Hidden Symmetries of the Principal Chiral Model and a Nonstandard Loop Algebra

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Abstract. We examine the precise structure of the loop algebra of 'dressing' symmetries of the Principal Chiral Model, and discuss a new infinite set of abelian symmetries of the field equations which preserve a symplectic form on the space of solutions.

1. The symmetries of the classical two-dimensional Principal Chiral Model, one of the original toy models for nonabelian gauge theories, have been intensively studied for over 15 years. Since the symmetry algebras are infinite dimensional, requiring their preservation at the quantum level provides strong constraints on the quantum theory. In particular, it has been suspected that an investigation of their representation theory would yield some clue about the bound state spectrum of nonabelian gauge theories. The aims of this paper are (a) to explain the rather subtle structure of the celebrated loop algebra of hidden symmetries of PCM, summarising the important points of our longer paper [1], and (b) to discuss some new abelian symmetries of the model. We will show that the latter symmetries preserve a certain natural symplectic form on the space of solutions, which leads us to expect them to be of central importance for an algebraic quantization of the theory. Recently there has been a renewed effort to understand the symmetries of PCM because of their suggested relevance for string theory [2].

2. The classical two-dimensional PCM is the differential equation

$$\partial_-(g^{-1}\partial_+ g) + \partial_+(g^{-1}\partial_- g) = 0 \ , \tag{1}$$

where g is a mapping from two-dimensional Minkowski space to U(N). The algebraic structure of the hidden symmetry transformations associated with the infinite set of non-local conserved currents [3, 4] of the PCM was first discussed by Dolan [5], who determined the algebra

$$[J^a_r, J^b_s] = \sum_c f^{ab}_c J^c_{r+s} \ , \qquad r,s \geq 0 \ , \tag{2}$$

where f^{ab}_c are structure constants of the Lie algebra of U(N) in a basis $\{T^a\}$; $[T^a, T^b] = \sum_c f^{ab}_c T^c$. By using, for the infinite set, a compact generating function form of transformation,

$$g \mapsto g\left(I - Y(x,\lambda) T Y^{-1}(x,\lambda)\right) \ , \tag{3}$$

Dolan's inductive arguments were streamlined in [6], allowing a direct verification of the closure of the commutation relations (2) on the fields g. Here T is a constant infinitesimal antihermitian matrix and $Y(x,\lambda)$ satisfies the PCM Lax-pair [3, 7],

$$\left(\partial_\pm + \frac{1}{1\pm\lambda} A_\pm\right) Y = 0 \ , \qquad A_\pm = g^{-1}\partial_\pm g \ , \tag{4}$$

in virtue of which (3) may easily be shown to generate symmetries of equation (1). Y is singular on the Riemann sphere only at $\lambda = \pm 1$ and satisfies the reality/boundary conditions

$$Y^\dagger(\lambda^*) = Y^{-1}(\lambda) \ , \quad Y(x, \lambda = \infty) = I \ , \quad Y(x, \lambda = 0) = g^{-1} \ , \quad Y(x_0, \lambda) = I \ , \tag{5}$$

where x_0 is some fixed point.

A contour integral representation of the transformation (3) is obtained in [1], corresponding to similar representations in the literature (see, for example, [2, 9]):

$$g \mapsto g \left(I - \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{Y(x,\lambda')\epsilon(\lambda')Y^{-1}(x,\lambda')}{\lambda'} d\lambda' \right) \ , \tag{6}$$

where the contour $\mathcal{C}$ is the union of two contours $\mathcal{C}_\pm$ around $\lambda = \pm 1$ (such that $\lambda = 0$ remains outside both of them). If the Lie-algebra-valued infinitesimal parameter of the transformation $\epsilon(\lambda)$ is taken to be proportional to $\lambda^r T^a$, $r \in \mathbf{Z}$, the integral may be evaluated (for $r < 0$ by deforming $\mathcal{C}$ to a contour around 0; for $r > 0$ to a contour around ∞; and for $r = 0$ to a pair of contours around 0 and ∞) and an algebra of symmetries

$$[J_r^a, J_s^b] = \sum_c f_c^{ab} J_{r+s}^c \ , \qquad r, s \in Z \ , \tag{7}$$

may be identified, verifying the determination of these commutation relations by [8]. A careful consideration of the symmetries (6) shows, however, that *finding the commutation relations (7) is not sufficient to identify the symmetry algebra with the standard loop algebra.* In particular:

a) Whereas for the standard loop algebra we require that the infinite sum $\sum \alpha_{ra} J_r^a$ should be admitted if and only if $\sum \alpha_{ra} \lambda^r T^a$ is convergent for $|\lambda| = 1$, (thus defining a map from the unit circle into the Lie algebra of U(N)), the infinite linear combinations of the J_r^a allowed in our case are entirely different.

b) There are transformations of the form (6) which cannot be expressed as linear combinations of the J_r^a.

c) Possibly most importantly, in the algebra associated with the transformations (6), the elements J_r^a with $r < 0$ are in fact in the closure of the linear span of the transformations $\{J_r^a; r \geq 0\}$. This means that the elements with J_r^a with $r < 0$ are not strictly linearly independent of those with $r \geq 0$.

3. Let G_- (resp. G_+) be the group of smooth maps from $\mathcal{C}$ to U(N) which are the boundaries of maps analytic inside (resp. outside) $\mathcal{C}$, i.e. analytic in the region $\{|\lambda - 1| < \delta\} \cup \{|\lambda + 1| < \delta\}$ (resp. $\{|\lambda - 1| > \delta\} \cap \{|\lambda + 1| > \delta\}$), where $\delta < 1$ is some radius. We denote the corresponding Lie algebra $\mathcal{G}_-$ (resp. $\mathcal{G}_+$). Y satisfying (4) clearly takes values in G_+ .

The symmetry algebra associated with nontrivial transformations of the form (6) is $\mathcal{G}_-$, a nonstandard loop algebra. This is explained fully in [1], but essentially it is because we clearly can take $\epsilon(\lambda)$ in (6) to be an arbitrary infinitesimal element of $\mathcal{G}_-$; taking it to be an infinitesimal element of $\mathcal{G}_+$ gives a trivial transformation.

Choosing $\epsilon(\lambda) \in \mathcal{G}_-$, the natural way to expand it is in a Taylor series in $\lambda + 1$ (or alternatively in $\lambda - 1$). Taking $\epsilon(\lambda)$ to be proportional to $(\lambda + 1)^n T^a$, for $n \geq 0$, we define a set of transformations $\{K_n^a\}$ satisfying the algebra

$$[K_n^a, K_m^b] = \sum_c f_c^{ab} K_{n+m}^c \ , \qquad n, m \geq 0 \ . \tag{8}$$

The transformations J_r^a satisfying the relations (7) are obtained in terms of the K_n^a by considering the expansion of λ^r in powers of $\lambda+1$ (valid in $|\lambda+1|<\delta$). This gives

$$J_r^a = \sum_{n=0}^{r}(-1)^{n+r}\binom{r}{n}K_n^a \ , \qquad r\geq 0 \ , \tag{9a}$$

$$J_r^a = \sum_{n=0}^{\infty}(-1)^{r}\binom{n-r-1}{-r-1}K_n^a \ , \qquad r<0 \ . \tag{9b}$$

Standard formulae for sums of binomial coefficients may be used to verify that the commutation relations (8) $\Rightarrow$ (7). Just as the J's for non-negative r can be expressed as finite sums of the K's, the latter can likewise be expanded as a finite linear combination of the former:

$$K_n^a = \sum_{r=0}^{n}\binom{n}{r}J_r^a \ . \tag{10}$$

When we substitute (10) into the right hand side of (9b), we find that we cannot reorder the summations to express this infinite sum as a linear combination of the J_r^a's with $r\geq 0$. In other words, if in the standard loop algebra we define elements K_n^a via (10), the infinite sum on the RHS of (9b) is not in the algebra, while it is in the nonstandard loop algebra $\mathcal{G}_-$.

In general, infinite linear combinations of the K's cannot be written as linear combinations of the J_r^a. Elements of $\mathcal{G}_-$ can however be *approximated* (to arbitrary accuracy) by finite sums of the J_r^a, $r\geq 0$, as required by Runge's theorem (see, for example, [10]). This notwithstanding, the elements J_r^a are not a spanning set for the algebra $\mathcal{G}_-$, as they are for the standard loop algebra; the spanning set for the algebra $\mathcal{G}_-$ is the set $\{K_n^a\}$. To see immediately that the elements $\{J_r^a\}$ are not a spanning set for the algebra $\mathcal{G}_-$, one need only consider an element of $\mathcal{G}_-$ proportional to $\ln\lambda$, defined with a cut from 0 to ∞ along half of the imaginary axis.

4. We now describe some new PCM symmetries. The PCM potentials $A_\pm$ satisfy the equations of motion

$$\partial_\mp A_\pm = \pm\tfrac{1}{2}[A_+,A_-] \ . \tag{11}$$

These imply that the eigenvalues of A_+ (resp. A_-) are independent of x^- (resp. x^+). In other words, A_+ and A_- are similarity transformations of diagonal antihermitean matrices $A(x^+)$ and $B(x^-)$ respectively,

$$\begin{aligned} A_+ &= s_0(x^+,x^-)A(x^+)s_0^{-1}(x^+,x^-) \ , \\ A_- &= \tilde{s}_0(x^+,x^-)B(x^-)\tilde{s}_0^{-1}(x^+,x^-) \ , \end{aligned} \tag{12}$$

where $s_0,\tilde{s}_0$ are U(N)-valued fields. The construction of [1] produced solutions of this form, with A,B free fields and $s_0,\tilde{s}_0$ satisfying certain equations; the equations for s_0 are

$$\begin{aligned} \partial_+ s_0 s_0^{-1} &= [t,A_+] \ , \\ \partial_- s_0 s_0^{-1} &= \tfrac{1}{2}(s_0Bs_0^{-1}-A_-) \ , \end{aligned} \tag{13}$$

where t, an auxiliary Lie-algebra-valued field, satisfies

$$\partial_- t = \tfrac{1}{4}(s_0Bs_0^{-1}-A_-)+\tfrac{1}{2}[t,A_-] \ . \tag{14}$$

The latter equation is sufficient for the consistency of the system (13).

Now, *if $f(x^+)$ is an arbitrary infinitesimal diagonal antihermitean matrix depending only on x^+, the transformation*

$$g \mapsto g\left(I + s_0 f(x^+) s_0^{-1}\right) \tag{15}$$

is a symmetry of the PCM. To prove this we note that under an arbitrary infinitesimal transformation of the form $g \mapsto g(I+\Phi)$, where Φ takes values in the Lie-algebra, we have $A_\pm \mapsto A_\pm + D_\pm \Phi$, where $D_\pm$ denotes the covaraint derivative defined by $D_\pm\Phi = \partial_\pm \Phi + [A_\pm, \Phi]$. It is straightforward to check that $\Phi = s_0 f(x^+) s_0^{-1}$ satisfies $\partial_- D_+ \Phi + \partial_+ D_- \Phi = 2\partial_-\partial_+\Phi + [A_+, \partial_-\Phi] + [A_-, \partial_+\Phi] = 0$ in virtue of (11,13,14) and therefore generates a symmetry. A cumbersome calculation shows that *these new symmetries form an infinite-dimensional abelian algebra*; this calculation is made redundant by the considerations of [1].

5. When considering symmetries of classical equations of a field theory, special importance is attached to symmetries (vector fields on the space of solutions) which preserve a symplectic form on the space of solutions. As noted above, associated with any solution of the PCM is a gauge potential with components $A_\pm$. On the space of gauge potentials on a two-manifold there is a natural symplectic structure [11],

$$\omega = \int Tr\,(\delta A \wedge \delta A) = \int dx^+ dx^-\; Tr\,(\delta A_+ \wedge \delta A_-)\,. \tag{16}$$

This symplectic form plays a central role in Chern-Simons theory [12]; see for example [13]. We consider the pullback of this symplectic form to the space of solutions of the PCM. This has also been considered in [14].

The symplectic form ω is known to be gauge invariant. Moreover, the potentials satisfying (11) are pure-gauge; so, in particular, all symmetries of the equations are gauge transformations. This seems to mean that all symmetry transformations must leave the symplectic form invariant, implying that it is totally degenerate (i.e. zero). This is an incorrect argument, however. PCM solutions form a *subset* of all pure gauge potentials and gauge transformations which are genuine PCM symmetries are *field-dependent*, whereas the invariance of ω is under field-independent gauge transformations.

Let us consider the symplectic form ω on a space on which the x^+ coordinate is compactified, and $x^- \in [a,b]$. An infinitesimal transformation of PCM solutions given by $g \mapsto g(I+\Phi)$ corresponds to the vector field

$$V = (D_+\Phi)\frac{\delta}{\delta A_+} + (D_-\Phi)\frac{\delta}{\delta A_-} \tag{17}$$

on the space of potentials, and we have

$$i_V\omega = \int dx^+ dx^-\; Tr\,((D_+\Phi)\delta A_- - (D_-\Phi)\delta A_+)\,. \tag{18}$$

Integrating by parts, and using the fact that all PCM potentials have zero curvature, which means that $D_+\delta A_- - D_-\delta A_+ = 0$, we obtain

$$i_V\omega = \int dx^+\; Tr(\Phi\delta A_+)\Big|_{x^-=b}^{x^-=a}\quad . \tag{19}$$

Using the first equation of (12) to write A_+ in terms of s_0 and A, this yields

$$i_V\omega = \int dx^+\; Tr\left(s_0^{-1}\Phi s_0 \delta A + [A, s_0^{-1}\Phi s_0] s_0^{-1}\delta s_0\right)\Big|_{x^-=b}^{x^-=a}\quad . \tag{20}$$

For the new PCM symmetries described in section 4, we have $\Phi = s_0 f(x^+) s_0^{-1}$, where $[f, A] = 0$, and f is field independent. It follows that for such symmetries we have $i_V\omega = \delta \int dx^+ Tr\,(f(x^+)A(x^+))$, implying that $\delta(i_V\omega) = 0$, i.e. the symplectic form is preserved under these symmetries.

The new PCM symmetries of section 4 thus preserve a symplectic form. The loop algebra symmetries are not believed to have such a property (see, e.g., [15]). Our symplectic structure on the space of PCM solutions is not the standard one. Usually the symplectic form is derived from a Lagrangian, and the standard PCM Lagrangian does not give the above symplectic form. If, however, in the Lagragian approach, one of the light-cone coordinates is regarded as 'time', the symplectic forms coincide. (This refers to the standard Lagrangian for the PCM, not the so-called 'dual' formulation [7].) Although this choice appears not to be a 'physical' one, we have some hopes that the new symmetries described in this letter, as well as the other constructions of [1], will shed some light on algebraic quantization of the PCM.

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