Problem

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Abstract

Using a Miura-Gardner-Kruskal type construction, we show that the Camassa-Holm equation has an infinite number of *local* conserved quantities. We explore the implications of these conserved quantities for global well-posedness.

1. Introduction.

Much interest has been developing in the Camassa-Holm (CH) equation

$$m_t = -2mu_x - m_x u \ , \qquad m = u - u_{xx} \ . \tag{1}$$

This equation first appeared in work of Fuchssteiner and Fokas [1] as an example of a bihamiltonian system, but more recently it was rediscovered by Camassa and Holm [2] as a model for shallow water waves. In addition, Misiołek [3] has shown that it describes a geodesic flow on the group Diff(S^1). In these and other regards, the CH equation has much in common with the KdV equation, but there are also a number of significant differences. In particular, when considered as an evolution equation on a suitable Sobolev space, KdV is *globally well-posed*, while CH is in general not [2, 4, 5]; the first derivative of a solution of CH can become infinite in finite time.

In this paper we present analogs for CH of two pieces of KdV theory. The first concerns the construction of the conserved quantities. Bihamiltonian structure implies that CH has an infinite number of conserved quantities. Explicitly, (1) can be written either in the form

$$
\begin{aligned}
m_t &= -B_1 \frac{\delta H_2}{\delta m} \ , \qquad (2)\\
B_1 = \partial_x - \partial_x^3 \ , &\qquad H_2 = \frac{1}{2}\int (u^3 + u u_x^2)\ dx \ ,
\end{aligned}
$$

or in the form

$$
\begin{aligned}
m_t &= -B_2 \frac{\delta H_1}{\delta m} \ , \qquad (3)\\
B_2 = \partial_x m + m\partial_x \ , &\qquad H_1 = \frac{1}{2}\int (u^2 + u_x^2)\ dx \ .
\end{aligned}
$$

Since B_1, B_2 are a *hamiltonian pair* [6], the bi-infinite sequence of functionals $\ldots, H_{-1}, H_0, H_1, \ldots$ defined by

$$
B_2 \frac{\delta H_n}{\delta m} = B_1 \frac{\delta H_{n+1}}{\delta m} \ , \qquad n \in \mathbf{Z} \ , \tag{4}
$$

are conserved quantities in involution with respect to the Poisson brackets determined by either B_1 or B_2. In [2], formulae were given for H_0, H_{-1}, H_{-2}, viz.

$$
H_0 = \int m\ dx \ , \qquad H_{-1} = \int \sqrt{m} dx \ , \qquad H_{-2} = -\frac{1}{4}\int \left(\frac{m_x^2}{4m^{5/2}} + \frac{1}{\sqrt{m}} \right)\ dx \tag{5}
$$

(there is a typo in the coefficients of H_{-2} in [2]). Unfortunately, however, the need to invert either B_1 or B_2 each time (4) is used makes it very hard to use this to generate further explicit formulae for the H_n, or to prove anything about them. The first result of this paper is an alternative derivation of a bi-infinite sequence of conserved quantities for CH, which we believe, but do not prove, to be equivalent to the H_n. The advantage of our method is that it shows directly that half of the conserved quantities (those we believe equivalent to H_{-n} for $n > 0$) are *local*, i.e. integrals of some function of the fields m and their x-derivatives. The existence of two constructions for the conserved quantities is familiar from KdV theory; for KdV the bihamiltonian structure gives the Lenard recursion [7], but locality is much easier to show via the Miura-Green-Kruskal (MGK) construction [8]. The new derivation we give for a CH is precisely an analog of MGK for KdV.

Our second result concerns the relevance of the conserved quantities for the initial value problem. For KdV the relevant results are due to Lax [9], who showed, for the periodic problem, that the conserved quantities bound Sobolev norms. We give a similar result for CH, restricted, however, to solutions with $m > 0$. In this case global well-posedness has been proved for CH [4].

One more point needs to be explained in this introduction. In most of the paper we do not work directly with the CH equation, but rather with the *associated Camassa-Holm* (ACH) equation, introduced in [10] and related to CH by a change of coordinates (see also [11] and [12]; in the latter work CH is actually related to the KdV hierarchy by a similar change of coordinates). In section 2 we describe the relationship between CH and ACH, so that the results proved for ACH can be immediately translated into results for CH. In section 3, we give the MGK-type construction for ACH, proving there exist an infinite number of local conserved quantities of ACH and hence also of CH. And in section 4, we show how these conserved quantities bound certain norms for certain classes of solutions of ACH, and hence also for CH.

2. CH and ACH

We concentrate our attention on two types of solution of CH, (1) solutions with $m > 0$ satisfying $m \to h^2$ as $|x| \to \infty$, where h is a positive constant, and (2) solutions with $m > 0$, and u, m periodic in x, with period independent of t. In general when $m > 0$ we can define $p = \sqrt{m}$, and the first equation of (1) becomes $p_t = -(up)_x$. This implies we can define new coordinates t_0, t_1 via

$$dt_0 = p\ dx - pu\ dt\ , \qquad dt_1 = dt\ . \tag{6}$$

More precisely, these define t_0, t_1 up to translations; choosing the origin to coincide with the origin of x, t coordinates we have

$$t_0 = \int_0^x p(x', t)\ dx' - \int_0^t u(0, t')p(0, t')\ dt'\ , \qquad t_1 = t\ . \tag{7}$$

Transforming to the new coordinates gives the associated Camassa-Holm (ACH) equation:

$$\dot{p} = -p^2 u', \qquad u = p^2 - p\left(\frac{\dot{p}}{p}\right)'\ . \tag{8}$$

Here a prime denotes differentiation with respect to t_0, and a dot differentiation with respect to t_1. The two classes of solutions of CH introduced above correspond, respectively, to (1) solutions of ACH with $p > 0$, satisfying $p \to h$ as $|t_0| \to \infty$, and (2) solutions of ACH with $p > 0$ and p, u periodic in t_0, with period independent of t_1. In the latter case, a solution of CH with period T corresponds to a solution of ACH with period $S = \int_0^T p(x, t)\ dx$ (which is independent of t), and in the opposite direction a solution of ACH with period S corresponds to a solution of CH with period $T = \int_0^S (1/p(t_0, t_1))\ dt_0$ (which is independent of t_1).

Evidently, a solution of CH in one of the two classes under consideration exists for all t if and only if the corresponding solution of ACH does. There is also a correspondence between conserved quantities of the two equations (we thank Andy Hone for explaining this to us, citing it as a result of Rogers, see [13]). Suppose $\mathcal{X}, \mathcal{T}$ are functions of m, u and their x, t derivatives, such that $\mathcal{X}_t = \mathcal{T}_x$ follows from (1). Then $\int \mathcal{X}\ dx$ is a conserved quantity of CH. Using the relations $\partial_x = p\partial_{t_0}$ and $\partial_t = \partial_{t_1} - pu\partial_{t_0}$, $\mathcal{X}$ and $\mathcal{T}$ can be rewritten as functions of p, u and their t_0, t_1 derivatives, and it is straightforward to check that $\partial_{t_1}(\mathcal{X}/p) = \partial_{t_0}(\mathcal{T} + u\mathcal{X})$. Thus $\int (\mathcal{X}/p)\ dt_0$ is a conserved quantity of ACH. This procedure can be reversed, and we obtain a correspondence between conserved quantities of the two equations. We will call a conserved quantity of CH (ACH) *local* if it is of the form $\int \mathcal{X}\ dx$ ($\int \tilde{\mathcal{X}}\ dt_0$), where $\mathcal{X}$ ($\tilde{\mathcal{X}}$) is a function of m (p) and its x (t_0) derivatives alone. The general correspondence just described can be checked to reduce to a correspondence of local conserved quantities.

The results we have just presented allow us to study ACH instead of CH. From the form (8) of ACH it is not clear in what sense this is an evolution equation, but it turns out that when we restrict to either of the classes of solutions introduced above it can be written in a simple evolutionary form. To see this we first eliminate u from (8). Gathering all the terms with t_1 derivatives on one side, the resulting equation can be written

$$\left(\partial_{t_0}^2 - \frac{p''}{2p} + \frac{p'^2}{4p^2} - \frac{1}{p^2}\right)\frac{\dot{p}}{\sqrt{p}} = 2\sqrt{p}\ p'\ . \tag{9}$$

To write this in evolutionary form (i.e. to have an explicit expression for $\dot{p}$) we need to solve the second order ordinary differential equation

$$\left(\partial_{t_0}^2 - \frac{p''}{2p} + \frac{p'^2}{4p^2} - \frac{1}{p^2}\right) y = f \tag{10}$$

(here f, p are given and y is the unknown). For class (1) of solutions, $f = 2\sqrt{p}\, p' \to 0$ as $|t_0| \to \infty$, and we need a (hopefully unique) solution y obeying a similar condition. Similarly, for class (2) of solutions, f is periodic, and we need a (hopefully unique) periodic solution y. Remarkably, for arbitrary p it is possible to explicitly solve the homogeneous problem corresponding to (10) (i.e. the case $f = 0$), and thence by standard methods solve the two inhomogeneous problems. Explicitly, in the homogeneous case, (10) has two linearly independent solutions

$$y_\pm(t_0) = \sqrt{p(t_0)} \exp\left(\pm \int^{t_0} \frac{ds_0}{p(s_0)}\right) , \tag{11}$$

and the two evolutionary forms of ACH are given by

$$\dot{p}(t_0, t_1) = -\int_{-\infty}^{\infty} p(t_0, t_1) p(s_0, t_1) p'(s_0, t_1) \exp\left(-\left|\int_{s_0}^{t_0} \frac{du_0}{p(u_0, t_1)}\right|\right) ds_0 , \tag{12}$$

$$\dot{p}(t_0, t_1) = -\int_0^S p(t_0, t_1) p(s_0, t_1) p'(s_0, t_1) \frac{\cosh\left(\frac{1}{2}\int_0^S \frac{du_0}{p(u_0,t_1)} - \left|\int_{s_0}^{t_0 - \left[\frac{t_0}{S}\right] S} \frac{du_0}{p(u_0,t_1)}\right|\right)}{\sinh\left(\frac{1}{2}\int_0^S \frac{du_0}{p(u_0,t_1)}\right)} ds_0 \tag{13}$$

respectively. In the latter, S denotes the period of p as a function of t_0, and $\left[\frac{t_0}{S}\right]$ denotes the largest integer not exceeding $\frac{t_0}{S}$; it is an enjoyable exercise to check that the expression $\cosh(\ldots)$ appearing in (13) is continuous when t_0 is an integer multiple of S, for any s_0.

3. The MGK construction for ACH

It was shown in [10] that (8) has a strong Bäcklund transformation

$$p \to p - 2s', \qquad u \to u + \frac{2\dot{s}}{p(p - 2s')} , \tag{14}$$

where s satisfies

$$s' = -\frac{s^2}{2p\lambda} + \frac{\lambda}{2p} + \frac{p}{2} , \tag{15}$$

$$\dot{s} = -s^2 + \frac{s\dot{p}}{p} + \lambda(\lambda + u) \tag{16}$$

(λ is a parameter). It is straightforward to check that

$$\dot{\left(\frac{s}{p}\right)} = \lambda \left(2s - \frac{\dot{p}}{p}\right)' , \tag{17}$$

from which we deduce that $\int (s/p)\, dt_0$ is a conserved quantity. But s is dependent on λ; so if we can find a consistent expansion of s in powers of λ, then in fact each term in the expansion of $\int (s/p)\, dt_0$ will yield a conserved quantity. Two such consistent expansions of s are:

$$s = \sum_{n=1}^{\infty} s_n \lambda^{\frac{n}{2}} \qquad \text{with } s_1 = p \tag{18}$$

and

$$s = \sum_{n=0}^{\infty} r_n \lambda^{1-n} \qquad \text{with } r_0 = 1 \; . \tag{19}$$

Using (18) and substituting in (15) gives the recursion

$$s_{n+1} = -s_n' + \frac{1}{2p}\left(\delta_{n2} - \sum_{i=0}^{n-2} s_{i+2}s_{n-i}\right) \qquad n = 1, 2, \ldots \quad . \tag{20}$$

With the aid of a symbolic manipulator it is easy to compute the first few of the s_n; we find that s_n/p is a total derivative for n even, but that for n odd we obtain nontrivial conserved quantities (both these statements can easily be proved). Writing $K_n = \int (s_{2n-1}/p)\; dt_0$, $n = 2, 3, \ldots$, and integrating by parts to reduce the order of derivatives appearing, we find (up to unimportant overall constants):

$$\begin{aligned}
K_2 &= \int \left(\frac{p'^2}{p^2} + \frac{1}{p^2}\right) dt_0 \; , \\
K_3 &= \int \left(4\frac{p''^2}{p^2} - 3\frac{p'^4}{p^4} + 10\frac{p'^2}{p^4} + \frac{1}{p^4}\right) dt_0 \; , \\
K_4 &= \int \left(24\frac{p'''^2}{p^2} + 72\frac{p''^3}{p^3} - 228\frac{p'^2 p''^2}{p^4} + 135\frac{p'^6}{p^6} + 84\frac{p''^2}{p^4} - 259\frac{p'^4}{p^6} + 105\frac{p'^2}{p^6} + \frac{3}{p^6}\right) dt_0 \; .
\end{aligned} \tag{21}$$

Following the procedure of converting conserved quantities of ACH to those of CH, as described in section 2 , it can be seen that K_2 gives H_{-2} as given in (5) up to an overall constant. Similarly, lengthy calculations show that the conserved quantity of CH determined by K_3 is actually H_{-3}, up to an overall constant; the explicit form of H_{-3} is

$$H_{-3} = \frac{1}{8}\int \left(\frac{m_{xx}^2}{m^{7/2}} - \frac{35}{16}\frac{m_x^4}{m^{11/2}} + \frac{5}{2}\frac{m_x^2}{m^{7/2}} + \frac{1}{m^{3/2}}\right) dx \; . \tag{22}$$

We conjecture that in fact K_n is equivalent to H_{-n} for all $n = 2, 3, \ldots$ (up to multiplication by overall constants). In any case, it is clear that the K_n are local, and therefore so are the corresponding conserved quantities of CH.

Using now the series (19) for s and substituting in (16) gives the recursion

$$r_{n+1} = \frac{1}{2}\left(u\delta_{n0} - \dot{r}_n + \frac{r_n \dot{p}}{p} - \sum_{i=0}^{n-1} r_{i+1}r_{n-i}\right) \qquad n = 0, 1, \ldots \quad . \tag{23}$$

Explicit expressions for the first few r_n are:

$$\begin{aligned}
r_1 &= \frac{1}{2}\left(u + \frac{\dot{p}}{p}\right) \; , \\
r_2 &= \frac{1}{8}\left(\left(\frac{\dot{p}}{p}\right)^2 - 2\dot{u} - 2\left(\frac{\dot{p}}{p}\right)^{\cdot} - u^2\right) \; , \\
r_3 &= \frac{1}{16}\left(2\left(\frac{\dot{p}}{p}\right)^{\cdot\cdot} + 2\left(u - \left(\frac{\dot{p}}{p}\right)\right)\left(\frac{\dot{p}}{p}\right)^{\cdot} - u\left(\frac{\dot{p}}{p}\right)^2 + 2\ddot{u} + 4u\dot{u} + u^3\right) \; .
\end{aligned} \tag{24}$$

The resulting conserved quantities $\int (r_n/p)\ dt_0$ are in general not local, since these expressions involve t_1-derivatives. r_0 gives rise to the conserved quantity $T = \int (1/p)\ dt_0$ of ACH mentioned in section 2. Lengthy calculations, not reproduced here, show that the conserved quantities of CH determined by r_1, r_2, r_3 are H_0, H_1, H_2 respectively (up to multiplication by overall constants). We conjecture that r_n gives rise to the conserved quantity H_{n-1} of CH for all $n > 0$.

In the next section we will use the first series of conserved quantities, and it will be useful to have a number of facts about them available. The following results are easy to prove:

1. For $n > 1$, s_n is a polynomial in $\frac{1}{p}, p', p'', \ldots$; s_n is odd under $p \to -p$.
2. Assigning weight 1 to $\frac{1}{p}$ and weight $n-1$ to $p^{(n)}$, s_n is a sum of terms of weight $n-2$.
3. Each term in s_n contains at most $n-1$ derivatives (so, for example, in s_7 a $(p''')^2/p$ term is allowed, but not a $(p''')^2(p')^2/p$ term).

These results concern s_n. In constructing the conserved quantities $K_n = \int (s_{2n-1}/p)\ dt_0$ $(n \geq 2)$ we are allowed to integrate by parts to reduce the order of derivatives appearing. In general, we can continue to do this until the highest derivative appearing in any term appears nonlinearly. After this, we have $K_n = \int \mathcal{K}_n\ dt_0$, where:

1. $\mathcal{K}_n$ is a polynomial in $\frac{1}{p}, p', p'', \ldots$, even under $p \to -p$ and divisible by $\frac{1}{p^2}$.
2. Assigning weight 1 to $\frac{1}{p}$ and weight $n-1$ to $p^{(n)}$, $\mathcal{K}_n$ is a sum of terms of weight $2n-2$.
3. Each term in $\mathcal{K}_n$ contains at most $2n-2$ derivatives, and in each term the highest derivative appears nonlinearly.

It follows that the highest order derivative appearing in $\mathcal{K}_n$ is $p^{(n-1)}$, and this appears only in a term proportional to $(p^{(n-1)})^2/p^2$. We will assume in what follows that the coefficient of this term is always nonzero, in which case we can without loss of generality take it to be positive. $\mathcal{K}_n$ also has a term with no derivatives, proportional to $1/p^{2n-2}$. Since this term is not affected in any way by the procedure of integration by parts, it also appears in s_{2n-1}/p, and from the recursion (20) it is possible to show its coefficient is nonzero. This guarantees the nontriviality of the series of conserved quantities. But it also implies that the K_n as we have defined them are actually infinite in the case of solutions with $p \to h$ as $|t_0| \to \infty$. This can be rectified by adding a suitable constant to $\mathcal{K}_n$; for example K_2 should be modified to $\int_{-\infty}^{\infty} \left(\frac{p'^2}{p^2} + \frac{1}{p^2} - \frac{1}{h^2} \right)\ dt_0$. Similarly the conserved quantity T should be modified to $\int_{-\infty}^{\infty} \left(\frac{1}{p} - \frac{1}{h} \right)\ dt_0$.

4. Bounds on norms from the conserved quantities

In this section we consider only the periodic case of ACH. By rescaling p, t_0, t_1 we can without loss of generality take $S = 1$. The $p > 0$ condition is awkward to work with, so we eliminate it via the substitution $p(t_0, t_1) = e^{v(t_0, t_1)}$. The evolution equation (13) becomes

$$\dot{v}(t_0, t_1) = -\int_0^1 e^{2v(s_0,t_1)} v'(s_0, t_1) \frac{\cosh\left(\frac{1}{2} \int_0^1 e^{-v(u_0,t_1)}\ du_0 - \left| \int_{s_0}^{t_0 - [t_0]} e^{-v(u_0,t_1)}\ du_0 \right| \right)}{\sinh\left(\frac{1}{2} \int_0^1 e^{-v(u_0,t_1)}\ du_0 \right)}\ ds_0\ . \quad (25)$$

Usual contraction mapping methods can be used to prove the local existence of solutions for this equation. For convenience we write out the first few local conserved quantities in terms of the new field v:

$$
\begin{aligned}
T &= \int_0^1 e^{-v}\ dt_0\ , \\
K_2 &= \int_0^1 \Big((v')^2 + e^{-2v} \Big)\ dt_0\ , \\
K_3 &= \int_0^1 \Big(4(v'')^2 + (v')^4 + 10e^{-2v}(v')^2 + e^{-4v} \Big)\ dt_0\ , \\
K_4 &= \int_0^1 \Big(24(v''')^2 + 60(v')^2(v'')^2 + 3(v')^6 \\
&\qquad + e^{-2v}\Big(84(v'')^2 - 63(v')^4\Big) + 105e^{-4v}(v')^2 + 3e^{-6v} \Big)\ dt_0\ .
\end{aligned}
\tag{26}
$$

To obtain these formulas we have performed some integration by parts, so that once again all terms in the densities have their highest derivative appearing nonlinearly. A direct check that T is conserved under the flow (25) is a long but ultimately rewarding exercise. The following analog of Lax's theorem now holds:

Theorem: Let v be a smooth function of period 1, and n a positive integer; the quantities

$$
\max\{|v(t_0)|, |v'(t_0)|, \ldots, |v^{(n-1)}(t_0)|\}, \qquad \int (v^{(n)}(t_0))^2\ dt_0
$$

can be bounded in terms of $T, K_2, \ldots, K_{n+1}$.

Proof: The cases $n = 1, 2$ needs to be checked individually; for larger n we can proceed by induction, exactly as in Lax [9], because for $n \geq 4$ the density of K_n is at most quadratic not only in $v^{(n-1)}$ but also in $v^{(n-2)}$. (For K_4 this is evident from the explicit formula, but is fortuitous because there are no *a priori* reasons to exclude a term proportional to $(v'')^3$; for $n > 4$ it follows since all terms in the density of K_n can have at most $2n - 2$ derivatives.)

For $n = 1$, writing $Q = \int_0^1 (v')^2\ dt_0$, we evidently have $Q < K_2$. Also, since for any t_0, s_0

$$
v(t_0) = v(s_0) + \int_{s_0}^{t_0} v'(u_0)\ du_0\ ,
$$

it follows from the Schwarz inequality that

$$
(v(t_0))^2 \leq 2\Big((v(s_0))^2 + Q \Big)\ .
$$

Taking s_0 to be any point such that $e^{-v(s_0)} = \int_0^1 e^{-v(u_0)}\ du_0 = T$ (such a point exists by the mean value theorem), we have at once that for all t_0

$$
(v(t_0))^2 \leq 2\Big((\ln T)^2 + Q \Big) \leq 2\Big((\ln T)^2 + K_2 \Big)\ .
$$

This gives a bound for $\max\{|v(t_0)|\}$. Moving now to $n = 2$, again it is clear that $Q_2 \equiv \int_0^1 (v'')^2\ dt_0 \leq K_3/4$, and an argument similar to that given before shows that for all t_0, s_0

$$
(v'(t_0))^2 \leq 2\Big((v'(s_0))^2 + \tfrac{K_3}{4} \Big)\ .
$$

Choosing s_0 such that $(v'(s_0))^2 = Q < K_2$ gives a bound on $\max\{|v'(t_0)|\}$. •

Thus in particular we see how the conserved quantities of (25) give bounds on the Sobolev norms of its solutions. It just remains to reverse two steps, to go from (25) to periodic ACH (13), and thence to the periodic case of CH. Both steps are immediate. For the step to ACH, we have $p = e^v$, and thus $p^{(r)} = e^v(v^{(r)}+$ a polynomial in lower derivatives of v). Thus, for example, bounds on $v, v', \ldots, v^{(n-1)}$ give bounds on $p, p', \ldots, p^{(n-1)}$, and in greater generality the theorem stated above is true with v replaced by p. Similarly to go from ACH to CH, we have $m = p^2$ and $\partial_x = p\partial_{t_0}$; thus $\partial_x^r m = p^2(2p^{r-1}p^{(r)}+$ a polynomial in lower derivatives of p). Immediately we see that bounds on $p, p', \ldots, p^{(n-1)}$ give bounds on m and its first $n-1$ derivatives with respect to x, and in greater generality the theorem above is true with $v(t_0)$ replaced by $m(x)$.

In fact the theorem above has other implications too. The bound on $|v|$ gives not only a bound on $|m|$ but also on $|\frac{1}{m}|$. Thus for a solution of CH with $m > 0$ we can find $\epsilon > 0$ such that $m > \epsilon$ for all time. This improves the result of Constantin and Escher [4] that a solution with $m > 0$ at some time has $m > 0$ for all time.

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