THE HAMILTON-JACOBI DIFFERENCE EQUATION *

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Abstract. We study a system of difference equations which, like Hamilton's equations, preserves the standard symplectic structure on $\mathbf{R}^{2m}$. In particular, we construct a differential-difference equation which we call *the Hamilton-Jacobi difference equation*, the analog of the Hamilton-Jacobi equation for our discrete system. We solve the Hamilton-Jacobi difference equation in a simple case.

The Hamilton-Jacobi equation is of great importance in analytic dynamics [1]. Here an analogous difference equation is derived for canonical systems of difference equations. It is shown that the general solution of the Hamilton-Jacobi difference equation is equivalent to the general solution of the canonical system of difference equations to which it is related.

We consider a sequence $(X^{(0)},Y^{(0)}),(X^{(1)},Y^{(1)}),\dots,(X^{(n)},Y^{(n)}),\dots$ in $\mathbf{R}^m\times\mathbf{R}^m$, generated from the initial point $(X^{(0)},Y^{(0)})$ via the difference equations

$$(1)\qquad \begin{aligned} X_k^{(n+1)} &= X_k^{(n)} + h\frac{\partial H_n(X^{(n)},Y^{(n+1)})}{\partial Y_k^{(n+1)}} \\ Y_k^{(n+1)} &= Y_k^{(n)} - h\frac{\partial H_n(X^{(n)},Y^{(n+1)})}{\partial X_k^{(n)}} \end{aligned}\ .$$

Throughout, $k=1,\dots,m$, and $n=0,1,2\dots$; $X_k^{(n)}$, $Y_k^{(n)}$ denote the k-th components of $X^{(n)}$, $Y^{(n)}$, and the "Hamiltonian" functions $H_n:\mathbf{R}^m\times\mathbf{R}^m\to\mathbf{R}$ are assumed differentiable. The equations (1) define $(X^{(n+1)},Y^{(n+1)})$ implicitly for given $(X^{(n)},Y^{(n)})$; we assume that for small enough h there exists a unique solution of these equations such that the mapping $(X^{(n)},Y^{(n)})\mapsto(X^{(n+1)},Y^{(n+1)})$ is a diffeomorphism which is a continuous deformation of the identity (below we also make the assumption that the map $(X^{(n)},Y^{(n)})\mapsto(X^{(n)},Y^{(n+1)})$ is a diffeomorphism). In the important case $H_n(X^{(n)},Y^{(n+1)})=\frac{1}{2}Y^{(n+1)}\cdot Y^{(n+1)}+V_n(X^{(n)})$, the equations (1) are explicit.

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Denoting by d the exterior derivative on $\mathbf{R}^m \times \mathbf{R}^m$ (the $(X^{(n)}, Y^{(n)})$ are determined by $(X^{(0)}, Y^{(0)})$ and in this way are functions on $\mathbf{R}^m \times \mathbf{R}^m$), it is straightforward to check that the equations (1) are equivalent to the equations

$$(2)\quad X^{(n+1)} \cdot dY^{(n+1)} + Y^{(n)} \cdot dX^{(n)} = d\left(Y^{(n+1)} \cdot X^{(n)} + hH_n(X^{(n)}, Y^{(n+1)})\right).$$

(Here a dot denotes the standard inner product on $\mathbf{R}^m$.) From (2) it follows that the standard symplectic structure on $\mathbf{R}^m \times \mathbf{R}^m$ is preserved, i.e.

$$\sum_{k=1}^{m} dY_k^{(n+1)} \wedge dX_k^{(n+1)} = \sum_{k=1}^{m} dY_k^{(n)} \wedge dX_k^{(n)}. \tag{3}$$

We therefore refer to the system of difference equations (1) as *canonical*, following the usage in [2]. The system (1) is not the only possible discretization of Hamilton's equations, and displays an apparently undesirable asymmetry between X and Y. We will comment on this further at a later stage.

Let us make the canonical change of variables $(X^{(n)}, Y^{(n)}) \mapsto (U^{(n)}, V^{(n)})$, defined by

$$\begin{aligned} V_k^{(n)} &= -\frac{\partial W_n(X^{(n)}, U^{(n)}, h)}{\partial U_k^{(n)}} \\ Y_k^{(n)} &= \frac{\partial W_n(X^{(n)}, U^{(n)}, h)}{\partial X_k^{(n)}} \end{aligned}, \tag{4}$$

which can alternatively be written as an equation for differentials

$$Y^{(n)} \cdot dX^{(n)} - V^{(n)} \cdot dU^{(n)} = dW_n(X^{(n)}, U^{(n)}, h). \tag{5}$$

(Again, assumptions on W are necessary to guarantee that the implicit equations (4) define a change of coordinates. We gloss over these.) Eliminating $X^{(n)}, Y^{(n)}$ from (5) using (2), we obtain

$$\begin{aligned} U^{(n+1)} \cdot dV^{(n+1)} &+ V^{(n)} \cdot dU^{(n)} \\ &= d\left(V^{(n+1)} \cdot U^{(n)} + hS_n(U^{(n)}, V^{(n+1)}, h)\right), \end{aligned} \tag{6}$$

or, equivalently,

$$\begin{aligned} U_k^{(n+1)} &= U_k^{(n)} + h\frac{\partial S_n(U^{(n)}, V^{(n+1)}, h)}{\partial V_k^{(n+1)}} \\ V_k^{(n+1)} &= V_k^{(n)} - h\frac{\partial S_n(U^{(n)}, V^{(n+1)}, h)}{\partial U_k^{(n)}} \end{aligned}, \tag{7}$$

where we have denoted

$$
\begin{aligned}
(8)\, hS_n(U^{(n)},V^{(n+1)},h) &= W_{n+1}(X^{(n+1)},U^{(n+1)},h) - W_n(X^{(n)},U^{(n)},h) \\
&+ V^{(n+1)}\cdot(U^{(n+1)}-U^{(n)}) \\
&- Y^{(n+1)}\cdot(X^{(n+1)}-X^{(n)}) + hH_n(X^{(n)},Y^{(n+1)})\ .
\end{aligned}
$$

In the equality (8) it is assumed that all the variables appearing on the RHS are written in terms of $U^{(n)}, V^{(n+1)}$ using the expressions for the change of variables and the difference equations. Equation (8) can also be written (using (1),(4) and (6)) in the form

$$
\begin{aligned}
&hS_n(U^{(n)},V^{(n+1)},h) + h\sum_{k=1}^{m} \frac{\partial W_{n+1}(X^{(n+1)},U^{(n+1)},h)}{\partial U_k^{(n+1)}} \frac{\partial S_n(U^{(n)},V^{(n+1)},h)}{\partial V_k^{(n+1)}} \\
(9)\quad &= W_{n+1}(X^{(n+1)},U^{(n+1)},h) - W_n(X^{(n)},U^{(n)},h) + hH_n(X^{(n)},Y^{(n+1)}) \\
&\qquad - h\sum_{k=1}^{m} \frac{\partial W_{n+1}(X^{(n+1)},U^{(n+1)},h)}{\partial X_k^{(n+1)}} \frac{\partial H_n(X^{(n)},Y^{(n+1)})}{\partial Y_k^{(n+1)}}\ .
\end{aligned}
$$

We see that the canonical system of difference equations (1) is transformed to another canonical system of difference equations (7) by the canonical change of variables (4).

We now consider the conditions necessary for a canonical change of variables to bring us to a system in which we have.

$$
(10)\qquad S_n(U^{(n)},V^{(n+1)},h) = 0.
$$

When (10) holds, the system (7) can be solved exactly, to give

$$
(11)\qquad U^{(n)} = U^{(0)}, \qquad V^{(n)} = V^{(0)}.
$$

Thus the form of the canonical change of variables (4) is

$$
(12)\qquad
\begin{aligned}
V_k^{(0)} &= -\frac{\partial W_n(X^{(n)},U^{(0)},h)}{\partial U_k^{(0)}} \\
Y_k^{(n)} &= \frac{\partial W_n(X^{(n)},U^{(0)},h)}{\partial X_k^{(n)}}
\end{aligned}
\quad ,
$$

where the functions $W_n(X^{(n)},U^{(0)},h)$ satisfy the difference equation

$$
\begin{aligned}
0 &= W_{n+1}(X^{(n+1)},U^{(0)},h) - W_n(X^{(n)},U^{(0)},h) + hH_n(X^{(n)},Y^{(n+1)}) \\
(13)\quad &- h\sum_{k=1}^{m} \frac{\partial W_{n+1}(X^{(n+1)},U^{(0)},h)}{\partial X_k^{(n+1)}} \frac{\partial H_n(X^{(n)},Y^{(n+1)})}{\partial Y_k^{(n+1)}}\ .
\end{aligned}
$$

The differential-difference equation (13) will be called the *Hamilton-Jacobi difference equation*. The solution $W_n = W_n(X^{(n)}, U^{(0)}, h)$, $n = 0, 1, 2, \ldots$, of (13), depending on m arbitrary constants $U^{(0)}$, will be called the general integral of (13) provided the system of implicit equations

$$V_k^{(0)} = -\frac{\partial W_n(X^{(n)}, U^{(0)}, h)}{\partial U_k^{(0)}} \tag{14}$$

(the first equation of (12)) can be solved for $X^{(n)}$. A sufficient condition for this is the nonvanishing of the determinant of the matrix of second derivatives

$$\frac{\partial^2 W_n(X^{(n)}, U^{(0)}, h)}{\partial U_k^{(0)} \partial X_l^{(n)}} \qquad \begin{matrix} k = 0, 1, \ldots, m \\ l = 0, 1, \ldots, m \end{matrix} \quad . \tag{15}$$

Thus we arrive at the following result: *if we know the general integral of the Hamilton-Jacobi difference equation (13), then finding the general solution of the canonical system of difference equations (1) is reduced to solving the system of implicit equations (12).*

Let us now consider the special "linear" case

$$\begin{aligned} H_n(X^{(n)}, Y^{(n+1)}) &= X^{(n)T} A_0^{(n)} X^{(n)} + X^{(n)T} A_1^{(n)} Y^{(n+1)} \\ &+ Y^{(n+1)T} A_2^{(n)} Y^{(n+1)}, \end{aligned} \tag{16}$$

where $A_0^{(n)}, A_1^{(n)}, A_2^{(n)}$ are $m \times m$ matrices, with $A_0^{(n)}, A_2^{(n)}$ symmetric. Equation (13) can be written

$$\begin{aligned} 0 &= W_{n+1}(X^{(n+1)}, U^{(0)}, h) - W_n(X^{(n)}, U^{(0)}, h) + hH_n(X^{(n)}, Y^{(n+1)}) \\ &- h \sum_{k=1}^{m} Y_k^{(n+1)} \frac{\partial H_n(X^{(n)}, Y^{(n+1)})}{\partial Y_k^{(n+1)}} , \end{aligned} \tag{17}$$

using the second equation of (12). Since the H_n of equation (16) is the sum of three terms of fixed homogeneity in the components of $Y^{(n+1)}$, this gives

$$\begin{aligned} 0 &= W_{n+1}(X^{(n+1)}, U^{(0)}, h) - W_n(X^{(n)}, U^{(0)}, h) \\ &+ h(X^{(n)T} A_0^{(n)} X^{(n)} - Y^{(n+1)T} A_2^{(n)} Y^{(n+1)}) . \end{aligned} \tag{18}$$

In this equation it is understood that the vectors $X^{(n+1)}, Y^{(n+1)}$ should be written in terms of $X^{(n)}, Y^{(n)}$ via the equations of motion, and that $Y^{(n)}$ is determined by the function $W_n(X^{(n)}, U^{(0)}, h)$ via the second equation of (12). Since the equations are rather lengthy in the general case, we restrict here to the simple case $A_0^{(n)} = A_2^{(n)} = \frac{1}{2}I$, $A_1^{(n)} = 0$, where we obtain, after some algebra, the system

$$\begin{aligned} 0 &= W_{n+1}\left((1-h^2)X^{(n)} + hY^{(n)}, U^{(0)}, h\right) - W_n(X^{(n)}, U^{(0)}, h) \\ &+ \tfrac{1}{2}h(1-h^2)X^{(n)}\cdot X^{(n)} + h^2 X^{(n)}\cdot Y^{(n)} - \tfrac{1}{2}hY^{(n)}\cdot Y^{(n)}\ , \end{aligned} \tag{19}$$

$$Y_k^{(n)} = \frac{\partial W_n(X^{(n)}, U^{(0)}, h)}{\partial X_k^{(n)}}\ .$$

Making the ansatz

$$W_n(X^{(n)}, U^{(0}, h) = Q(X^{(n)}, U^{(0}, h) - nh\alpha, \qquad \alpha \ \text{ constant}, \tag{20}$$

which is appropriate whenever H_n is independent of n, we finally obtain

$$\begin{aligned} \alpha &= \left(\frac{Q\left((1-h^2)X^{(n)} + hY^{(n)}, U^{(0)}, h\right) - Q(X^{(n)}, U^{(0)}, h)}{h}\right) \\ &+ \tfrac{1}{2}(1-h^2)X^{(n)}\cdot X^{(n)} + hX^{(n)}\cdot Y^{(n)} - \tfrac{1}{2}Y^{(n)}\cdot Y^{(n)}\ , \end{aligned} \tag{21}$$

$$Y_k^{(n)} = \frac{\partial Q(X^{(n)}, U^{(0)}, h)}{\partial X_k^{(n)}}\ .$$

For the case $m = 1$ a solution of this can be found: Taking $U^{(0)} = \alpha$, it is a simple but laborious calculation to check that

$$\begin{aligned} Q(X^{(n)}, \alpha, h) &= \frac{h}{4}X^{(n)2} - \frac{1}{2}\sqrt{1 - \frac{h^2}{4}} \\ &\quad \left(X^{(n)}\sqrt{C^2\alpha - X^{(n)2}} + \alpha C^2 \sin^{-1}\left(\frac{X^{(n)}}{C\sqrt{\alpha}}\right)\right) \end{aligned} \tag{22}$$

satisfies equation (21), where C is a positive constant defined by

$$\frac{1}{C^2} = \frac{\theta}{2h}\sqrt{1 - \frac{h^2}{4}}, \qquad \theta = \cos^{-1}\left(1 - \frac{h^2}{2}\right). \tag{23}$$

Writing $\beta = V^{(0)}$, we have (from equation (14)):

$$\begin{aligned} \beta &= -\frac{\partial W_n(X^{(n)}, \alpha, h)}{\partial \alpha} \\ &= nh - \frac{\partial Q(X^{(n)}, \alpha, h)}{\partial \alpha} \\ &= nh + \frac{h}{\theta}\sin^{-1}\left(\frac{X^{(n)}}{C\sqrt{\alpha}}\right)\ , \end{aligned} \tag{24}$$

giving

$$X^{(n)} = -C\sqrt{\alpha}\sin\left(n\theta - \frac{\beta\theta}{h}\right). \tag{25}$$

We also have

$$Y^{(n)} = \frac{\partial Q(X^{(n)},\alpha,h)}{\partial X^{(n)}} = \frac{h}{2}X^{(n)} - \sqrt{1-\frac{h^2}{4}}\sqrt{C^2\alpha - X^{(n)2}}, \tag{26}$$

which, using equation (25) and the fact $\sin(\theta/2) = h/2$, gives

$$Y^{(n)} = -C\sqrt{\alpha}\cos\left(n\theta - \frac{\beta\theta}{h} - \frac{\theta}{2}\right) . \tag{27}$$

It is straightforward to check that equations (25) and (27) give the general solution of the canonical system (1) for the case $m = 1$ and $H_n(X^{(n)}, Y^{(n+1)}) = \frac{1}{2}\left(X^{(n)2} + Y^{(n+1)2}\right)$. The quantity $X^{(n)2}+Y^{(n)2}-hX^{(n)}Y^{(n)} = C^2\alpha(1-h^2/4)$ is conserved in this case. It should be admitted that the above solution (22) of the Hamilton-Jacobi difference equation (21) was actually constructed by obtaining the general solution of (1) and working backwards. At the present time the differential-difference equation (21) is more of formal interest than it is useful for calculations.

We conclude with a discussion of the $h \to 0$ limit. Denoting in the obvious fashion by $X(t), Y(t), H(t, X, Y)$ etc. the limits (as $h \to 0$ and $n \to \infty$ with $nh = t$) of $X^{(n)}, Y^{(n)}, H_n(X^{(n)}, Y^{(n+1)})$ etc., the canonical system of difference equations (1) tends in the limit to the Hamiltonian system

$$\frac{dX_k}{dt} = \frac{\partial H(t,X,Y)}{\partial Y_k} , \qquad \frac{dY_k}{dt} = -\frac{\partial H(t,X,Y)}{\partial X_k} . \tag{28}$$

In the limit we have

$$\begin{aligned} &\frac{1}{h}\left(W_{n+1}(X^{(n+1)},U^{(0)},h) - W_n(X^{(n)},U^{(0)},h)\right) \\ &\qquad \longrightarrow \quad \frac{\partial W(t,X,U)}{\partial t} + \sum_{k=1}^{m} \frac{\partial W(t,X,U)}{\partial X_k}\frac{dX_k}{dt} , \end{aligned} \tag{29}$$

so, dividing equation (13) by h, taking the limit, and using the first equation of (28), we have

$$\frac{\partial W(t,X,U)}{\partial t} + H(t,X,Y) = 0, \tag{30}$$

where the vector Y is defined by

$$Y_k = \frac{\partial W(t, X, U)}{\partial X_k}. \tag{31}$$

The general integral of the Hamilton-Jacobi equation (30)-(31) enables us to completely integrate the Hamiltonian system (28), the change of variables

$$V_k = -\frac{\partial W(t, X, U)}{\partial U_k}\ , \qquad Y_k = \frac{\partial W(t, X, U)}{\partial X_k} \tag{32}$$

transforming the system to the system

$$\frac{dU_k}{dt} = 0\ , \qquad \frac{dV_k}{dt} = 0\ . \tag{33}$$

Thus in the $h \to 0$ limit we recover the standard Hamilton-Jacobi theory for continuous Hamiltonian systems.

As was mentioned towards the start of this paper, the system (1) is only one possible discretization of Hamilton's equations (28). The system (1) displays an asymmetry between X and Y, and also, in the general case, only defines $X^{(n+1)}$ and $Y^{(n+1)}$ implicitly. We currently do not know how to provide a discretization of Hamilton's equations that is canonical, in the sense that equation (3) holds, but does not have the two drawbacks just mentioned. In particular, the most obvious discretization

$$\begin{aligned} X_k^{(n+1)} &= X_k^{(n)} + h\frac{\partial H_n(X^{(n)}, Y^{(n)})}{\partial Y_k^{(n)}} \\ Y_k^{(n+1)} &= Y_k^{(n)} - h\frac{\partial H_n(X^{(n)}, Y^{(n)})}{\partial X_k^{(n)}} \end{aligned}$$

is explicit and symmetric in X and Y, but not canonical.

The work in this paper is relevant to the study of *symplectic integrators* [3], numerical techniques for the integration of hamiltonian systems, preserving the symplectic structure. The relationship will be discussed in another publication. We thank the referee for pointing out the need to comment on the asymmetry of the system (1).

REFERENCES

[1] See, for example, H.Goldstein, *Classical Mechanics*, Addison-Wesley, 2nd edition, 1980.

[2] A.Khalanai and D.Veksler, *Kachestvennaia Teoriia Impul'snukh Sistem*, Mir, Moscow, 1971.

[3] See, for example, P.J.Channell and C.Scovel, Symplectic Integration of Hamiltonian Systems, *Nonlinearity*, **3** (1990), 231-259.