Oct 1995

Zero Curvature Formulations of Dual Hierarchies

Jeremy Schiff

Department of Mathematics and Computer Science

Bar Ilan University

Ramat Gan 52900, Israel

Abstract

Zero curvature formulations are given for the "dual hierarchies" of standard soliton equation hierarchies, recently introduced by Olver and Rosenau, including the physically interesting Fuchssteiner-Fokas-Camassa-Holm hierarchy.

PACS Numbers: 03.40.Kf, 47.20.Ky, 52.35.Sb

Through the years since the discovery of the notion of “integrability” in PDEs, quite a number of integrable PDEs have been discovered, most of which remain obscure for lack of any physical significance. For nearly fifteen years now, the equation

$$u_t + 2\kappa u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1}$$

derived by Fuchssteiner and Fokas [1] has enjoyed such obscurity; but in a recent paper of Camassa and Holm [2], this equation was rediscovered, and looks likely to be of some importance. Like Fuchssteiner and Fokas, Camassa and Holm showed that, for $\kappa = 0$, (1) has *bihamiltonian structure*: if we write $m = u - u_{xx}$, then (1) takes the form

$$m_t \;= -\; J_1 \frac{\delta H_2}{\delta m} \;= -\; J_2 \frac{\delta H_1}{\delta m}, \tag{2}$$

where

$$J_1 = \partial - \partial^3, \qquad J_2 = \partial m + m\partial \tag{3}$$

are two compatible hamiltonian operators, and

$$H_2 = \tfrac{1}{2}\int_{-\infty}^{\infty} (u^3 + uu_x^2)dx, \qquad H_1 = \tfrac{1}{2}\int_{-\infty}^{\infty} (u^2 + u_x^2)dx. \tag{4}$$

The novelty of Camassa and Holm's work was that they gave a *physical derivation* of (1). Furthermore, for $\kappa = 0$, they found solutions to (1) which they named “peakons” (travelling wave solutions with a corner at their peak); these take the simple form

$$u = c\exp(-|x - ct|). \tag{5}$$

More generally they showed that

$$u = \sum_{i=1}^{N} p_i(t)\exp(-|x - q_i(t)|) \tag{6}$$

gives an N-peakon solution, provided $\{p_i(t), q_i(t)\}$ solves hamilton's equations for the hamiltonian

$$H_A = \tfrac{1}{2}\sum_{i,j=1}^{N} p_i p_j \exp(-|q_i - q_j|). \tag{7}$$

Camassa and Holm proved this hamiltonian system is integrable, and gave its solution for N=2. For $\kappa \neq 0$, solutions of (1) have been investigated numerically in [3].

A little prior to Camassa and Holm's work, Rosenau and Hyman [4] made the remarkable observation that a large class of nonlinear PDEs with nonlinear dispersion terms exhibited "compacton" solutions, *viz.* solitons with compact spatial support. Rosenau [5] further showed that this phenomenon can also occur in integrable PDEs; in particular, if we replace (x, t) in the Fuchssteiner-Fokas-Camassa-Holm equation (1) by (ix, it), we find the equation

$$u_t + 2\kappa u_x + u_{xxt} + 3uu_x + 2u_x u_{xx} + uu_{xxx} = 0, \tag{8}$$

and this admits, for $\kappa = 0$, the compacton solution

$$u = c\cos(x - ct) \qquad |x - ct| \le \frac{\pi}{2}. \tag{9}$$

(The compacton solutions of (8) are actually unstable; but it serves to illustrate that compactons can occur in the framework of integrability; in addition it seems further equations in its hierarchy have acceptable properties. I thank Philip Rosenau for information on this point.)

In the wake of this work, two apparently widely applicable constructions of integrable PDEs with nonlinear dispersion terms have been given. The first, due to Rosenau [6], consists of applying Lagrange transformations to soliton-bearing integrable PDEs, such as the KdV and MKdV equations. The philosophy here is that the standard solitons in such equations, despite being of infinite spatial extent, carry finite mass and/or momentum, and hence must be of compact support when measured in mass and/or momentum units. The second construction, due to Olver and Rosenau [7] (again a rediscovery of Fuchssteiner and Fokas' earlier work [1]; the reader should also see the modern work [8] of Fokas), starts from the observation that the two hamiltonian operators J_1, J_2 given in (3) look like recombinations of terms from the two standard hamiltonian operators of the KdV equation (see, for example, [9]). In fact it turns out that if a bihamiltonian integrable hierarchy has one hamiltonian operator which is a constant coefficient differential operator, and another hamiltonian operator which is a linear combination of a constant coefficient differential operator and another operator which scales homogeneously with non-zero degree when the fields are rescaled, then by recombining terms from these hamiltonian operators one can construct a new hierarchy. In [7] this procedure is followed to construct dual hierarchies of the KdV, MKdV, Broer-Kaup-Kupershmidt and Ito hierarchies (the NLS hierarchy is also dualized by a variant of the general procedure). The aim of this paper is to provide yet another method of constructing dual hierarchies, reproducing the results of [7]. This time the initial observation is the similarity of the linear system associated with the Camassa-Holm equation (the linear system is given in equation (6) of [2]), and the linear system associated with the KdV equation. We will see that *zero curvature formulations* of dual hierarchies can be obtained by a simple modification of the well-known zero curvature formulations of the standard soliton equation hierarchies.

The original purpose of this work was twofold. First, for standard soliton equation hierarchies, the zero curvature formulation is a springboard for revealing many other properties of the hierarchies. In particular, in the zero curvature formulation one sees a natural group action on the space of solutions (the group of "dressing transformations"), which, when it can be made explicit, gives rise to a host of solutions of the hierarchies (for a compact overview of how the group of dressing transformations gives rise to the tau-function formalism for the MKdV equation, see [10]). Alas, while the zero curvature formulations of dual hierarchies are only slight variations of those for standard hierarchies, this slight variation complicates the explicit realization of dressing transformations, and we have been unable, as of yet, to compute explicit dressing transformations and generate solutions this way. The second hope in undertaking this work was that, while Olver and Rosenau's construction [7] *cannot* be extended to, for example, the Boussinesq ($SL(3)$ KdV) equation (one hamiltonian operator is a constant coefficient differential operator, as required, but the other is the sum of a constant coefficient differential operator and another term that does not scale homogeneously under any rescaling of the fields), it was hoped that the zero curvature formulations would suggest an extension. Extensive experiments in this direction — which will not be reported here — have so far yielded only negative results. It seems quite possible that dual hierarchies can only be constructed for a handful of soliton equation hierarchies, and not for all the various infinite chains of hierarchies, like the $SL(N)$ KdV hierarchies [11], that exist.

The content of this note is therefore limited to presenting zero curvature formulations of the existing dual hierarchies. It is to be hoped that these will be of use in further studies of these hierarchies, and in finding solutions. We will see some minor immediate benefits of our labor; in particular, we will see that the dual Broer-Kaup-Kupershmidt hierarchy and the dual Ito hierarchy are equivalent, and we clarify a little further the structure of the dual NLS hierarchy. Also, of course, the zero curvature forms we will present, can be used to derive "standard" Lax pairs for the dual hierarchies, via a simple procedure we will illustrate.

Zero Curvature Formulations. The notion of a zero curvature formulation for a soliton equation dates back to the work [12] (and other works in the Soviet literature). In [12] it was observed that several equations of physical interest could be written in the form

$$\partial_t A = \partial_x B + [B, A], \tag{10}$$

where A, B are functions of x, t valued in the Lie algebra of the $SL(2)$ loop group, that is, A, B are traceless, 2×2 matrix valued functions of x, t, λ. Equation (10) reduces to the desired soliton equation by specifying a very particular dependence on the "spectral parameter" λ. In greater generality, the majority of (if not all) soliton equation hierarchies can be written in the form

$$\partial_{t_r} A = \partial_x B_r + [B_r, A], \tag{11}$$

where $A, \{B_r\}$ (r runs over an appropriate index set) are functions of $x, \{t_r\}, \lambda$, valued in some matrix Lie algebra, with a certain specified λ dependence. The classic example is the KdV hierarchy, for which $r \in \{1, 3, 5, ...\}$ and

$$\begin{aligned} A &= \begin{pmatrix} 0 & 1 \\ u(x,t)+\lambda & 0 \end{pmatrix} \\ B_r &= \begin{pmatrix} 0 & 0 \\ \lambda^{\frac{r+1}{2}} & 0 \end{pmatrix} + \begin{pmatrix} \text{polynomial of} \\ \text{degree } \frac{r-1}{2} \text{ in } \lambda \end{pmatrix}. \end{aligned} \tag{12}$$

(Note that requiring consistency of equations (11) almost fully determines the matrices B_r from the information in (12). To precisely determine the B_r's one should add on (a) the conditions $\partial_{t_r} B_s = \partial_{t_s} B_r + [B_r, B_s]$, and (b) certain homogeneity conditions. For brevity, we shall overlook these details in this note.) The utility of this formulation of the hierarchy is that equations (11) are invariant under *gauge transformations*

$$\begin{aligned} A &\to \xi\, A\, \xi^{-1} + \partial_x \xi\, \xi^{-1} \\ B_r &\to \xi\, B_r\, \xi^{-1} + \partial_{t_r} \xi\, \xi^{-1}, \end{aligned} \tag{13}$$

where ξ is a function of $x, \{t_r\}, \lambda$, valued in the appropriate Lie group. The group of such gauge transformations that leave the specified λ dependence of $A, \{B_r\}$ unchanged is the group of dressing transformations [13] referred to above.

Other gauge transformations that are of interest are those that map one hierarchy to another, known as Miura maps. To illustrate, the MKdV hierarchy is given by (11), with $r \in \{1, 3, 5, ...\}$ again, and

$$\begin{aligned} A &= \begin{pmatrix} j(x,t) & 1 \\ \lambda & -j(x,t) \end{pmatrix} \\ B_r &= \begin{pmatrix} 0 & 0 \\ \lambda^{\frac{r+1}{2}} & 0 \end{pmatrix} + \begin{pmatrix} \text{polynomial of} \\ \text{degree } \frac{r-1}{2} \text{ in } \lambda \end{pmatrix}. \end{aligned} \tag{14}$$

Choosing

$$\xi \;=\; \begin{pmatrix} 1 & 0 \\ j(x,t) & 1 \end{pmatrix}, \tag{15}$$

the MKdV choice of $A, \{B_r\}$ (14) is mapped to the KdV choice (12), with $u = j_x + j^2$. (Note that in [10], Wilson works with an apparently different zero curvature formulation of MKdV; he is simply using a different matrix representation of the $SL(2)$ loop group).

As for all the hierarchies we will consider in this note, the $r = 1$ equations for KdV and MKdV are trivial. In each case $B_1 = A$, and the flow equations are $u_{t_1} = u_x$ and $j_{t_1} = j_x$ respectively. From these equations we see t_1 can be identified with x. The first nontrivial equations are obtained from $r = 3$: for KdV,

$$B_3 = \begin{pmatrix} \frac{1}{4}u_x & \lambda - \frac{1}{2}u \\ \lambda^2 + \frac{1}{2}\lambda u + \frac{1}{4}u_{xx} - \frac{1}{2}u^2 & -\frac{1}{4}u_x \end{pmatrix} \quad \text{yielding} \quad u_{t_3} = \tfrac{1}{4}u_{xxx} - \tfrac{3}{2}uu_x, \qquad (16)$$

and for MKdV,

$$B_3 = \begin{pmatrix} \lambda j + \frac{1}{4}(j_{xx} - 2j^3) & \lambda - \frac{1}{2}(j_x + j^2) \\ \lambda^2 + \frac{1}{2}\lambda(j_x - j^2) & -\lambda j - \frac{1}{4}(j_{xx} - 2j^3) \end{pmatrix} \quad \text{yielding} \quad j_{t_3} = \tfrac{1}{4}j_{xxx} - \tfrac{3}{2}j^2 j_x. \qquad (17)$$

The procedure for extracting "standard" Lax pairs from zero curvature formulations is as follows. Equations (11) are consistency conditions for the equations

$$\begin{aligned} \partial_x \psi &= A\psi \\ \partial_{t_r} \psi &= B_r \psi. \end{aligned} \qquad (18)$$

(Here $\psi(x,t)$ is a vector in an appropriate representation of the appropriate Lie algebra.) For KdV and $r = 3$, we eliminate ψ_2 from (18) to arrive at the KdV Lax pair:

$$\begin{aligned} \psi_{1xx} &= (u + \lambda)\psi_1 \\ \psi_{1t_3} &= \tfrac{1}{4}u_x\psi_1 + (\lambda - \tfrac{1}{2}u)\psi_{1x}. \end{aligned} \qquad (19)$$

The zero curvature formulations of the other standard hierarchies relevant to this paper follow; all are associated with the $SL(2)$ loop algebra.

1. The Broer-Kaup-Kupershmidt (BKK) hierarchy. (This hierarchy was brought to prominence by Kupershmidt in [14], where it was attributed to Broer and Kaup. It seems, however, that it should also be attributed to Whitham. It is frequently just referred to as a "Boussinesq-type" hierarchy.)

$$\begin{aligned} A &= \begin{pmatrix} \lambda + v(x,t) & 1 \\ u(x,t) & -\lambda - v(x,t) \end{pmatrix} \\ B_r &= \begin{pmatrix} \lambda^r & 0 \\ 0 & -\lambda^r \end{pmatrix} + \begin{pmatrix} \text{polynomial of} \\ \text{degree } r-1 \text{ in } \lambda \end{pmatrix}, \qquad r = 1, 2, 3, \ldots \end{aligned} \qquad (20)$$

Lowest nontrivial equation:

$$B_2 = \begin{pmatrix} \lambda^2 + \frac{1}{2}v_x - v^2 & \lambda - v \\ \lambda u - \frac{1}{2}u_x - uv & -\lambda^2 - \frac{1}{2}v_x + v^2 \end{pmatrix} \qquad (21)$$

yields

$$\begin{aligned} v_{t_2} &= (\tfrac{1}{2}v_x - v^2 + \tfrac{1}{2}u)_x \\ u_{t_2} &= (-\tfrac{1}{2}u_x - 2uv)_x. \end{aligned} \tag{22}$$

2. The NLS hierarchy.

$$\begin{aligned} A &= \begin{pmatrix} \lambda & \psi(x,t) \\ \bar{\psi}(x,t) & -\lambda \end{pmatrix} \\ B_r &= \begin{pmatrix} \lambda^r & 0 \\ 0 & -\lambda^r \end{pmatrix} + \begin{pmatrix} \text{polynomial of} \\ \text{degree } r-1 \text{ in } \lambda \end{pmatrix}, \qquad r = 1, 2, 3, ... \end{aligned} \tag{23}$$

Lowest nontrivial equation:

$$B_2 = \begin{pmatrix} \lambda^2 - \frac{1}{2}\psi\bar{\psi} & \lambda\psi + \frac{1}{2}\psi_x \\ \lambda\bar{\psi} - \frac{1}{2}\bar{\psi}_x & -\lambda^2 + \frac{1}{2}\psi\bar{\psi} \end{pmatrix} \tag{24}$$

yields

$$\begin{aligned} \psi_{t_2} &= \tfrac{1}{2}\psi_{xx} - \psi^2\bar{\psi} \\ \bar{\psi}_{t_2} &= -\tfrac{1}{2}\bar{\psi}_{xx} + \psi\bar{\psi}^2. \end{aligned} \tag{25}$$

Miura map to the BKK hierarchy:

$$\xi = \begin{pmatrix} \frac{1}{\sqrt{\psi}} & 0 \\ 0 & \sqrt{\psi} \end{pmatrix} \tag{26}$$

giving $v = -\psi_x/2\psi$, $u = \psi\bar{\psi}$. (The relationship of NLS to the BKK hierarchy, and many other "NLS-type" equations, was given in [15]).

3. The Ito hierarchy. [16]

$$\begin{aligned} A &= \begin{pmatrix} 0 & 1 \\ \frac{p(x,t)}{\lambda} + q(x,t) + \lambda & 0 \end{pmatrix} \\ B_r &= \begin{pmatrix} 0 & 0 \\ \lambda^{\frac{r+1}{2}} & 0 \end{pmatrix} + \begin{pmatrix} \text{polynomial of} \\ \text{degree } \frac{r-1}{2} \text{ in } \lambda \end{pmatrix} + f(x,t)A, \qquad r = 1, 3, 5, ... \end{aligned} \tag{27}$$

Lowest nontrivial equation:

$$B_3 = \begin{pmatrix} \frac{1}{4}q_x & \lambda - \frac{1}{2}q \\ \lambda^2 + \frac{1}{2}\lambda q + (\frac{1}{4}q_{xx} - \frac{1}{2}q^2 + p) - \frac{1}{2\lambda}pq & -\frac{1}{4}q_x \end{pmatrix} \tag{28}$$

yields

$$\begin{aligned} q_{t_3} &= \tfrac{1}{4}q_{xxx} - \tfrac{3}{2}qq_x + p_x \\ p_{t_3} &= -pq_x - \tfrac{1}{2}qp_x \end{aligned} \tag{29}$$

(the substitution $p = r^2$ returns the standard form of the Ito equation). The Ito hierarchy is just one of what we shall call the *generalized Ito hierarchies* (which in turn are just a subset of the hierarchies discussed in [17]). For any nonnegative integer M there exists a hierarchy with

$$A = \begin{pmatrix} 0 & 1 \\ \lambda + S & 0 \end{pmatrix}, \qquad S = \sum_{n=0}^{M} s_n(x,t)\lambda^{-n}, \tag{30}$$

and B_r $(r = 1, 3, 5, ...)$ specified by the requirement that its upper right hand entry is a polynomial of degree $(r-1)/2$ in λ with leading order coefficient 1. For $M = 0$ this is the KdV hierarchy, for $M = 1$ the usual Ito hierarchy, and for $M = 2$ it is a simple exercise to show the lowest nontrivial equation in the hierarchy takes form:

$$\begin{aligned} s_{0t_3} &= \tfrac{1}{4}s_{0xxx} - \tfrac{3}{2}s_0 s_{0x} + s_{1x} \\ s_{1t_3} &= s_{2x} - s_1 s_{0x} - \tfrac{1}{2}s_0 s_{1x} \\ s_{2t_3} &= -s_2 s_{0x} - \tfrac{1}{2}s_0 s_{2x}. \end{aligned} \tag{31}$$

For the KdV and MKdV hierarchies, and for the 3 hierarchies just listed, *the zero curvature formulations of the dual hierarchies are obtained by rescaling entries in the matrices A by appropriate powers of* λ, and adjusting the matrices $\{B_r\}$ to maintain consistency. For example for the Fuchssteiner-Fokas-Camassa-Holm hierarchy (the dual of KdV), we take

$$A = \begin{pmatrix} 0 & 1 \\ u(x,t)/\lambda + 1 & 0 \end{pmatrix}, \tag{32}$$

i.e. A has the same form as for the KdV hierarchy, but with its $(1,2)$ entry rescaled by a factor λ^{-1}. Choosing B_r of the form "polynomial of degree $(r-1)/2$ in λ plus a multiple of A", it is straightforward to obtain the flows

$$u_{t_r} = \left[(\partial_x u + u\partial_x)(\tfrac{1}{2}\partial_x^3 - 2\partial_x)^{-1}\right]^{\frac{r-1}{2}} u_x. \tag{33}$$

These are the flows of the Fuchssteiner-Fokas-Camassa-Holm hierarchy; in particular, setting $r = 1$, and defining v via $u = \frac{1}{2}v_{xx} - 2v$, we obtain

$$2v_{t_3} - \tfrac{1}{2}v_{xxt_3} = 6vv_x - v_x v_{xx} - \tfrac{1}{2}vv_{xxx}, \tag{34}$$

a simple rescaling of equation (1). Note the form of B_r is also obtained from that of the KdV hierarchy, by rescaling its $(2,1)$ entry by a factor λ^{-1}.

For the dual of MKdV we take

$$A = \begin{pmatrix} j(x,t)/\sqrt{\lambda} & 1 \\ 1 & -j(x,t)/\sqrt{\lambda} \end{pmatrix}. \tag{35}$$

This is of the same as for MKdV (14), after a rescaling of the $(1,1)$ and $(2,2)$ entries by $\lambda^{-\frac{1}{2}}$, and of the $(2,1)$ entry by λ^{-1}. The appropriate choice for B_r this time cannot be found from a rescaling of the MKdV form: in particular, powers of $\sqrt{\lambda}$ appear in the off-diagonal terms of B_r. B_r can however be completely specified (up to an unimportant overall factor) by the requirement that the sum of its off-diagonal elements be a polynomial in λ of degree $(r-1)/2$. For $r=1$ we have $B_1 = A$, as usual, and for $r=2$ we find

$$B_3 = \begin{pmatrix} \sqrt{\lambda}(j - \frac{1}{4}m_{xx}) + \frac{1}{\sqrt{\lambda}}js & \lambda + s + \frac{1}{2}\sqrt{\lambda}m_x \\ \lambda + s - \frac{1}{2}\sqrt{\lambda}m_x & -\sqrt{\lambda}(j - \frac{1}{4}m_{xx}) - \frac{1}{\sqrt{\lambda}}js \end{pmatrix}, \tag{36}$$

where m is related to j by $j = \frac{1}{4}m_{xx} - m$, and $s = \frac{1}{2}(\frac{1}{4}m_x^2 - m^2)$. This gives the flow equation

$$j_{t_3} = \left[\frac{j}{2}\left(\frac{m_x^2}{4} - m^2\right)\right]_x, \qquad j = \frac{1}{4}m_{xx} - m \tag{37}$$

(c.f. [7]). The general flow is

$$j_{t_r} = \left[\partial_x j \partial_x^{-1} j (\tfrac{1}{4}\partial_x^2 - 1)^{-1}\right]^{\frac{r-1}{2}} j_x \tag{38}$$

Note that there is no obvious Miura map from the dual of MKdV to the dual of KdV. The small modifications we have made to the matrices A in each case have been sufficient to destroy this, and in general Miura maps do not survive the dualization procedure. We will see that it is also the case that new Miura maps can emerge. It remains an interesting open question as to whether there exists a modification of the Fuchssteiner-Fokas-Camassa-Holm equation.

We now list the zero curvature forms for the duals of the other hierarchies listed above.

1. The dual BKK hierarchy. We take

$$\begin{aligned} A &= \begin{pmatrix} 1 + v(x,t)/\lambda & 1/\sqrt{\lambda} \\ u(x,t)/\sqrt{\lambda} & -1 - v(x,t)/\lambda \end{pmatrix} \\ B_r &= \begin{pmatrix} \beta(1 + v(x,t)/\lambda) - \frac{1}{2}\beta_x & \beta/\sqrt{\lambda} \\ \sqrt{\lambda}\gamma + u\beta/\sqrt{\lambda} & -\beta(1 + v(x,t)/\lambda) + \frac{1}{2}\beta_x \end{pmatrix} \qquad r = 1, 2, 3, ..., \end{aligned} \tag{39}$$

where

$$\begin{aligned} \beta &= \lambda^{r-1} + \sum_{n=0}^{r-2} \beta_n(x,t)\lambda^n \\ \gamma &= \sum_{n=0}^{r-2} \gamma_n(x,t)\lambda^n. \end{aligned} \tag{40}$$

It is straightforward to check this gives the flow

$$\partial_{t_r}\begin{pmatrix} v \\ u \end{pmatrix} = [\mathcal{S}\mathcal{R}^{-1}]^{r-1}\begin{pmatrix} v_x \\ u_x \end{pmatrix} \tag{41}$$

where

$$\begin{aligned} \mathcal{S} &= \begin{pmatrix} 0 & \partial_x v \\ 2v & \partial_x u + u\partial_x \end{pmatrix} \\ \mathcal{R} &= \begin{pmatrix} 1 & \frac{1}{2}\partial_x^2 - \partial_x \\ -\partial_x - 2 & 0 \end{pmatrix}. \end{aligned} \tag{42}$$

$\mathcal{S}\mathcal{R}^{-1}$ is (up to simple rescalings) the recursion operator found in [7], but in [7] it is factored as $\tilde{\mathcal{S}}\tilde{\mathcal{R}}^{-1}$, where

$$\begin{aligned} \tilde{\mathcal{S}} = \mathcal{S}\begin{pmatrix} \partial_x & 0 \\ 0 & 2 \end{pmatrix} &= \begin{pmatrix} 0 & 2\partial_x v \\ 2v\partial_x & 2(\partial_x u + u\partial_x) \end{pmatrix} \\ \tilde{\mathcal{R}} = \mathcal{R}\begin{pmatrix} \partial_x & 0 \\ 0 & 2 \end{pmatrix} &= \begin{pmatrix} \partial_x & \partial_x^2 - 2\partial_x \\ -\partial_x^2 - 2\partial_x & 0 \end{pmatrix}. \end{aligned} \tag{43}$$

To write the $r = 2$ flow in local form, in [7] the substitution

$$\begin{pmatrix} v \\ u \end{pmatrix} = \tilde{\mathcal{R}}\partial_x^{-1}\begin{pmatrix} V \\ U \end{pmatrix} = \begin{pmatrix} V - 2U + U_x \\ -2V - V_x \end{pmatrix} \tag{44}$$

is introduced, giving the flow

$$\partial_{t_2}\begin{pmatrix} V - 2U + U_x \\ -2V - V_x \end{pmatrix} = \begin{pmatrix} 2(UV - 2U^2 + UU_x)_x \\ (-8UV + V^2 - 2UV_x)_x \end{pmatrix}. \tag{45}$$

From the zero curvature approach it is rather more natural to perform the substitution

$$\begin{pmatrix} v \\ u \end{pmatrix} = \begin{pmatrix} \tilde{V} \\ \frac{1}{2}\tilde{U}_{xx} - 2\tilde{U} - 2\tilde{V} - \tilde{V}_x \end{pmatrix} \tag{46}$$

giving the flow

$$\partial_{t_2}\begin{pmatrix} \tilde{V} \\ \frac{1}{2}\tilde{U}_{xx} - 2\tilde{U} - 2\tilde{V} - \tilde{V}_x \end{pmatrix} = \begin{pmatrix} (\tilde{U}\tilde{V})_x \\ (\frac{1}{2}\tilde{U}\tilde{U}_{xx} + \frac{1}{4}\tilde{U}_x^2 - (\tilde{U}\tilde{V})_x + \tilde{V}^2 - 2\tilde{U}\tilde{V} - 3\tilde{U}^2)_x \end{pmatrix}, \tag{47}$$

or, equivalently,

$$\partial_{t_2}\begin{pmatrix} \tilde{V} \\ \frac{1}{2}\tilde{U}_{xx} - 2\tilde{U} \end{pmatrix} = \begin{pmatrix} (\tilde{U}\tilde{V})_x \\ (\frac{1}{2}\tilde{U}\tilde{U}_{xx} + \frac{1}{4}\tilde{U}_x^2 + \tilde{V}^2 - 3\tilde{U}^2)_x \end{pmatrix}, \tag{48}$$

an equation that will appear again later. Note that the relationship of U, V and $\tilde{U}, \tilde{V}$ is given by

$$\begin{aligned} V &= \tilde{V} - \tfrac{1}{2}\tilde{U}_x + \tilde{U} \\ U &= \tfrac{1}{2}\tilde{U}. \end{aligned} \tag{49}$$

(The origin of this variable $\tilde{U}$ is that β introduced in (41)-(42) takes the form $\lambda + \tilde{U}$ for $r = 2$.)

2. The dual NLS hierarchy. To obtain the dual NLS hierarchy of [7], take

$$A = \begin{pmatrix} 1 & \psi(x,t)/\sqrt{\lambda} \\ \bar{\psi}(x,t)/\sqrt{\lambda} & -1 \end{pmatrix}$$
$$B_r = \begin{pmatrix} \alpha & \sqrt{\lambda}\beta \\ \sqrt{\lambda}\gamma & -\alpha \end{pmatrix} \qquad r = 1, 2, 3, ..., \tag{50}$$

where

$$\alpha = \lambda^{r-1} + \sum_{n=0}^{r-2} \alpha_n(x,t)\lambda^n$$
$$\beta = \sum_{n=0}^{r-2} \beta_n(x,t)\lambda^n \tag{51}$$
$$\gamma = \sum_{n=0}^{r-2} \gamma_n(x,t)\lambda^n$$

to get the flows

$$\partial_{t_r} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} = \left[\begin{pmatrix} \psi\partial_x^{-1}\bar{\psi} & -\psi\partial_x^{-1}\psi \\ -\bar{\psi}\partial_x^{-1}\bar{\psi} & \bar{\psi}\partial_x^{-1}\psi \end{pmatrix} \begin{pmatrix} \frac{1}{2}\partial_x - 1 & 0 \\ 0 & \frac{1}{2}\partial_x + 1 \end{pmatrix}^{-1} \right]^{r-1} \begin{pmatrix} 2\psi \\ -2\bar{\psi} \end{pmatrix}. \tag{52}$$

Note the unusual form of the $r = 1$ flow. For $r = 2$ we set $\psi = \frac{1}{2}v_x e^{2x}$, $\bar{\psi} = \frac{1}{2}w_x e^{-2x}$, to get the flow

$$\partial_{t_2}\partial_x \begin{pmatrix} v \\ w \end{pmatrix} = vw \begin{pmatrix} v_x \\ -w_x \end{pmatrix}, \tag{53}$$

with conserved quantity $v_x w_x$ as noted in [7].

The forms (50)-(51) are not the most natural guess for the zero curvature formulation in this case; for example we note that the ansatz does not permit choosing $B_1 = A$, which is why the $r = 1$ equation obtained above is nonstandard. We therefore consider the more general ansatz

$$A = \begin{pmatrix} 1 & \psi(x,t)/\sqrt{\lambda} \\ \bar{\psi}(x,t)/\sqrt{\lambda} & -1 \end{pmatrix}$$
$$B_r = \begin{pmatrix} \alpha & \beta/\sqrt{\lambda} \\ \gamma/\sqrt{\lambda} & -\alpha \end{pmatrix} \qquad r = 1, 2, 3, ..., \tag{54}$$

where

$$\alpha = \lambda^{r-1} + \sum_{n=0}^{r-2} \alpha_n(x,t)\lambda^n$$
$$\beta = \sum_{n=0}^{r-1} \beta_n(x,t)\lambda^n \tag{55}$$
$$\gamma = \sum_{n=0}^{r-1} \gamma_n(x,t)\lambda^n.$$

(50)-(51) is just this with the restriction $\beta_0 = \gamma_0 = 0$. It turns out that the ansatz (54)-(55) is consistent provided $\beta_0 = \psi\mathcal{M}(x,t)$, $\gamma_0 = \bar{\psi}\mathcal{M}(x,t)$ for some function $\mathcal{M}(x,t)$ (there is a small further freedom; consistency only determines the α_n's up to a constant, which can however be fixed by a homogeneity condition). In particular, we can now choose $B_1 = A$ (for this $\beta_0 = \psi$, $\gamma_0 = \bar{\psi}$) to recover a standard $r = 1$ flow equation. For the choice $\beta_0 = \psi$, $\gamma_0 = \bar{\psi}$ it is straightforward to find the $r = 2$ flow equation:

$$\partial_{t_2}\partial_x \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} v_{xx} + wvv_x \\ w_{xx} - wvw_x \end{pmatrix}. \tag{56}$$

Defining new coordinates x', t_2' via $\partial_{x'} = \partial_x$, $\partial_{t_2'} = \partial_{t_2} - \partial_x$, we see (56) is equivalent to (53). This reflects a simple general symmetry of the system (11): adding cA (c constant) to each B_r can be exactly cancelled by a change of coordinates from x, t_r to x', t_r' defined by

$$\begin{aligned} \partial_{x'} &= \partial_x \\ \partial_{t_r'} &= \partial_{t_r} - c\partial_x. \end{aligned} \tag{57}$$

Taking $\beta_0 = \psi$, $\gamma_0 = \bar{\psi}$ in the ansatz (54)-(55) is, taking into account the freedom to add a constant to α_0,equivalent to adding A to each of the B_r's of the ansatz (50)-(51), thus explaining the relationship of (53) and (56). (Note this freedom we have just mentioned does not exist in B_1, so the two $r = 1$ equations we have obtained above are *not* related by a change of coordinates of the form (57)!)

At this stage it is maybe appropriate to mention another general symmetry of (11), for the case where A, B_r are traceless 2 by 2 matrices. Writing

$$\begin{aligned} A &= A_+E^+ + A_-E^- + A_0E^0 \\ B_r &= B_{r+}E^+ + B_{r-}E^- + B_{r0}E^0, \end{aligned} \tag{58}$$

where

$$E^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad E^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad E^0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{59}$$

(11) is symmetric under

$$
\begin{aligned}
A_+ \to \lambda^\alpha A_+ \qquad & A_- \to \lambda^{-\alpha} A_- \qquad & A_0 \to A_0 \\
B_{r+} \to \lambda^\alpha B_{r+} \qquad & B_{r-} \to \lambda^{-\alpha} B_{r-} \qquad & B_{r0} \to B_{r0},
\end{aligned}
\tag{60}
$$

for any constant α. In both the dual BKK and dual NLS hierarchies, the form of A has been obtained by scaling the $(1,1)$ and $(2,2)$ entries of A in the standard hierarchy by λ^{-1}, and the $(1,2)$ and $(2,1)$ entries by $\lambda^{-1/2}$. The symmetry (60) allows one to express the rescaling of the components of A necessary to pass from the standard to the dual hierarchies in a variety of equivalent ways. We will exploit this shortly.

3.The dual Ito and generalized Ito hierarchies. For the dual of the Ito hierarchy we take

$$
\begin{aligned}
A &= \begin{pmatrix} 0 & 1 \\ 1 + q(x,t)/\lambda + p(x,t)/\lambda^2 & 0 \end{pmatrix} \\
B &= \begin{pmatrix} -\frac{1}{2}\beta_x & \beta \\ \beta(1 + \frac{p}{\lambda} + \frac{q}{\lambda^2}) - \frac{1}{2}\beta_{xx} & \frac{1}{2}\beta_x \end{pmatrix}, \qquad r = 1, 3, 5, ...,
\end{aligned}
\tag{61}
$$

with

$$
\beta = \lambda^{\frac{r-1}{2}} + \sum_{n=0}^{\frac{r-3}{2}} \beta_n(x,t)\lambda^n. \tag{62}
$$

The matrix A has been obtained from that of the standard Ito hierarchy by the same scaling used to obtain Fuchssteiner-Fokas-Camassa-Holm from KdV, that is, the $(2,1)$ entry of A has been multiplied by λ^{-1}. For $r = 3$ we have $\beta = \lambda + b(x,t)$, and we find that we can take $p = \frac{1}{2}b_{xx} - 2b$, to obtain the flow

$$
\begin{aligned}
\left(\tfrac{1}{2}b_{xx} - 2b\right)_{t_3} &= q_x + b_{xx}b_x + \tfrac{1}{2}bb_{xxx} - 6bb_x \\
q_{t_3} &= 2qb_x + bq_x.
\end{aligned}
\tag{63}
$$

On substituting $q = w^2$ this becomes

$$
\begin{aligned}
\left(\tfrac{1}{2}b_{xx} - 2b\right)_{t_3} &= \left(w^2 + \tfrac{1}{4}b_x^2 + \tfrac{1}{2}bb_{xx} - 3b^2\right)_x \\
w_{t_3} &= (wb)_x,
\end{aligned}
\tag{64}
$$

i.e. we have recovered the lowest nontrivial flow in the dual BKK hierarchy, equation (48).

The relationship between the dual BKK hierarchy and the dual Ito hierarchy we have just seen is an instance of a Miura map “born” after dualization. Exploiting the symmetry (60) of (11), with $\alpha = \frac{1}{2}$, we observe that the BKK hierarchy has a zero curvature formulation with

$$
A = \begin{pmatrix} 1 + v(x,t)/\lambda & 1 \\ u(x,t)/\lambda & -1 - v(x,t)/\lambda \end{pmatrix}. \tag{65}
$$

Via gauge transformation with

$$\xi = \begin{pmatrix} 1 & 0 \\ 1 + v(x,t)/\lambda & 1 \end{pmatrix} \tag{66}$$

this is brought to the form

$$A = \begin{pmatrix} 0 & 1 \\ 1 + (u + 2v + v_x)/\lambda + v^2/\lambda^2 & 0 \end{pmatrix}, \tag{67}$$

which is of the form used in (61), and therefore defines a Miura map from the dual BKK hierarchy to the dual Ito hierarchy. When written out in the variables we have used to write the lowest nontrivial flows of the dual BKK and dual Ito hierarchies, the Miura map becomes an equivalence.

Finally, we note that the generalized Ito hierarchies can be dualized in the same way as the KdV and Ito hierarchies, that is we take

$$A = \begin{pmatrix} 0 & 1 \\ 1 + S & 0 \end{pmatrix}, \qquad S = \sum_{n=0}^{M} s_n(x,t)\lambda^{-n-1}, \tag{68}$$

with B_r ($r = 1, 3, 5, ...$) specified by the requirement that its upper right hand entry is a polynomial of degree $(r-1)/2$ in λ with leading order coefficient 1. Thus we have at least one infinite family of integrable hierarchies that affords dualization.

Acknowledgements

I thank Philip Rosenau for interesting me in his work, and the Rashi foundation for support as a Guastella fellow. Thanks also to Peter Olver and Philip Rosenau for bringing the second reference of [1] to my attention, and to the referee for the first reference of [1], and also the reference [8].

References

[1] B.Fuchssteiner and A.S.Fokas, *Physica* **4D**, 47 (1981); B.Fuchssteiner, *Prog.Th.Phys.* **65**, 861 (1981).

[2] R.Camassa and D.D.Holm, *Phys.Rev.Lett.* **71**, 1661 (1993).

[3] F.Cooper and H.Shepard, *Solitons in the Camassa-Holm Shallow Water Equation*, Los Alamos/University of New Hampshire preprint (June 1995).

[4] P.Rosenau and J.M.Hyman, *Phys.Rev.Lett.* **70**, 564 (1993).

[5] P.Rosenau, *Phys.Rev.Lett.* **73**, 1737 (1994).

[6] P.Rosenau, *On Soliton-Compacton Duality*, Tel-Aviv University preprint (undated).

[7] P.J.Olver and P.Rosenau, *Tri-Hamiltonian Soliton-Compacton Duality*, University of Minnesota/Tel-Aviv University preprint (April 1995).

[8] A.S.Fokas, *Acta Appl.Math.* **39**, 295 (1995).

[9] P.J.Olver, *Applications of Lie Groups in Differential Equations*, Springer-Verlag (1986).

[10] G.Wilson, *Phil.Trans.Roy.Soc.Lond.* **A 315**, 393 (1985).

[11] V.G.Drinfel'd and V.V Sokolov, *Jour.Sov.Math.* **30**, 1975 (1985).

[12] M.J.Ablowitz, D.J.Kaup, A.C.Newell and H.Segur, *Phys.Rev.Lett* **31**, 125 (1973).

[13] V.E.Zakharov and A.B.Shabat, *Func.Anal.Appl.* **13**, 166 (1979).

[14] B.A.Kupershmidt, *Comm.Math.Phys.* **99**, 51 (1985).

[15] J.Schiff, Institute for Advanced Study preprint (Aug 1992), hep-th/9210029.

[16] M.Ito, *Phys.Lett.* **91A**, 335 (1982).

[17] M.Antonowicz and A.P.Fordy, *Physica* **28D**, 345 (1987).