

# The Camassa–Holm Equation: A Loop Group Approach

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**Abstract.** A map is presented that associates with each element of a loop group a solution of an equation related by a simple change of coordinates to the Camassa–Holm (CH) Equation. Certain simple automorphisms of the loop group give rise to Bäcklund transformations of the equation. These are used to find 2-soliton solutions of the CH equation, as well as some novel singular solutions.

## 1 Introduction

Substantial interest is accumulating in the Camassa–Holm (CH) equation:

$$u_t = 2f_x u + f u_x, \quad u = \frac{1}{2} f_{xx} - 2f. \quad (1)$$

This equation has been believed to be integrable for many years [9, 11], but only recently has it been widely studied, following the work of Camassa and Holm [3] showing that it describes shallow water waves. Camassa and Holm found that this equation exhibits “peakons”, i.e. solitary wave solutions with discontinuous first derivative at their crest (here  $f$  is regarded as the fundamental field; for peakon solutions  $u$  is just a moving delta function). Multippeakon solutions can be found, and are related to an integrable finite dimensional Hamiltonian system, which has been exhaustively studied [2, 16]. Both numerical and analytic studies [4, 5] suggest that for suitable initial data (1) describes the decomposition of the initial data into peakon components; in particular for analytic initial data with  $u$  of mixed sign, the first derivative of  $f$  develops a discontinuity in finite time. In addition to peakon solutions, (1) is known to have analytic soliton solutions tending to finite nonzero depth at spatial infinity [7, 13]; these converge to peakon solutions as the depth at infinity tends to zero.

A number of papers have appeared explaining various aspects of the integrable structure of CH [8, 10, 15, 18, 20]. Despite these results, much work remains to be done, particular in regard to generating explicit solutions. The aim of the current paper is to present the analog, for CH, of a cornerstone of KdV theory, the *Segal–Wilson map* [22]. The Segal–Wilson map associates with each element of a loop group a solution (possibly with singularities) of KdV. In [23], Wilson gave a very explicit version of this map, writing down a huge class of solutions of the modified KdV equation. Here I will give an analogous formula for CH; more precisely, I give a map from a loop group to the space of solutions of an equation related by a simple

change of coordinates to the CH equation, which I will call the associated Camassa-Holm (ACH) equation. The application I will make of this result is the construction of Bäcklund transformations (BTs) for the ACH equation. In the case of the KdV equation, it is known that BTs have their origins in simple automorphisms of the relevant loop group [21], and by looking at similar transformations here, BTs can be derived for ACH. These facilitate the construction of new solutions of CH, along with a new formula for 2-soliton solutions. (A formula for 2-soliton solutions has already been given in [1], using a different approach, with which I will not compare here.)

This paper is structured in a logically incorrect fashion, but one which I hope will enable others working on the CH equation to read the results obtained without going into details of the loop group construction. Section 2 contains all the results that do not require some understanding of loop groups: Here I define the ACH equation, explore its elementary solutions and properties, give two BTs for ACH (derived later by loop group techniques), and use the BTs to study less elementary solutions of ACH, and the corresponding solutions of CH. Section 3 contains the details of the map from a loop group to solutions of ACH, the main result of the paper. Finally, Section 4 contains the derivation of the two BTs using loop group methods, which logically should precede much of the material in section 2.

The reader will see in section 2 that this paper only studies solutions of CH for which  $u$  is of constant sign (only then is the transformation to ACH defined). I have tried hard to find a loop group construction that gives rise to mixed sign solutions of CH, but without success. This, along with the results of [5], leads me to conjecture (1) is in some sense “more” integrable for solutions of constant sign than for solutions of mixed sign (in [5] it is shown that if a solution of (1) is of constant sign at some time, then it remains so). The exact sense of this remains to be clarified, but it is certainly clear that the CH equation presents an interesting challenge to the integrable systems community.

A few more introductory points: First, in the current paper I limit myself to the study of (1), and not the related equation obtained by replacing the definition of  $u$  in (1) by  $u = \frac{1}{2}f_{xx} + 2f$ , which admits a compacton solution [13, 17, 19]. Second, note the choice of coefficients I have made in (1) differs slightly from that in [3]; in particular for my choice of coefficients positive elevation peakons move to the left. Third, I note that the change of coordinates from CH to ACH is suggested in [10]. And finally, I draw the reader’s attention to the papers [6], which study the periodic problem for the CH equation.

## 2 The ACH Equation and its Bäcklund Transformations

*Proposition 1.* There exists a one to one correspondence between  $C^\infty$  solutions of (1) with  $u$  positive and  $C^\infty$  solutions of

$$\dot{p} = p^2 f', \quad f = \frac{p}{4} \left( \frac{\dot{p}}{p} \right)' - \frac{p^2}{2}, \quad (2)$$

with  $p$  positive. Here  $p, f$  are functions of  $t_0, t_1$ , a prime denotes differentiation with respect to  $t_0$ , and a dot differentiation with respect to  $t_1$ . Equation (2) will be referred to as the *Associated Camassa-Holm (ACH) Equation*.

*Proof.* Suppose we have a solution to (1) with  $u$  positive, and let  $p = \sqrt{u}$ . Then  $p_t = (pf)_x$ . It follows that we can define a new set of coordinates  $t_0, t_1$  (the reason for this notation will become clear in section 3) via

$$dt_0 = p dx + p f dt, \quad dt_1 = dt. \quad (3)$$

In the new coordinates equation (1) becomes (2). To go from a solution of (2) to a solution of (1), we note that the change of coordinates (3) implies

$$\begin{pmatrix} \frac{\partial t_0}{\partial x} & \frac{\partial t_0}{\partial t} \\ \frac{\partial t_1}{\partial x} & \frac{\partial t_1}{\partial t} \end{pmatrix} = \begin{pmatrix} p & pf \\ 0 & 1 \end{pmatrix},$$

and so

$$\begin{pmatrix} \frac{\partial x}{\partial t_0} & \frac{\partial x}{\partial t_1} \\ \frac{\partial t}{\partial t_0} & \frac{\partial t}{\partial t_1} \end{pmatrix} = \begin{pmatrix} \frac{1}{p} & -f \\ 0 & 1 \end{pmatrix}.$$

Given a solution  $p, f$  of (2), with  $p$  non-vanishing, we find  $x$  as a function of  $t_0, t_1$  by integrating

$$\frac{\partial x}{\partial t_0} = \frac{1}{p}, \quad \frac{\partial x}{\partial t_1} = -f. \quad (4)$$

By the first equation of (2), these equations are integrable. We clearly can identify  $t$  and  $t_1$ ; and since  $\frac{\partial x}{\partial t_0} > 0$  it follows that the map between  $x$  and  $t_0$  for fixed  $t = t_1$  is one to one (and  $C^\infty$ ). Thus we can express  $t_0, t_1$  in terms of  $x, t$  to obtain a solution of (1) (with  $u = p^2$ ). It is clear that this sets up a one to one correspondence between “positive” solutions of the two equations.

*Note.* Clearly a solution of (2) with  $p$  negative also gives rise to a solution of (1). A solution of (2) for which  $p$  has zeros will in general give rise to a number of solutions of (1): in integrating (4) we obtain a relationship between  $t_0$  and  $x$  (for fixed  $t_1 = t$ ) which is many to one. We will see an example of this below. In the other direction, a solution of (1) with  $u$  always negative can be used to give a solution with  $u$  always positive by the replacements  $u \rightarrow -u$ ,  $f \rightarrow -f$ ,  $t \rightarrow -t$ , and from this we can obtain a solution of (2). But there is no apparent way to obtain a solution of (2) from a solution of (1) in which  $u$  is allowed to change sign. Finally, the correspondence also extends to solutions with point singularities; below we will see an example of this too.

*Proposition 2.*  $p(x, t) = \phi(x - ct)$  ( $c \neq 0$ ) solves (2) if

$$(\phi')^2 = -\frac{4}{c}\phi^3 + \alpha\phi^2 + \beta\phi + 4, \quad (5)$$

where  $\alpha, \beta$  are arbitrary constants. In particular we have “soliton solutions”

$$\phi(z) = A \operatorname{sech}^2 \left( \sqrt{\frac{A}{c}} z \right) + h, \quad A = \frac{c}{h^2} - h, \quad (6)$$

(here  $h \neq 0$  and  $h^3 \operatorname{sgn}(c) < |c|$ ), and (singular) “rational solutions”

$$\phi(z) = c^{1/3} - \frac{c}{z^2}. \quad (7)$$

*Proof.* This is a straightforward computation. Equation (5) is familiar from the theory of the KdV equation, but the nonzero constant term implies that if  $\phi' \rightarrow 0$  at spatial infinity, then  $\phi$  cannot go to zero there, as seen in the forms of both the soliton and rational solutions. In the soliton solutions the parameter  $h$  is the asymptotic height; for the rational solutions the asymptotic height is  $c^{1/3}$ , a limiting case of the heights allowed for soliton solutions.

*Corresponding Solutions of CH.* It is a straightforward but arduous matter to translate the solutions of ACH just presented to solutions of CH, following the procedure in the proof of

proposition 1. From the soliton solutions of ACH we obtain the soliton solutions of CH, which for  $c > 0$  take the form

$$\begin{aligned} f &= -\frac{h^2}{2} + c \left( \frac{1}{A \operatorname{sech}^2 X + h} - \frac{1}{h} \right), \\ u &= (A \operatorname{sech}^2 X + h)^2, \end{aligned} \quad (8)$$

where  $A = (c/h^2) - h$ , and  $X$  ( $= \sqrt{A/c}(t_0 - ct_1)$ ) is determined from

$$x - \left( \frac{c}{h} + \frac{h^2}{2} \right) t = \sqrt{\frac{1}{1 - h^3/c}} X + \frac{1}{2} \ln \left( \frac{me^{2X} + 1}{e^{2X} + m} \right), \quad m = \left( \sqrt{\frac{c}{h^3}} - \sqrt{\frac{c}{h^3} - 1} \right)^2. \quad (9)$$

The speed of the solution as a solution of CH is  $\tilde{c} = c/h + h^2/2$ , differing from the speed  $c$  of the solution as a solution of ACH. To understand the nature of this solution, it is useful to look at the relation of  $x - \tilde{c}t$  and  $X$  in the limits  $X \rightarrow \pm\infty$  and  $X \rightarrow 0$ : For  $X \rightarrow \pm\infty$

$$x - \tilde{c}t = \sqrt{\frac{1}{1 - h^3/c}} X \pm \ln \left( \sqrt{\frac{c}{h^3}} - \sqrt{\frac{c}{h^3} - 1} \right) + o(1),$$

and for  $X \rightarrow 0$

$$x - \tilde{c}t = \left( \sqrt{\frac{1}{1 - h^3/c}} - \sqrt{1 - h^3/c} \right) X + o(X).$$

In figure 1, a plot of  $x - \tilde{c}t$  against  $X$  is given for the value  $h^3/c = 1/3$ , as well as plots of  $f/h^2$  and  $u/h^2$  against  $x - \tilde{c}t$  for this value. The limit  $h \rightarrow 0$  with  $\tilde{c}$  constant is — according to [13] — the peakon limit (or, rather, since we are looking at right moving solutions, the anti-peakon limit); I leave it as an interesting exercise to the reader to show that in this limit we indeed obtain the anti-peakon solution

$$f = -\tilde{c} \exp(-2|x - \tilde{c}t|), \quad u = 2\tilde{c}\delta(|x - \tilde{c}t|). \quad (10)$$

Turning now to the rational solutions (7), for these  $\phi$  changes sign twice (at  $z = \pm c^{1/3}$ ) and has a singularity (at  $z = 0$ ). Following the method of translation back to solutions of CH Leads to

$$f = \tilde{c} \frac{1 - \frac{1}{3}X^2}{X^2 - 1}, \quad u = \frac{2}{3}\tilde{c} \left( 1 - \frac{1}{X^2} \right)^2, \quad (11)$$

where  $X$  ( $= zc^{-1/3}$ ) is determined by

$$x - \tilde{c}t = X + \frac{1}{2} \ln \left| \frac{X - 1}{X + 1} \right|, \quad (12)$$

and  $\tilde{c} = \frac{3}{2}c^{2/3}$ . The map from  $X$  to  $x - \tilde{c}t$  is three to one, so here a single solution of ACH gives three solutions of CH, corresponding to the ranges on which  $\phi$  is of definite sign, viz.  $X < -1$ ,  $-1 < X < 1$  and  $X > 1$ . In the middle range the solution has a singularity at  $X = 0$ . The three solutions are illustrated in figure 2. For  $X < -1$  and  $X > 1$ , the solutions for  $u$  take the form of a “kink”, with the limit at one end being approached polynomially and at the other end exponentially; the solution for  $f$  is finite at one end, and diverges exponentially at the other end. For the range  $-1 < X < 1$ , the solution for  $u$  is a “spikon”

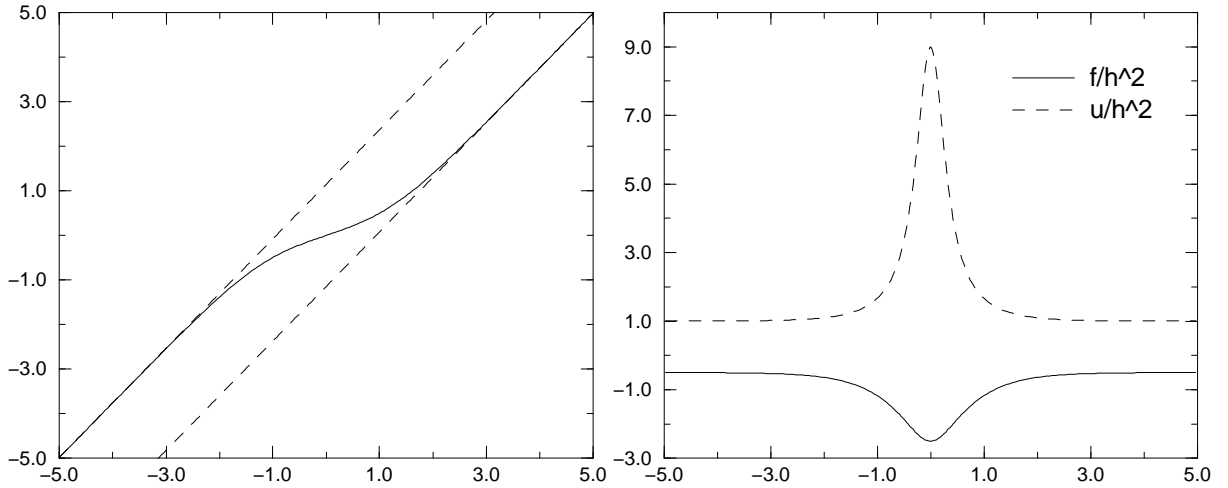


Figure 1: The soliton solution, for  $h^3/c = 1/3$ . On the left, a plot of  $x - \tilde{c}t$  versus  $X$  is displayed; the straight lines show the  $X \rightarrow \pm\infty$  behavior. On the right the functions  $u/h^2$  and  $f/h^2$  are plotted against  $x - \tilde{c}t$ .

(an infinitely peaked soliton), and  $f$  has a cusp at  $x - \tilde{c}t = 0$  and exponentially diverges at infinity. (For  $x - \tilde{c}t \approx 0$  we find  $f \approx -1 - 2 \cdot 3^{-1/3}(x - \tilde{c}t)^{2/3}$ , and there is a cusp at  $x - \tilde{c}t = 0$ , not a simple corner as appears from the low resolution plot in figure 2.) One might hope to “splice” together the two solutions for  $X < -1$  and  $X > 1$  to form a finite height peakon solution for  $f$  with polynomial decay at infinity; this does not seem to be possible.

*Note.* I have not, as of yet, explored the solutions of (1) corresponding to the cnoidal wave solutions of (2).

The great advantage of ACH over CH is that ACH has standard Bäcklund transformations. These are presented here (and can be verified by direct computations), but their derivations will be given in section 4.

*Proposition 3.* The ACH equation (2) has the Bäcklund transformation  $p \rightarrow p - s'$  where

$$s' = -\frac{s^2}{p\theta} + p\theta \left( \frac{1}{p^2} + \frac{1}{\theta} \right), \quad (13)$$

$$\dot{s} = -s^2 + \frac{\dot{p}}{p}s + \theta(\theta - 2f). \quad (14)$$

This is a strong BT (in the sense that (13) and (14) are only consistent if  $p, f$  obey (2)).  $\theta$  is a parameter,  $\theta \neq 0$ . Under the BT

$$f \rightarrow f - \frac{\dot{s}}{p(p - s')}. \quad (15)$$

Applying this BT to the constant solution with  $p = h$  and  $f = -h^2/2$ , we find we can take

$$s = h\theta \sqrt{\frac{1}{h^2} + \frac{1}{\theta}} \tanh \left( \sqrt{\frac{1}{h^2} + \frac{1}{\theta}} (t_0 + \theta h t_1 + C) \right) \quad (16)$$

or

$$s = h\theta \sqrt{\frac{1}{h^2} + \frac{1}{\theta}} \coth \left( \sqrt{\frac{1}{h^2} + \frac{1}{\theta}} (t_0 + \theta h t_1 + C) \right), \quad (17)$$

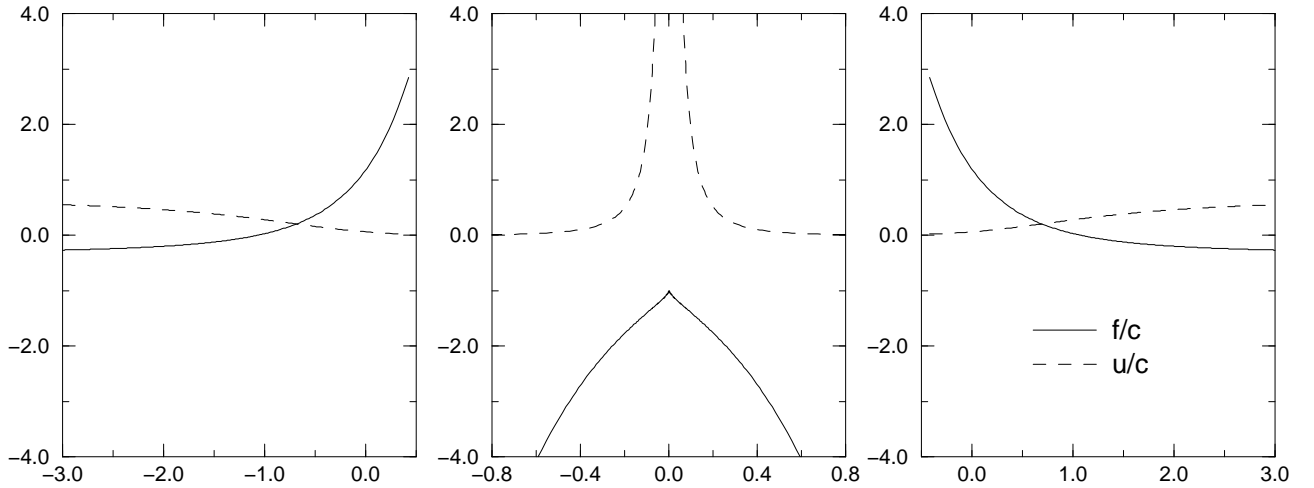


Figure 2: The solutions of CH obtained from rational solution of ACH, for the ranges  $X < -1$ ,  $-1 < X < 1$  and  $X > 1$  (from left to right).

where  $C$  is a constant of integration. Taking the choice (16) for  $s$ , with  $C = 0$  and  $\theta = -c/h$ , returns the soliton solution (6). The choice (17) gives a singular solution. Repeated application of the BT is simplified by the following nonlinear superposition formula:

*Proposition 4.* If application of the above BT with parameter  $\theta_1$  to a solution  $p$  of (2) yields the new solution  $p_1 = p - s'_1$ , and application with parameter  $\theta_2$  yields the new solution  $p_2 = p - s'_2$ , then

$$p_{12} = p - \left( \frac{(\theta_1 - \theta_2)(\theta_1\theta_2 - s_1s_2)}{\theta_2s_1 - \theta_1s_2} \right)' \quad (18)$$

is also a solution of (2) (arising from repeated application of the BT with parameters  $\theta_1$  and  $\theta_2$  — in either order — to  $p$ ). The corresponding  $f$  takes the form

$$f_{12} = f - \theta_1\theta_2(\theta_1 - \theta_2) \left( \frac{p}{p_{12}} \right) \left( \frac{s_1 - s_2}{(\theta_2s_1 - \theta_1s_2)^2} \right). \quad (19)$$

*Proof (outline only).* The formulae (18) and (19) are best proven by direct verification using a symbolic manipulator. As regards their derivation, the starting point is commutativity of BTs with different parameter values, for which an argument can be given at the loop group level (see section 4). Given this, suppose applying the BT with parameter  $\theta_2$  to  $p_1$  gives  $p_{12} = p_1 - s'_{12}$ , and applying the BT with parameter  $\theta_1$  to  $p_2$  gives  $p_{21} = p_2 - s'_{21}$ . If  $p_{12} = p_{21}$ , then  $(s_1 + s_{12})' = (s_2 + s_{21})'$ , and this suggests looking at the possibility that  $s_1 + s_{12} = s_2 + s_{21}$ . Since from the BT we have expressions for the derivatives of  $s_1, s_2, s_{12}, s_{21}$ , we can differentiate this relationship to find other algebraic relationships from which  $s_{12}$  and  $s_{21}$  can be determined (due to their length, I do not reproduce these calculations here). This is the origin of the formula (18).

2-soliton solutions of ACH are now easily found using the superposition formula on the constant solution  $p = h$ ,  $f = -h^2/2$ . Taking  $h > 0$ ,  $\theta_2 < \theta_1 < -h^2$ ,  $s_1$  of the form (16) and  $s_2$  of the form (17) we obtain

$$p = h - h(\theta_1 - \theta_2) \cdot \frac{(\theta_2 + h^2) + (\theta_1 - \theta_2) \tanh^2 y_2 - (\theta_1 + h^2) \tanh^2 y_1 \tanh^2 y_2}{\left( \sqrt{\theta_1(\theta_2 + h^2)} - \sqrt{\theta_2(\theta_1 + h^2)} \tanh y_1 \tanh y_2 \right)^2}, \quad (20)$$

where  $y_1 = \sqrt{1/h^2 + 1/\theta_1}(t_0 + \theta_1 h t_1 + C_1)$  and  $y_2 = \sqrt{1/h^2 + 1/\theta_2}(t_0 + \theta_2 h t_1 + C_2)$ . For  $\theta_2 < \theta_1 < -h^2$  it is simple to check this is nonsingular, and moreover that  $p > h$ , so in particular  $p$  has no zeros and  $f$  is nonsingular too. For  $f$  we find the formula

$$f = -\frac{h^2}{2} - (\theta_1 - \theta_2) \cdot \frac{\theta_2(\theta_2 + h^2) + (\theta_1 - \theta_2)(h^2 + \theta_1 + \theta_2) \tanh^2 y_2 - \theta_1(\theta_1 + h^2) \tanh^2 y_1 \tanh^2 y_2}{\left(\sqrt{\theta_2(\theta_2 + h^2)} - \sqrt{\theta_1(\theta_1 + h^2)} \tanh y_1 \tanh y_2\right)^2 - (\theta_1 - \theta_2)^2 \tanh^2 y_2}. \quad (21)$$

To translate these results into 2-soliton solutions of CH we follow the procedure of proposition 1:  $f$  is as given in (21) and  $u = p^2$ , where  $p$  is as given in (20), but now  $t_1$  must be replaced by  $t$ , and  $t_0$  is a parameter, related to the coordinates  $x, t$  by

$$x = \int_0^{t_0} \frac{1}{p(t'_0, t_1)} dt'_0 - \int_0^{t_1} f(0, t'_1) dt'_1 \quad (22)$$

(which solves (4)). I cannot see how to evaluate these integrals analytically, but they can be approximated in various limits, as well as evaluated numerically. In figure 3, snapshots of a 2-soliton solution of ACH are given, and in figure 4 the corresponding pictures of the corresponding solution of CH are shown. One important feature of the passage from ACH to CH is that the speeds of the soliton components change; for ACH the speeds are  $c_i = -\theta_i h$ ,  $i = 1, 2$ , and for CH they are  $\tilde{c}_i = c_i/h + h^2/2 = -\theta_i + h^2/2$ .

I proceed to a second BT for ACH, which will also be derived later by loop group techniques.

*Proposition 5.* The ACH equation (2) has the strong Bäcklund transformation

$$p \rightarrow p \left[ \left( 1 - \frac{pBB'}{\tau} \right)^2 - \frac{B^4}{\tau^2} \right], \quad (23)$$

where  $B$  and  $\tau$  satisfy the equations.

$$(pB')' = B \left( \frac{1}{p} + \frac{p}{\theta} \right) \quad \dot{B} = \left( \frac{-\dot{p}}{2p} \right) B + \theta p B' \quad (24)$$

$$\tau' = \frac{pB^2}{\theta} \quad \dot{\tau} = \theta(p^2 B'^2 - B^2) \quad (25)$$

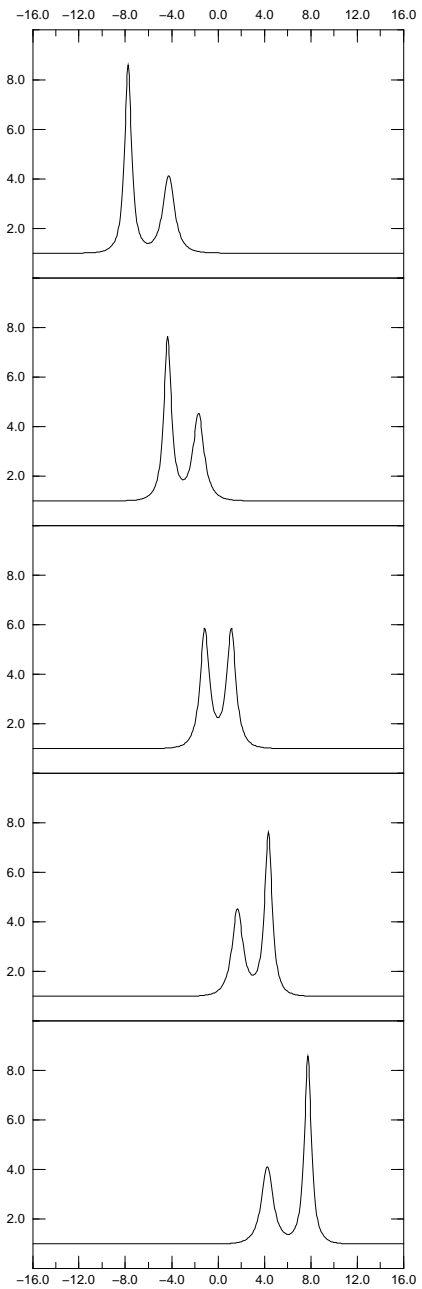
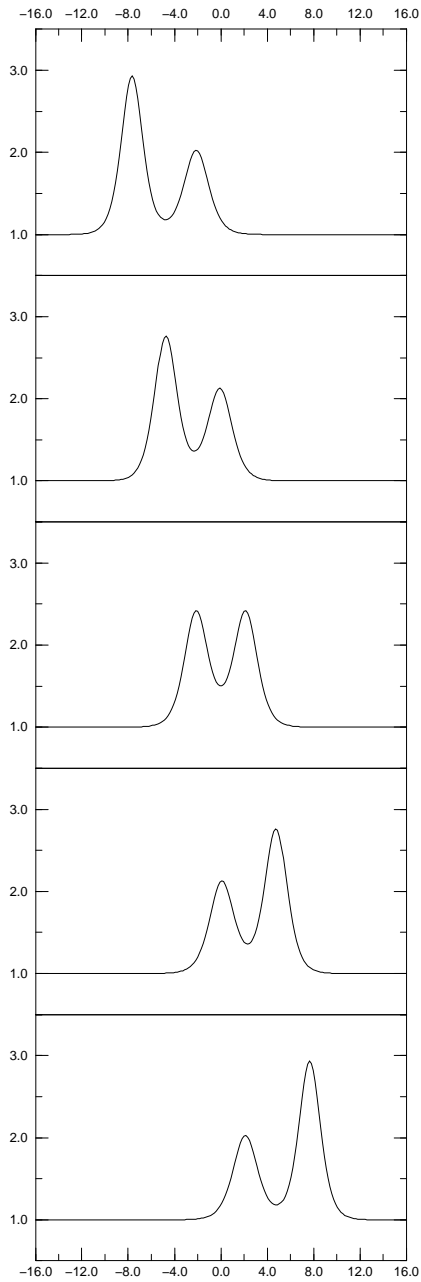
and  $\theta$  is a parameter,  $\theta \neq 0$ .

This BT looks more involved than the previous one, but it is actually very similar to implement. The apparently difficult part of implementation is solving the first equation of (24), a second order linear equation determining the  $x$ -dependence of  $B$ . This is, however, just a linearization of the Riccati equation (13) that appears in the previous BT: if  $B$  solves the first equation of (24) then  $s = p\theta B'/B$  solves (13).

As an example of the use of this BT, consider its action on the trivial solution  $p = h$ ,  $f = -h^2/2$  ( $h$  constant). One allowed choice of  $B, \tau$  is

$$B = K_1 \exp \left( \sqrt{\frac{1}{h^2} + \frac{1}{\theta}} (t_0 + h\theta t_1) \right)$$

$$\tau = K_1^2 \left( \frac{h}{2\theta \sqrt{\frac{1}{h^2} + \frac{1}{\theta}}} \exp \left( 2\sqrt{\frac{1}{h^2} + \frac{1}{\theta}} (t_0 + h\theta t_1) \right) + K_2 \right),$$



From left to right:

Figure 3: The 2-soliton solution of ACH given by equation (20), with  $h = 1$ ,  $\theta_1 = -2$ ,  $\theta_2 = -3$ ,  $C_1 = C_2 = 0$ . Plots are of  $p$  as a function of  $t_0$ , for  $t_1 = -3, -2, -1, 0, 1, 2$  (from top to bottom).

Figure 4: The corresponding 2-soliton solution of CH. Plots are of  $u$  as a function of  $x$ , for  $t = -3, -2, -1, 0, 1, 2$  (from top to bottom).



where  $K_1, K_2$  are constants, and it is straightforward to show that this returns the 1-soliton solution with speed  $-h\theta$ . A more general possibility for  $B, \tau$  is

$$\begin{aligned} B &= K_1 \cosh \left( \sqrt{\frac{1}{h^2} + \frac{1}{\theta}} (t_0 + h\theta t_1 + K_2) \right) \\ \tau &= \frac{K_1^2 h}{2\theta} \left( t_0 - \frac{\theta}{h} (2\theta + h^2) t_1 + K_3 + \frac{1}{2\sqrt{\frac{1}{h^2} + \frac{1}{\theta}}} \sinh \left( 2\sqrt{\frac{1}{h^2} + \frac{1}{\theta}} (t_0 + h\theta t_1 + K_2) \right) \right), \end{aligned}$$

where  $K_1, K_2, K_3$  are constants, with resulting solution of ACH

$$p = h - 4h \left( 1 + \frac{\theta}{h^2} \right) \frac{\left( \frac{2\theta}{h^2} \right) (1 + \cosh y_2) + y_1 \sinh y_2}{(y_1 + \sinh y_2)^2}, \quad (26)$$

where  $y_1 = 2\sqrt{\frac{1}{h^2} + \frac{1}{\theta}} (t_0 - \frac{\theta}{h} (2\theta + h^2) t_1 + K_3)$ ,  $y_2 = 2\sqrt{\frac{1}{h^2} + \frac{1}{\theta}} (t_0 + \theta h t_1 + K_2)$ . This solution, which is illustrated in figure 5 (for  $\theta < 0$ ) and figure 6 (for  $\theta > 0$ ), should presumably be considered as the superposition of a soliton and a simple rational solution. For almost every value of  $t_1$ , it has a singularity at a single value of  $t_0$ , and two zeros, giving singularities of the corresponding function  $f$ . (The possible exceptions to this are the two values of  $t_1$  defined by the relations  $y_1 = \pm 2\sqrt{\theta(\theta + h^2)}/h^2$ ,  $y_2 = -\sinh^{-1} y_1$ ; when these relations hold, the ratio in (26) is undefined.)

Let us look more closely at the solution in the case  $\theta > 0$ . As can be seen in figure 6, as  $t_1$  increases, the singularity of  $p$  passes through the zeros of  $p$ . The implications of this for the corresponding solutions of CH are quite dramatic. Because of the 2 zeros of  $p$  there are 3 corresponding solutions of CH, but let us focus on the solution corresponding to the  $t_0$  region between the two zeros of  $p$ . This solution evidently has the remarkable feature that for  $t$  below one critical time  $t_c^1$  and above another critical time  $t_c^2$  (determined as explained above), the solution for  $u = p^2$  is analytic, but for  $t_c^1 < t < t_c^2$ , the solution has a singularity. The singularity moves in from  $x = +\infty$  and out to  $x = -\infty$  as  $t$  increases between the two critical values. The reason a singularity can develop and disappear this way is that the dynamics of  $u$  is driven by the dynamics of  $f$  and  $f$  blows up at both  $\pm\infty$ . It turns out that for the solution (26) it is possible to analytically compute the integrals required to change coordinates back from ACH to CH, and in figure 7 the interesting solution of CH just described is illustrated (though this illustration does not adequately capture the most important feature, that for  $t < t_c^1$  and  $t > t_c^2$  there is *no* singularity). For completeness, I note that for  $K_2 = K_3 = 0$  the critical times are given by

$$t_c^2 = -t_c^1 = \frac{1}{2\theta(\theta + h^2)} \left( 1 + \frac{\sinh^{-1} z}{z} \right), \quad z = \frac{2\sqrt{\theta(\theta + h^2)}}{h^2}.$$

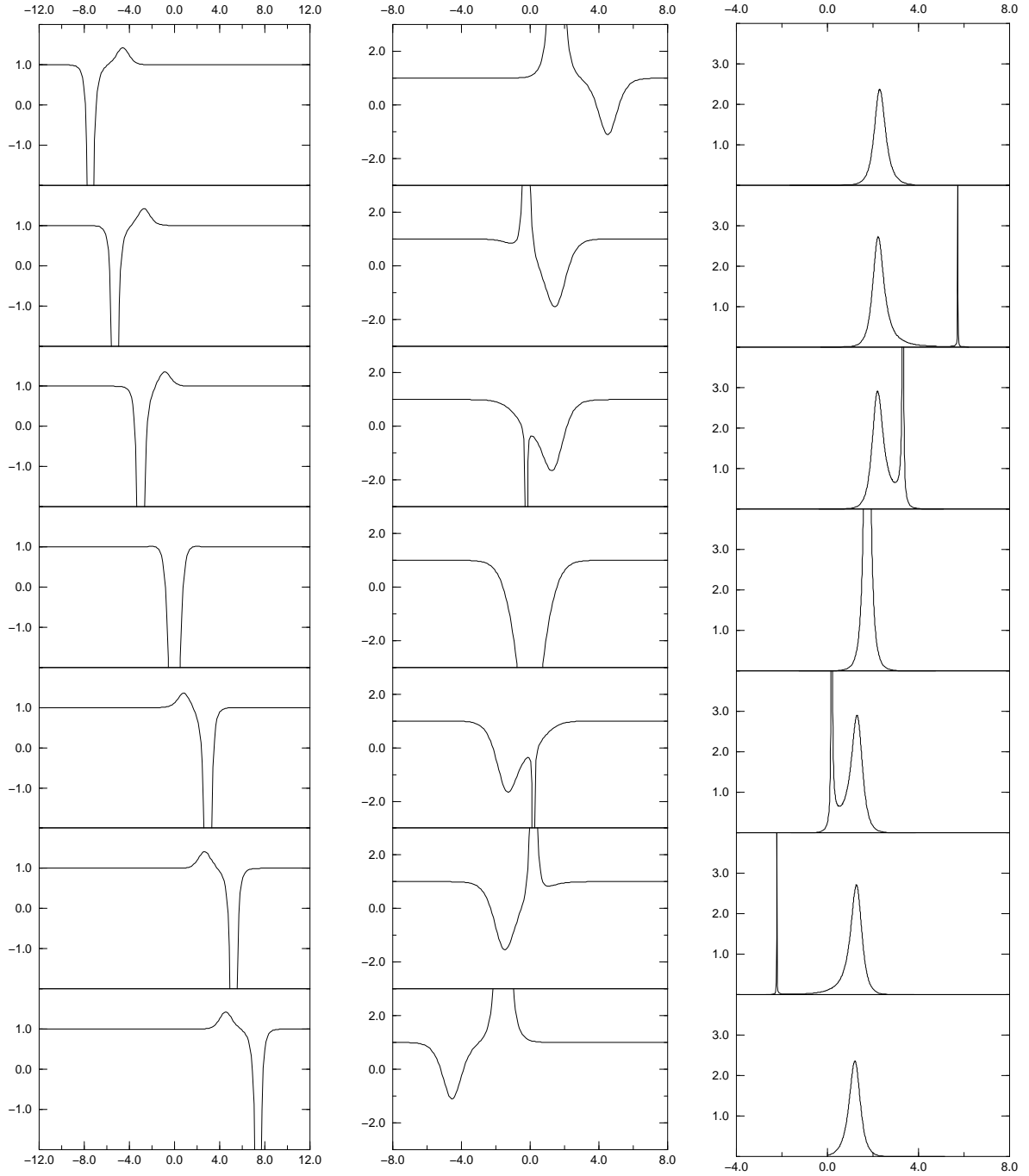
### 3 The Loop Group Construction for ACH

#### 3.1 The ACH and CH Hierarchies

The key to the map from a loop group to solutions of ACH is the following trivially-checked property:

*Proposition 6.* The ACH equation (2) has a zero curvature formulation

$$\partial_{t_1} Z_0 = \partial_{t_0} Z_1 + [Z_1, Z_0] \quad (27)$$



From left to right:

Figure 5: The solution (26) of ACH for  $h = 1$ ,  $\theta = -2$ ,  $K_2 = K_3 = 0$ . Plots of  $p$  against  $t_0$  for  $t_1 = -3, -2, -1, 0, 1, 2, 3$  (from top to bottom).

Figure 6: The solution (26) of ACH for  $h = 1$ ,  $\theta = 1$ ,  $K_2 = K_3 = 0$ . Plots of  $p$  against  $t_0$  for  $t_1 = -3, -0.5, -0.4, 0, 0.4, 0.5, 3$  (from top to bottom).

Figure 7: The solution of CH arising from the middle  $t_0$  range (the range with  $p < 0$ ) of the solution of ACH of figure 6. Plots of  $u$  against  $x$  for  $t = -0.5, -0.4055, -0.37, 0, 0.37, 0.4055, 0.5$  (from top to bottom). The solution is analytic for  $|t| > t_c^2 \approx 0.4058$ .

where

$$Z_0 = \begin{pmatrix} 0 & 1/p \\ p/\lambda + 1/p & 0 \end{pmatrix}, \quad (28)$$

$$Z_1 = \lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} -\dot{p}/2p & 0 \\ -2f & \dot{p}/2p \end{pmatrix}, \quad (29)$$

and  $f$  is given by the formula in (2).

There is an integrable hierarchy associated with the CH equation [3], so it is natural to investigate whether there is a hierarchy related to the ACH equation. Though I do not intend to explore it in full detail here, there is a hierarchy, which can be easily defined using a zero curvature formulation:

*Definition.* The  $n$ -th ACH equation ( $n \in \mathbf{Z}$ ,  $n \neq 0$ ) is the zero curvature equation

$$\partial_{t_n} Z_0 = \partial_{t_0} Z_n + [Z_n, Z_0] \quad (30)$$

where

$$Z_0 = \begin{pmatrix} 0 & 1/p(t_0, t_n) \\ p(t_0, t_n)/\lambda + 1/p(t_0, t_n) & 0 \end{pmatrix}, \quad (31)$$

and

- for  $n > 0$ ,  $Z_n$  is a polynomial in  $\lambda$  of order  $n$ , with highest degree term  $\lambda^n \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  and constant term with vanishing 1, 2 entry.
- for  $n < 0$ ,  $Z_n$  is a polynomial in  $1/\lambda$  of order  $1 - n$ . with highest degree term  $\lambda^{n-1} \begin{pmatrix} 0 & 0 \\ p(t_0, t_n) & 0 \end{pmatrix}$  and constant term proportional to  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ .

It is straightforward to check the consistency of the above definition. Of particular interest is the  $n = -1$  equation, which, writing  $r = 1/p$ , (and ignoring one constant of integration) becomes

$$\partial_{t_{-1}} r = \left( \frac{1}{4} r'' - \frac{3}{8} \frac{r'^2}{r} - \frac{1}{2} r^3 \right)'. \quad (32)$$

It is also straightforward to use the above definition to show that the  $n > 1$  ACH equations are related by a change of coordinates to the higher equations in the CH hierarchy, for which a zero curvature formulation is given in [20]. Less straightforward (but nevertheless possible) is to show the consistency of all the ACH equations, i.e. that we can look for functions  $p(\dots, t_{-1}, t_0, t_1, \dots)$  simultaneously satisfying all the equations in the hierarchy. This calculation is made unnecessary by the loop group construction that will shortly be given, which constructs solutions of the entire hierarchy.

### 3.2 The Loop Group $G$

Suppose  $\epsilon > 0$ , and denote by  $\mathcal{C}_0$  and  $\mathcal{C}_\infty$  respectively the circles  $\{|\lambda| = \epsilon\}$  and  $\{|\lambda| = 1/\epsilon\}$  in the Riemann sphere. Write  $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_\infty$ . The loop group we will need, which I denote  $G$ , is the group of smooth maps from  $\mathcal{C}$  into  $SL(2)$ . I denote by  $G_+$  the subgroup of  $G$  of maps which are the boundary values of analytic maps from  $\{\epsilon < |\lambda| < 1/\epsilon\}$  to  $SL(2)$ , and by

$G_-$  the subgroup of  $G$  of maps which are the boundary values of analytic maps  $S(\lambda)$  from  $\{|\lambda| < \epsilon\} \cup \{|\lambda| > 1/\epsilon\}$  to  $SL(2)$ , satisfying the boundary conditions

$$S(0) = \begin{pmatrix} 1/\alpha & 0 \\ \beta & \alpha \end{pmatrix}, \quad S(\infty) = \begin{pmatrix} \sqrt{1+\gamma^2} & \gamma \\ \gamma & \sqrt{1+\gamma^2} \end{pmatrix}, \quad (33)$$

for some  $\alpha, \beta, \gamma$  (with  $\alpha \neq 0$ ). (It is straightforward to check these conditions do define a group). Throughout I identify  $SL(2)$  with its fundamental representation. The key property of  $G$  we shall use is that a dense open subset of elements  $U \in G$  (given a certain natural topology) can be factorized in the form

$$U = S^{-1}Y \quad (34)$$

with  $S \in G_-$  and  $Y \in G_+$  (the so-called ‘‘Birkhoff factorization’’ property).

The corresponding splitting of the Lie algebra  $\mathcal{G}$  of  $G$  is described as follows. An element  $v \in \mathcal{G}$  has Fourier decompositions on both the circles  $\mathcal{C}_0$  and  $\mathcal{C}_\infty$ , i.e. we can write

$$\begin{aligned} v &= \sum_{n=-\infty}^{\infty} a_n \lambda^n & |\lambda| = \epsilon \\ &= \sum_{n=-\infty}^{\infty} b_n \lambda^n & |\lambda| = 1/\epsilon, \end{aligned} \quad (35)$$

where the coefficients  $a_n, b_n$  are in the Lie algebra  $sl(2)$ . Consider the terms containing negative powers of  $\lambda$  in the series valid on  $|\lambda| = \epsilon$ . Since the series  $\sum_{n=1}^{\infty} a_{-n} \lambda^{-n}$  converges for  $|\lambda|^{-1} = 1/\epsilon$ , it converges absolutely for  $|\lambda|^{-1} < 1/\epsilon$ , i.e. for  $|\lambda| > \epsilon$ , defining an analytic function there. Similarly the series  $\sum_{n=1}^{\infty} b_n \lambda^n$  converges absolutely for  $|\lambda| < 1/\epsilon$ , defining an analytic function there. And thus, for arbitrary  $t \in sl(2)$  (we will fix  $t$  shortly),

$$v_+ = \sum_{n=1}^{\infty} a_{-n} \lambda^{-n} + t + \sum_{n=1}^{\infty} b_n \lambda^n$$

defines an analytic function on  $\epsilon < |\lambda| < 1/\epsilon$ , and is also convergent on the boundaries of this region, thus defining an element of the Lie algebra  $\mathcal{G}_+$  of  $G_+$ . We have

$$\begin{aligned} v - v_+ &= (a_0 - t) + \sum_{n=1}^{\infty} (a_n - b_n) \lambda^n & |\lambda| = \epsilon \\ &= (b_0 - t) + \sum_{n=1}^{\infty} (b_{-n} - a_{-n}) \lambda^{-n} & |\lambda| = 1/\epsilon. \end{aligned}$$

Thus, irrespective of  $t$ ,  $v - v_+$  is the boundary value of a function analytic in  $\{|\lambda| < \epsilon\} \cup \{|\lambda| > 1/\epsilon\}$ . We choose  $t$  so that  $v_- = v - v_+$  is in the Lie algebra  $\mathcal{G}_-$  of  $G_-$ ; from (33), the extra conditions we need to satisfy are

$$(v - v_+)(\lambda = 0) = \begin{pmatrix} 1 - a & 0 \\ b & 1 + a \end{pmatrix}, \quad (v - v_+)(\lambda = \infty) = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix},$$

for some  $a, b, c$ , and a brief calculation shows that this can be done by taking

$$t = \begin{pmatrix} (b_0)_{11} & (a_0)_{12} \\ (b_0)_{21} - (b_0)_{12} + (a_0)_{12} & (b_0)_{22} \end{pmatrix}.$$

To summarize, we have shown the following:

*Proposition 7.* For all  $v \in \mathcal{G}$ , there exists a unique way to write  $v = v_+ + v_-$  with  $v_+ \in \mathcal{G}_+$  and  $v_- \in \mathcal{G}_-$ . If  $v$  has series expansions as in (35), then

$$\begin{aligned} v_+ &= \sum_{n=1}^{\infty} a_{-n} \lambda^{-n} + \begin{pmatrix} (b_0)_{11} & (a_0)_{12} \\ (b_0)_{21} - (b_0)_{12} + (a_0)_{12} & (b_0)_{22} \end{pmatrix} + \sum_{n=1}^{\infty} b_n \lambda^n & \epsilon \leq |\lambda| \leq 1/\epsilon \\ v_- &= \begin{pmatrix} (a_0)_{11} - (b_0)_{11} & 0 \\ (b_0)_{12} - (b_0)_{21} & (a_0)_{22} - (b_0)_{22} \end{pmatrix} + \sum_{n=1}^{\infty} (a_n - b_n) \lambda^n & |\lambda| \leq \epsilon \\ &= \begin{pmatrix} 0 & (b_0)_{12} - (a_0)_{12} \\ (b_0)_{12} - (a_0)_{12} & 0 \end{pmatrix} + \sum_{n=1}^{\infty} (b_{-n} - a_{-n}) \lambda^{-n} & |\lambda| \geq 1/\epsilon \end{aligned}$$

### 3.3 The Map from $G$ to Solutions of the ACH Hierarchy

*Proposition 8.* There exists a natural map from the loop group  $G$  to solutions (possibly with singularities) of the ACH hierarchy. This map descends to a map from the coset space  $G/G_+$  to solutions of the ACH hierarchy.

*Proof.* The proof I give of this follows the description of the Segal-Wilson map given in [21], which in turn follows the proof of a similar result for the KP hierarchy given by Mulase [14]. Similar ideas appear in [12].

Let  $\mathcal{M}$  denote the infinite dimensional affine manifold with coordinates  $\dots t_{-1}, t_0, t_1, \dots$ , and define a  $\mathcal{G}_+$  valued one-form  $\Omega$  on  $\mathcal{M}$  by

$$\Omega = \sum_{n=-\infty}^{\infty} \lambda^n \begin{pmatrix} 0 & 1 \\ 1 + \frac{1}{\lambda} & 0 \end{pmatrix} dt_n . \quad (36)$$

Since evidently  $d\Omega = \Omega \wedge \Omega = 0$ , the differential system

$$dU(t) = \Omega U(t) , \quad (37)$$

where  $U$  is a  $G$ -valued function on  $\mathcal{M}$ , is Frobenius integrable, with general solution

$$U(t) = MU(0) , \quad (38)$$

where

$$\begin{aligned} M &= \exp \left( \sum_{n=-\infty}^{\infty} \lambda^n \begin{pmatrix} 0 & 1 \\ 1 + \frac{1}{\lambda} & 0 \end{pmatrix} t_n \right) \\ &= \cosh \left( z \sqrt{1 + \frac{1}{\lambda}} \right) I + \sinh \left( z \sqrt{1 + \frac{1}{\lambda}} \right) \begin{pmatrix} 0 & (1 + 1/\lambda)^{-1/2} \\ (1 + 1/\lambda)^{1/2} & 0 \end{pmatrix} , \end{aligned}$$

and  $z = \sum_{n=-\infty}^{\infty} \lambda^n t_n$ . Let  $U = S^{-1}Y$ ,  $S \in G_-$ ,  $Y \in G_+$ , be the Birkhoff decomposition of  $U$ , as described in section 3.2. Substituting into (37) we find

$$-dS S^{-1} + dY Y^{-1} = S\Omega S^{-1} , \quad (39)$$

from which it follows that

$$dS S^{-1} = -(S\Omega S^{-1})_- \quad \text{and} \quad dY Y^{-1} = (S\Omega S^{-1})_+ , \quad (40)$$

where here I am using the notation of proposition 7 for the projections of an element of  $\mathcal{G}$  to  $\mathcal{G}_+$  and  $\mathcal{G}_-$ . If we write  $Z = dY Y^{-1}$ , then clearly  $dZ = Z \wedge Z$ , or, writing

$$Z = \sum_{n=-\infty}^{\infty} Z_n dt_n, \quad (41)$$

the components of  $Z$  satisfy the zero curvature equations

$$\partial_{t_m} Z_n - \partial_{t_n} Z_m = [Z_m, Z_n] \quad (42)$$

(c.f. (27),(30)). On the other hand, the second equation of (40) tells us that

$$Z_n = \left( \lambda^n S \begin{pmatrix} 0 & 1 \\ 1 + \frac{1}{\lambda} & 0 \end{pmatrix} S^{-1} \right)_+ . \quad (43)$$

I will now show that this fixes the form of the matrices  $Z_n$  appearing in the zero curvature equations (42) to the form of the matrices appearing in the zero curvature formulation of the ACH hierarchy given in section 3.1. From this it follows at once that given  $U(0) \in G$  to specify a solution of (37) we can find an associated solution of the ACH hierarchy, by computing in turn  $U$ ,  $S$  (or  $Y$ ) and then  $Z = (S\Omega S^{-1})_+ = dY Y^{-1}$ . This is the natural map of the proposition.

To show that (43) correctly fixes the form of the  $Z_n$ , consider first what we know about  $S$ .  $S$  is an element of  $G_-$  and hence has expansions

$$\begin{aligned} S &= \sum_{n=0}^{\infty} S_n \lambda^n, & S_0 &= \begin{pmatrix} 1/\alpha & 0 \\ \beta & \alpha \end{pmatrix}, & |\lambda| &\leq \epsilon \\ &= \sum_{n=0}^{\infty} \tilde{S}_n \lambda^{-n}, & \tilde{S}_0 &= \begin{pmatrix} \sqrt{1+\gamma^2} & \gamma \\ \gamma & \sqrt{1+\gamma^2} \end{pmatrix}, & |\lambda| &\geq 1/\epsilon. \end{aligned}$$

To see the content of (43), we use these formulae to expand  $\lambda^n S \begin{pmatrix} 0 & 1 \\ 1 + \frac{1}{\lambda} & 0 \end{pmatrix} S^{-1}$  in Laurent series valid in  $0 < |\lambda| \leq \epsilon$  and  $1/\epsilon \leq |\lambda| < \infty$ , and then use the projection formula of proposition 7. For example, for  $n = 0$ , we have, for  $0 < |\lambda| \leq \epsilon$ :

$$\begin{aligned} S \begin{pmatrix} 0 & 1 \\ 1 + \frac{1}{\lambda} & 0 \end{pmatrix} S^{-1} &= (S_0 + \lambda S_1) S \begin{pmatrix} 0 & 1 \\ 1 + \frac{1}{\lambda} & 0 \end{pmatrix} (S_0 + \lambda S_1)^{-1} + O(\lambda) \\ &= \frac{1}{\lambda} S_0 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} S_0^{-1} + S_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} S_0^{-1} + \left[ S_1 S_0^{-1}, S_0 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} S_0^{-1} \right] + O(\lambda) \\ &= \begin{pmatrix} O(1) & 1/\alpha^2 + O(\lambda) \\ \alpha^2/\lambda + O(1) & O(1) \end{pmatrix}, \end{aligned}$$

where here  $O(1)$  denotes only non-negative powers of  $\lambda$ , and for  $1/\epsilon \leq |\lambda| < \infty$

$$\begin{aligned} S \begin{pmatrix} 0 & 1 \\ 1 + \frac{1}{\lambda} & 0 \end{pmatrix} S^{-1} &= \tilde{S}_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tilde{S}_0^{-1} + O(1/\lambda) \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + O(1/\lambda). \end{aligned}$$

Using the projection formula we obtain

$$Z_0 = \left( S \begin{pmatrix} 0 & 1 \\ 1 + \frac{1}{\lambda} & 0 \end{pmatrix} S^{-1} \right)_+ = \begin{pmatrix} 0 & 1/\alpha^2 \\ \alpha^2/\lambda + 1/a^2 & 0 \end{pmatrix}, \quad (44)$$

of the required form with  $p = \alpha^2$ . The required results for  $Z_n$ ,  $n \neq 0$ , follow in a similar manner; for  $n > 0$  the expansion for  $0 < |\lambda| \leq \epsilon$  gives no contribution to the projection, and for  $n < 0$  the expansion for  $1/\epsilon \leq |\lambda| < \infty$  gives no contribution.

To conclude the proof, I note that the solutions generated of the ACH hierarchy may have singularities because the Birkhoff decomposition is only possible for open, dense subset of  $G$ . And, the reason the map descends to the coset  $G/G_+$  is that if we multiply  $U(0)$  on the right by  $g \in G_+$ , then  $U(t)$  and  $Y$  get similarly multiplied, but  $S$  is left unchanged, and therefore so is the solution of ACH.

## 4 The Derivation of BTs of ACH

As explained in [21] for the case of the KdV equation, BTs for ACH are associated with simple automorphisms of the loop group. The relevant automorphisms do not preserve the fibration of  $G$  over  $G/G_+$ , so a single solution of ACH, corresponding to a  $G_+$  coset in  $G$ , will typically get mapped by a BT into a family of solutions, corresponding to a family of cosets.

The aim in this section is to outline how the automorphisms

$$U(0) \rightarrow \sqrt{\frac{\lambda - \theta}{\lambda + 1}} \begin{pmatrix} 0 & 1 \\ 1 + 1/\lambda & 0 \end{pmatrix} U(0) \begin{pmatrix} 0 & \lambda/(\lambda - \theta) \\ 1 & 0 \end{pmatrix} \quad (45)$$

$$U(0) \rightarrow U(0) \left( I + \frac{M}{\lambda - \theta} \right), \quad M = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (46)$$

give rise, respectively, to the BTs of propositions 3 and 5. This is simply an exercise in Birkhoff factorization. We will need the first terms in the expansion of  $Y$  around  $\lambda = \theta$ ,  $\theta \neq 0$ . Without loss of generality (since we have not specified  $\epsilon$  in the definition of  $G$ ), we assume  $\theta$  is in the region of analyticity of  $Y$ , in which case we can write

$$Y(\lambda, t) = Y_0(t) \left[ I + Y_1(t)(\lambda - \theta) + O(\lambda - \theta)^2 \right],$$

$$Y_0(t) = \begin{pmatrix} A(t) & B(t) \\ C(t) & D(t) \end{pmatrix}, \quad Y_1(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & -a(t) \end{pmatrix}.$$

( $Y_0, Y_1$  of course have  $\theta$  dependence, but we treat  $\theta$  as fixed.) Substituting these expansions into the equations  $\partial_{t_0} Y = Z_0 Y$  and  $\partial_{t_1} Y = Z_1 Y$ , and using the forms of  $Z_0, Z_1$  given in proposition 6 (expanded around  $\lambda = \theta$ ), we find relations between the fields  $A, B, C, D, a, b, c$  and the field  $p$ ; amongst these relations are the equations

$$\begin{aligned} B' &= \frac{D}{p} & \dot{B} &= -\frac{\dot{p}}{2p} B + \theta D \\ D' &= B \left( \frac{p}{\theta} + \frac{1}{p} \right) & \dot{D} &= (\theta - 2f) B + \frac{\dot{p}}{2p} D \\ b' &= \frac{p B^2}{\theta^2} & \dot{b} &= D^2 - B^2. \end{aligned}$$

Note that if we eliminate  $D$  from this system and write  $\tau = \theta(1 + b)$  we recover the system of equations (24) and (25).  $A$  and  $C$  obey a set of equation identical to that for  $B$  and  $D$ .

*Derivation of the First BT.* Under the transformation (45), we have

$$\begin{aligned} U(t) &\rightarrow \sqrt{\frac{\lambda-\theta}{\lambda+1}} \begin{pmatrix} 0 & 1 \\ 1+1/\lambda & 0 \end{pmatrix} U(t) \begin{pmatrix} 0 & \lambda/(\lambda-\theta) \\ 1 & 0 \end{pmatrix} \\ &= \sqrt{\frac{\lambda-\theta}{\lambda+1}} \begin{pmatrix} 0 & 1 \\ 1+1/\lambda & 0 \end{pmatrix} S^{-1} \begin{pmatrix} -C/A & 1 \\ 1-\theta/\lambda & 0 \end{pmatrix}^{-1} \cdot \begin{pmatrix} -C/A & 1 \\ 1-\theta/\lambda & 0 \end{pmatrix} Y \begin{pmatrix} 0 & \lambda/(\lambda-\theta) \\ 1 & 0 \end{pmatrix}, \end{aligned}$$

where in the last line, I have factored  $U$ , and inserted the product of a certain matrix and its inverse, chosen so that the final expression is the product of a function analytic in  $\{\lambda < \epsilon\} \cup \{\lambda > 1/\epsilon\}$  and another function analytic in  $\{\epsilon < \lambda < 1/\epsilon\}$ . (This can be directly checked; the boundary condition obeyed by  $S$  at  $\lambda = 0$  must be used.) This does not quite complete the Birkhoff factorization; it is necessary to insert a further matrix — independent of  $\lambda$  — and its inverse in order to make sure the new  $S$  satisfies the required boundary conditions. The calculation is arduous, and I omit the details; the final result is

$$\begin{aligned} S &\rightarrow \frac{1}{\sqrt{A^2 - C^2}} \begin{pmatrix} A & 0 \\ -C & (A^2 - C^2)/A \end{pmatrix} \begin{pmatrix} -C/A & 1 \\ 1 - \theta/\lambda & 0 \end{pmatrix} \sqrt{\frac{\lambda+1}{\lambda-\theta}} S \begin{pmatrix} 0 & \lambda/(\lambda+1) \\ 1 & 0 \end{pmatrix} \\ Y &\rightarrow \frac{1}{\sqrt{A^2 - C^2}} \begin{pmatrix} A & 0 \\ -C & (A^2 - C^2)/A \end{pmatrix} \begin{pmatrix} -C/A & 1 \\ 1 - \theta/\lambda & 0 \end{pmatrix} Y \begin{pmatrix} 0 & \lambda/(\lambda-\theta) \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

Examining the behavior of  $S(0)$  we see that the induced transformation on  $\alpha = 1/(S(0))_{11}$  is

$$\alpha \rightarrow \frac{\sqrt{\theta(C^2 - A^2)}}{A} \frac{1}{\alpha},$$

and so (since  $p = \alpha^2$ ),

$$p \rightarrow \left( \frac{C^2}{A^2} - 1 \right) \frac{\theta}{p}.$$

As mentioned above,  $A$  and  $C$  obey the same equations as  $B$  and  $D$ , so we can eliminate  $C$  to write the transformation

$$p \rightarrow \left( \frac{p^2 A^2}{A^2} - 1 \right) \frac{\theta}{p} = \theta p \left( \frac{A^2}{A^2} - \frac{1}{p^2} \right).$$

To make contact with the form in which I have given the BT in section 2, two more manipulations are necessary. First, note that  $s = p\theta A'/A$  satisfies equation (13), and in terms of this the transformation becomes simply

$$p \rightarrow \frac{s^2 - \theta^2}{\theta p} = p - s'.$$

Second, it is necessary to use the  $t_1$  evolution equation for  $A$  to check that  $s$  satisfies (14); this is completely straightforward.

It just remains to confirm that BTs associated with different values of  $\theta$  commute. At first this appears not to be the case. If we denote the automorphism (45) of  $G$  by  $f(\theta)$ , then it is simple to check that the effect of first applying  $f(\theta_1)$  and then  $f(\theta_2)$  to  $U(0)$  is

$$U(0) \rightarrow U(0) \begin{pmatrix} \sqrt{\frac{\lambda-\theta_2}{\lambda-\theta_1}} & 0 \\ 0 & \sqrt{\frac{\lambda-\theta_1}{\lambda-\theta_2}} \end{pmatrix},$$



indicating that order is important. The source of the noncommutativity is that in general

$$\begin{pmatrix} 0 & \frac{\lambda}{\lambda-\theta_1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{\lambda}{\lambda-\theta_2} \\ 1 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & \frac{\lambda}{\lambda-\theta_2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{\lambda}{\lambda-\theta_1} \\ 1 & 0 \end{pmatrix} .$$

But the BT corresponds to the action of the automorphism on an whole  $G_+$  coset in  $G$ , and when this is taken into account commutativity is restored. This can be seen from the simple identity

$$\begin{pmatrix} 0 & \frac{\lambda}{\lambda-\theta_1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{\lambda}{\lambda-\theta_2} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{\lambda}{\lambda-\theta_2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{\lambda}{\lambda-\theta_1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(by inserting  $G_+$  elements — in fact constant  $SL(2)$  matrices in this case — we can make the necessary matrices commute).

*Derivation of the Second BT.* Under the transformation (46) we have

$$\begin{aligned} U(t) &\rightarrow U(t) \left( I + \frac{M}{\lambda - \theta} \right) \\ &= S^{-1} \left( I + \frac{N}{\lambda - \theta} \right)^{-1} \cdot \left( I + \frac{N}{\lambda - \theta} \right) Y \left( I + \frac{M}{\lambda - \theta} \right) , \end{aligned}$$

where once again I have inserted a matrix and its inverse to make the last expression the product of functions analytic in appropriate regions. The correct choice of  $N$ , which works for any  $M$  such that  $M^2 = 0$ , is

$$N = -Y_0 M (I + Y_1 M)^{-1} Y_0^{-1} \quad (47)$$

(this satisfies  $N^2 = 0$  and  $NY_0M = 0$ ). Again, this does not complete the Birkhoff factorization, and we need to insert a constant matrix and its inverse to restore the boundary condition for  $S$  at  $\lambda = 0$ . We thus get the transformation

$$\begin{aligned} S &\rightarrow \begin{pmatrix} \sqrt{1+h^2} & h \\ h & \sqrt{1+h^2} \end{pmatrix} \left( I + \frac{N}{\lambda - \theta} \right) S \\ Y &\rightarrow \begin{pmatrix} \sqrt{1+h^2} & h \\ h & \sqrt{1+h^2} \end{pmatrix} \left( I + \frac{N}{\lambda - \theta} \right) Y \left( I + \frac{M}{\lambda - \theta} \right) , \end{aligned}$$

where

$$h = \frac{N_{12}}{\sqrt{(\theta - N_{22})^2 - N_{12}^2}} . \quad (48)$$

The resulting transformation for  $p$  is given by

$$p \rightarrow p \left[ \left( 1 - \frac{N_{22}}{\theta} \right)^2 - \left( \frac{N_{12}}{\theta} \right)^2 \right] . \quad (49)$$

So far all the formulas given have been for an arbitrary choice of  $M$  in (46), with just the proviso that  $M^2 = 0$ . For the particular choice of  $M$  indicated we find  $N_{12} = B^2/(1+b)$  and  $N_{22} = BD/(1+b)$ . Contact is made with the presentation of the BT in section 2 via the system of equations for  $B, D, b$  given above. For this second BT, commutativity for different values of  $\theta$  is trivial.

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