Contest Design with Uncertain Performance and Costly Participation

Priel Levy and **David Sarne**
Department of Computer Science
Bar Ilan University, Israel
priel.levy@live.biu.ac.il, sarned@cs.biu.ac.il

Igor Rochlin
School of Computer Science
College of Management, Israel
igor.rochlin@gmail.com

Abstract

This paper studies the problem of designing contests for settings where a principal seeks to optimize the quality of the best performance obtained, and potential contestants only strategize about whether to participate in the contest, as participation incurs some cost. This type of contest can be mapped to various real-life settings (e.g., an audition, a beauty pageant, technology crowdsourcing). The paper provides a comparative game-theoretic based solution to two variants of the above underlying model: parallel and sequential contest, enabling a characterization of the equilibrium strategies in each. Special emphasis is placed on the case where the contestants are homogeneous which is often the case in real-life whenever the contestants are basically alike and their ranking in the contest is mostly influenced by some probabilistic factors (e.g., luck). Here, several (somehow counter-intuitive) properties of the equilibrium are proved, in particular for the sequential contest, leading to a comprehensive characterization of the principal preference between the two.

1 Introduction

Contests have been used since the dawn of man as a mechanism for inducing individual efforts. In recent years, contests are used not only as a means for determining the best contestant, but also for generating value. For example DARPA is offering Grand Challenges to promote the development of cutting-edge technologies, firms run contests to come up with new products (such as the LEGO Ideas contests [Schlagwein and Bjorn-Andersen, 2014]) and not-for-profit organizations are organizing contests for transformative solutions that benefit mankind (as with the Hult Prize [Prize, 2017]).

The majority of contest research deals with models where contestants can influence their performance in the contest through the effort or money they put [Baik, 1994; Moldovanu and Sela, 2006]. In various settings, however, this is not the case. For example in contests where performance is being determined subjectively by external judges based on some attributes the contestants have no control over or cannot change at the time of the contest (as in the case of a beauty pageant, or when casting a background actor for a scene). Or, in cases where performance is determined based on the contestant's skills and luck (e.g., in a fishing contest or a backgammon tournament) or purely based on luck (as in guessing the number of coins hidden in a closed treasure box). Similar inability of contestants to influence their performance in a contest may hold in technology crowdsourcing contests, whenever individuals' offerings are fully based on the technologies they have patented and the quality of the solution of each given technology to the contest initiator's need is a priori uncertain and requires some effort to reveal. Common to all the above examples, that the contestants' strategy is limited to participating or not participating in the contest. This latter decision becomes non-trivial whenever participation incurs some cost (e.g., physically getting to the contest, paying some participation fee).

In this paper we study contests of the above type, i.e., the contestants' performance in the contest is beyond the control of contestants at the time of the contest and participation is costly. Unlike most prior literature, that was primarily focused on parallel contest models, in this work we consider also the option for a sequential contest. In sequential contest, only one contestant performs at a time and its performance measure is known to the following contestants. Such design is commonly used in real-life, e.g., in Olympic sports such as platform diving, pole vault and javelin throw.

Contributions. The paper provides a comprehensive game-theoretic based analysis of the parallel and sequential contest model variants, specifying the conditions under which different kinds of equilibria hold. The analysis enables demonstrating that the preference of the contest to be used (parallel or sequential), from the contest organizer's point of view, can frequently change as a function of the setting parameters—even a slight variation in one of the parameters can result in several alternations in preference. For the case of homogeneous contestants, however, we manage to characterize some setting classes where parallel contest dominates a sequential one and vice versa. In particular, we prove a transition in preference from parallel to sequential contest as the ratio between the participation cost and the prize awarded increases. Interestingly (and somehow counter-intuitively), we find that in sequential contest with homogeneous contestants, despite the asymmetry imposed by the sequential process, in equilibrium both the contestant's strategies and their expected profit do not depend on the number of contestants nor the position of a contestant in the sequence. The latter finding suggests that the sequential competition is just as fair as the parallel one.

2 Related Work

Most literature in the area of contest design deals with effort-based contests, where the effort expended by contestants determines their probability of winning a prize [Nti, 1999; Moldovanu and Sela, 2006] (perhaps the most common is the Tullock contest [Buchanan *et al.*, 1983] in which the winning probability is the ratio of the contestant's effort and the total effort exerted by all contestants). Our model, as motivated above assumes contestants cannot influence their chance of winning at the time the contest takes place and their strategy space is limited to participating or opting not to participate in the contest. Among the few works that consider this kind of contest, the most relevant ones are Ghosh and Hummel (2012) and Ghosh and Kleinberg (2016). These works however are focused on the question of how to design a multi-prize scheme and are limited to the case of parallel contest (whereas ours considers also the option for a sequential one). More importantly, their model assumes contestants learn about their performance measure in the contest prior to having to decide on participation, whereas in our model there is no certainty concerning performance at the time the participation decision is made. This latter difference is fundamental, as it results in a completely different analysis (and solution structure).

Finally, our work differs from most existing contest-design work in a way that it considers the option of using a sequential contest and provides a comparative analysis of the sequential and parallel model variants. Prior work has focused primarily in simultaneous contest. The sequential contest is not completely absent in literature, however to the best of our knowledge has been analyzed only in the context of effort-based contest [Morgan, 2003; Fu and Lu, 2012; Stracke, 2013] with very little comparative analysis of the two.

3 The Model

The model considers a contest organizer and a set $A = \{A_1, ..., A_k\}$ of $k > 1$ heterogeneous potential contestants (denoted "agents" onwards). Agents are fully-rational and self-interested. Each agent A_i can either participate in the contest, incurring some cost c_i, or opt to avoid participating in the contest. The performance of an agent in the contest is a priori unknown and is being affected both by its inherent competence and various external factors (e.g., luck, weather conditions, refereeing). This is modeled by taking the performance of agent A_i to be determined according to some probabilistic function $f_i(x)$ (where $F_i(x)$ is the corresponding cumulative distribution function).

The goal of the organizer is to maximize the expected maximum performance obtained by agents in a contest it runs. In order to encourage participation in the contest the organizer offers a prize $M > 0$ to the agent ranked first (performance-wise) in the contest.[1] In case none of the agents choose to participate in the contest, the prize is randomly awarded to one of the agents and the performance as perceived by the organizer is set to some pre-set fallback performance v_0.[2] The choice of always awarding the prize is a standard modeling assumption in contest theory [Liu *et al.*, 2013; Faravelli, 2011] and corresponds to the case where agents can opt to expend some default minimal performance (e.g., submitting a recommendation code that chooses randomly in the Netflix challenge) without incurring any cost. The goal of each agent is to maximize its own expected profit, defined as the expected prize awarded to it minus the cost incurred if participating in the contest. It is assumed that $f_i(x)$, c_i and M are all common knowledge in the sense that they are known to all agents and to the organizer.

We consider two model variants, differing in the way the contest is designed. The first is based on simultaneous participation ("parallel contest"), i.e., each agent's participation decision takes part in parallel to the others'. The second is based on sequential participation ("sequential contest"). Here, each agent in its turn (according to some pre-defined order) gets to see the results of its predecessors (whether participated, and if so also their performance) and then decides whether to participate in the contest.

4 Analysis

The analysis is divided according to the contest type (parallel and sequential). Due to space limitations, we defer much of the mathematical manipulations used, as these are quite standard. A moderately more detailed (step-by-step) version of the analysis and proofs can be downloaded from:
`http://bit.ly/2mbm0Yl`

4.1 Parallel Contest

We use $\{P, \neg P\}$ to denote the actions available to each agent, where P stands for *participate* and $\neg P$ for *not participate*. Since the game in this case is simultaneous, every agent A_i's strategy can be captured by the probability p_i ($0 \le p_i \le 1$) it chooses action P ($\forall A_i \in A$).

Consider agent A_i. Given the strategy profile of all other agents $\{p_j | A_j \in A \wedge A_j \neq A_i\}$, the probability that the maximum performance obtained by others is equal to or less than y, denoted $\bar{F}_i(y)$, is:

$$\bar{F}_i(y) = \prod_{A_j \in A - \{A_i\}} \Big(p_j F_j(y) + (1 - p_j)\Big)$$

The expected profit of agent A_i if participating, denoted B_i^P, is thus given by:

$$B_i^P = -c_i + M \int_{y=-\infty}^{\infty} f_i(y) \bar{F}_i(y) dy \tag{1}$$

Similarly, the expected profit of agent A_i if not participating, denoted $B_i^{\neg P}$, is given by:

$$B_i^{\neg P} = \frac{M}{k} \prod_{A_j \in A - \{A_i\}} (1 - p_j) \tag{2}$$

The best response strategy of every agent A_i is thus to participate if $B_i^P > B_i^{\neg P}$ and not participate otherwise. A Bayesian Nash Equilibrium (BNE) solution $\{p_1, ..., p_k\}$ to the parallel contest should therefore satisfy: (a) for every agent A_i for which $p_i = 0$, $B_i^P \le B_i^{\neg P}$; (b) for every agent A_i for which $p_i = 1$, $B_i^P \ge B_i^{\neg P}$; and (c) for every agent A_i for which $0 < p_i < 1$, $B_i^P = B_i^{\neg P}$. It is possible that a given setting will have more than a single equilibrium solution (i.e., multi-equilibria), though the question of which of those will be used is beyond the scope of the current paper.

One specific setting where we can determine the nature of equilibrium in terms of the strategies to be used is where the

[1] Since performance is continuous, the chance of having two agents ranked first is negligible. Otherwise, a tie-breaking rule is required.

[2] As otherwise the expected maximum performance is undefined. Typically this will be zero.

agents differ only in their participation costs (e.g., in the case of an audition for a background actor where participants differ primarily in their costs of getting to the audition) as stated in the following proposition.

Proposition 1. *When $f_i(x) = f_j(x)\ \forall i, j, x$, the BNE obtained is: (a) strictly in pure strategies such that all agents participate, whenever $\frac{c_i}{M} \leq \frac{1}{k}\ \forall i$; (b) strictly in pure strategies such that all agents do not participate, whenever $\frac{c_i}{M} \geq \frac{k-1}{k}$ $\forall i$; and (c) in mixed or pure strategies, where different agents use different strategies, otherwise.*

Proof. For case (a), since $f_i(x) = f_j(x)\ \forall i, j$, all participants in the contest are equally likely to win. Therefore participation results in an expected profit of at least $\frac{M}{k} - c_i \geq 0$ and hence it is the best response for the agent if the number of other participants is at least one. If no other agent participates then if agent A_i participates its profit is $M - c_i$ and otherwise $\frac{M}{k}$. The difference between the two is: $M - c_i - \frac{M}{k} = \frac{M(k-1)}{k} - c_i \geq c_i(k-1) - c_i \geq 0$. Hence even in this case participation is the best strategy. For case (b), any individual agent A_i has an incentive not to participate, regardless of the strategy used by the others, as otherwise its expected profit is bounded from above either by $M - c_i \leq \frac{M}{k}$ in case no one else participates, or by $\frac{M}{k} - c_i < \frac{c_i}{k-1} - c_i \leq 0$ in case the number of the other participants is at least one ($k > 1$). In all other cases, there is an incentive for at least one agent to participate in the contest if all others opt not to participate, and similarly to opt not to participate if all others do participate. □

Finally, the organizer's expected profit is given by:

$$B^{org} = \int_{y=-\infty}^{\infty} max(y, v_0) \frac{d(\bar{F}(y))}{dy} dy \tag{3}$$

where $\bar{F}(y) = \prod_{A_j \in A} \Big(p_j F_j(y) + (1-p_j)\Big)$ is the probability that the maximum performance obtained in a contest involving k agents is less than y.

4.2 Sequential Contest

In the case of a sequential contest the (subgame perfect) BNE is fully in pure strategies, as agents have perfect information about the performance of preceding agents. An agent's strategy S is its choice of participation given the best performance obtained so far, formally captured by the function $S : \mathbb{R} \cup \emptyset \to \{P, \neg P\}$, where $\emptyset$ is the case where all former agents opted not to participate in the contest.

For exposition purposes we align the agents' participation order with their index. Consider an agent A_i when the best performance obtained so far is v and the strategies used by all other agents are given by $S_{-i} = \{S_1, ..., S_{i-1}, S_{i+1}, .., S_k\}$. We use the function $F_i^{next}(y)$ to denote the probability that given that the best performance reached once A_i participates is y the maximum performance obtained by all participants after A_i is equal to or less than y. The expected profit of A_i if participating, whenever receiving v, denoted $B_i^P(v)$, is thus:

$$B_i^P(v) = \begin{cases} M \int_{y=v}^{\infty} f_i(y) F_i^{next}(y) dy - c_i & v \neq \emptyset \\ M \int_{y=-\infty}^{\infty} f_i(y) F_i^{next}(y) dy - c_i & v = \emptyset \end{cases} \tag{4}$$

Similarly, the expected profit of A_i if not participating, whenever receiving v, denoted $B_i^{\neg P}(v)$, is given by:

$$B_i^{\neg P}(v) = \begin{cases} \frac{M}{k} & v = \emptyset \wedge S_j(\emptyset) = \neg P\ \ \forall j > i \\ 0 & otherwise \end{cases} \tag{5}$$

Theorem 1. *The best-response strategy of any agent A_i given the strategies of the other agents S_{-i} and the best performance obtained so far v is: (a) use a threshold r_i to determine whether or not to participate in case $v \neq \emptyset$ and there is r_i satisfying:*

$$\frac{c_i}{M} = \int_{y=r_i}^{\infty} f_i(y) F_i^{next}(y) dy \tag{6}$$

where:

$$F_i^{next}(y) = \begin{cases} \prod_{(S_j(y)=P) \wedge (j>i)} F_j(y) & i \leq k \\ 1 & i = k \end{cases} \tag{7}$$

(b) not participate in case $v \neq \emptyset$ and there is no r_i satisfying (6); (c) participate if $\frac{k-1}{k} \geq \frac{c_i}{M}$ and otherwise not participate, in case $v = \emptyset$ and $S_j(\emptyset) = \neg P\ \forall j > i$; (d) participate if $\frac{c_i}{M} \leq \int_{y=-\infty}^{\infty} f_i(y) F_i^{next}(y) dy$ and otherwise not participate, in case $v = \emptyset$ and $\exists j > i$ such that $S_j(\emptyset) = P$.

Proof. For the case where $v \neq \emptyset$, participating is better than not participating if $M \int_{y=v}^{\infty} f_i(y) F_i^{next}(y) dy - c_i \geq 0$ (according to (4) and (5)). Since $\int_{y=v}^{\infty} f_i(y) F_i^{next}(y) dy$ decreases in v, participating is the preferred choice for any performance v that is lesser than r_i for which (6) holds. This proves part (a). If there is no r_i satisfying (6) then it can either be because $\int_{y=r_i}^{\infty} f_i(y) F_i^{next}(y) dy$ is always greater than $\frac{c_i}{M}$ or always lesser than $\frac{c_i}{M}$. However, the term reaches zero for $r_i \to \infty$ and therefore it is necessarily the case where the expected gain from participating is lower than the cost incurred, resulting in preferring not to participate. This proves part (b).

The case of $v = \emptyset$ is different as the agent can potentially gain from not participating if no one else will choose to participate as well. According to (4) and (5), in case $S_j(\emptyset) = \neg P$ $\forall j > i$, participating is the preferred choice if $M - c_i \geq \frac{M}{k}$ and therefore the distinction between the two cases in part (c). Finally, when $v = \emptyset$ and $\exists j > i$ such that $S_j(\emptyset) = P$, the decision is based solely on the chance of winning, being the first to participate. The proof in this case is similar to the one provided for case (a), in the sense of equating gains and costs incurred, and therefore omitted. □

A BNE is thus of the form $((C_1^{\emptyset}, C_1^{\neg\emptyset}), ..., (C_k^{\emptyset}, C_k^{\neg\emptyset}))$, where $C_i^{\emptyset} \in \{P, \neg P\}$ corresponds to the case of being reached with a performance $\emptyset$ and $C_i^{\neg\emptyset} \in \{P, \neg P, r_i\}$ corresponds to the case of being reached with some real performance v, satisfying the conditions specified in Theorem 1.

We now turn to calculating the expected profit of the organizer. Let $B_i^{org}(v)$ denote the organizer's expected profit when the current participant is A_i (with $k - i + 1$ more agents to go) and the maximum performance of the preceding $i - 1$ agents is v. The value $B_i^{org}(v)$ is calculated based on A_i's position in the sequence. If A_i is the last agent (i.e., $i = k$) and does not participate ($S_i(v) = \neg P$) the organizer obtains the maximum between the default profit v_0 and the performance v (or trivially v_0 if $v = \emptyset$). In case A_i does participate ($S_i(v) = P$) the profit relies on the maximum between v_0 and the performance obtained by A_i: $B_i^{org}(v) = \int_{y=-\infty}^{\infty} max(y, v_0) f_i(y) dy$ if $v = \emptyset$ and $B_i^{org}(v) = \int_{y=v}^{\infty} max(y, v_0) f_i(y) dy + F_i(v) max(v, v_0)$ otherwise. If A_i is not the last in the sequence ($i < k$) then we replace the actual performances y and v in the above by the expected profits $B_{i+1}^{org}(y)$ and $B_{i+1}^{org}(v)$, as the contest is to be continued with the transition to agent A_{i+1}, resulting in: $B_i^{org}(v) = B_{i+1}^{org}(v)$ for $S_i(v) = \neg P$, $B_i^{org}(v) = \int_{y=-\infty}^{\infty} f_i(y) B_{i+1}^{org}(y) dy$ for $S_i(v) = P$ and $v = \emptyset$, and $B_i^{org}(v) = \int_{y=v}^{\infty} f_i(y) B_{i+1}^{org}(y) dy + F_i(v) B_{i+1}^{org}(v)$ otherwise. The organizer's expected profit from the contest, denoted B^{org}, is given by $B^{org} = B_1^{org}(\emptyset)$

5 Influencing Dynamics

The choice of the type of contest (sequential or parallel) to be used by the organizer is not trivial and the preference of type may alternate even with the slightest change in setting parameters. In this section we illustrate this by following the changes in the nature and structure of the equilibrium obtained, and consequently the organizer's preference, as the participation cost of one of the agents changes. The setting used includes three agents, where $c_1 = c_2 = 0.16$ and c_3 is the independent parameter. All three agents are characterized by a uniform performance distribution function between 0 and 1 (i.e., $f_1(x) = f_2(x) = f_3(x) = 1$ for $0 \le x \le 1$ and zero otherwise). The prize to be awarded to the winner is $M = 0.4$ and the fallback performance is $v_0 = 0$.

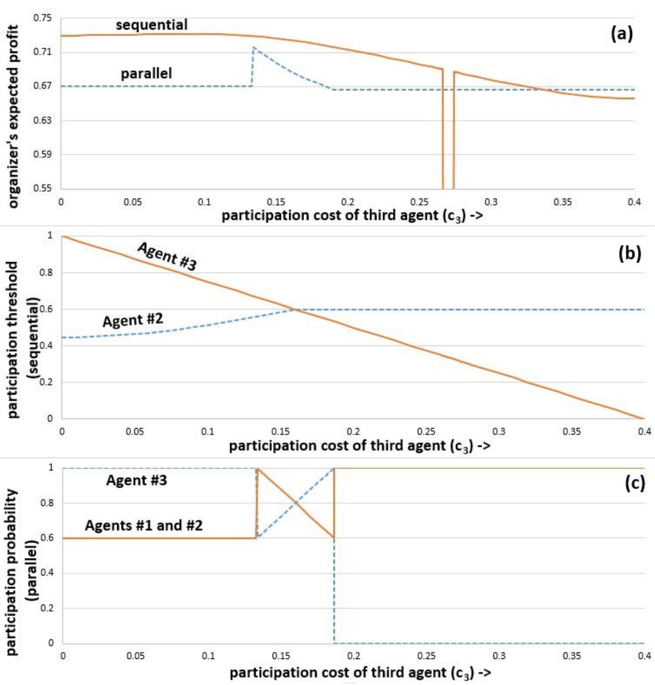

Figure 1: The influence of c_3 over: (a) organizer's expected profit; (b) participation thresholds when competing sequentially; and (c) participation probabilities in a parallel contest. See main text for the details of the setting used.

Figure 1 compares the organizer's expected profit in the above setting when using sequential and parallel contest, as a function of the participation cost c_3 of A_3 (Graph (a)). Graphs (b) and (c) depict the strategies used by the different agents in the equilibrium obtained, enabling a better understanding of the behaviors reflected in the organizer's expected profit. For exposition purposes we use (p_1, p_2, p_3) to denote the equilibrium in the parallel case, where p_i is the participation probability of agent A_i. For the sequential case we present the participation's thresholds r_2 and r_3.

We begin with the parallel contest. Here, there are several equilibria, and the curve used depicts the one in which A_1 and A_2 are using a symmetric strategy. This latter choice is made for two primary reasons: First, since A_1 and A_2 are symmetric, the most natural (and fair) equilibrium is the one where they use the same strategy (and gain the same profit). Second, in this example, the symmetric equilibrium is the one that maximizes the organizer's expected profit. For $c_3 < M/3 = 0.133$ we obtain an equilibrium $(0.6, 0.6, 1)$. Other equilibria are $(1, 0, 1)$ and $(0, 1, 1)$. In the interval $0.133 < c_3 < 0.186$ a mixed equilibrium of type (p, p, p_3), which could not hold for $c_3 < M/3$, replaces the $(p, p, 1)$ solution which is not stable anymore. Interestingly, in the transition that takes place at $c_3 = 0.133$ we observe another counter-intuitive phenomena according to which a decrease in the competence of one of the agents (an increase in the participation cost of A_3 in this case) results in an increase in the organizer's expected profit. In the interval $0.186 < c_3 \le M/2 = 0.2$ there are three equilibria, each characterized by having two of the agents participate in the contest and the third not participating, all resulting in an expected profit of $B^{org} = 2/3$. Finally, with $c_3 > 0.2$ we obtain the single equilibrium $(1, 1, 0)$, resulting once again in $B^{org} = 2/3$.

With the sequential contest, we obtain a single BNE. As expected, the threshold used by A_3 decreases as its participation cost increases. The decreasing competition from A_3 results in an increase in the threshold used by A_2. For $c_3 = 0.16$ all three agents are homogeneous and indeed $r_2 = r_3 = 0.6$ as calculated by (11). Once r_3 becomes less than 0.6, the threshold used by A_2 becomes fixed and equals 0.6. This can intuitively be explained by the fact that in the absence of A_3 the threshold used by A_2 is $r_2 = 0.6$ (according to (11)). Meaning that A_2's expected profit over those cases where its performance is greater than 0.6 is enough to justify participation. However adding A_3 with a strategy $r_3 < 0.6$ does not affect A_2's profit whenever performing better than 0.6, as A_3 opts not to participate in such cases.

Finally, when A_3's participation cost becomes big enough ($c_3 = 2M/3 = 0.267$), the BNE's nature changes to having all three agents opt not to participate (in which case the organizer's expected profit drops to zero). This is because as long as A_1 and A_2 do not participate, for any $c_3 > M - \frac{M}{3}$ not participating dominates participating for A_3. In such case, if A_1 is not participating then A_2 gains more from not participating and receiving $\frac{M}{3}$ than participating (and hence invoking also A_3's participation, though with a relatively small threshold r_3). Similar considerations apply to A_1. Interestingly, for $c_3 > 0.273$ these considerations do not hold anymore, as A_3 becomes so incompetent that A_2 actually finds it beneficial to deviate towards participating even when A_1 does not participate.

As can be seen in Graph (a) of the figure, there are three transition points in the organizer's preference of the type of contest to be used. Further transitions in preference occur with different orderings of the agents in the sequential contest model, as well as changes in c_1, c_2 and M.

6 Homogeneous Agents

In many real-life contest settings agents are homogeneous in the sense that they share the same participation cost ($c_i = c, \forall i$) and the probability distribution function according to which their performance is determined ($f_i(x) = f(x), \forall i$). This is very common whenever the differences in competence of the agents are minor thus their performance at the time of contest is mostly influenced by some probabilistic factors (e.g., luck, refereeing and weather conditions). For example, in a backgammon contest, participant's playing skills are typically similar and their performance mostly differ due to rolls of dice. Their costs of participating in the contest can be considered similar in case it is based on a standard participation fee. The analysis of the homogeneous case reveals several interesting properties related to the structure of the game equilibrium.

6.1 Parallel Contest

Since the agents are homogeneous we are interested in the symmetric equilibrium (of the form $p_i = p\,\forall i$), as it is the natural and fair one (offering all agents the same expected profit). When using the same p, Equations (1) and (2) can be substantially simplified and expressed as follows:

$$B_i^P = \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{1}{j+1} M p^j (1-p)^{k-j-1} - c \quad (8)$$

$$B_i^{\neg P} = \frac{M}{k}(1-p)^{k-1} \quad (9)$$

Theorem 2 provides a comprehensive characterization of the BNE in this case, as a function of the ratio $\frac{c}{M}$, including a closed form solution for p.

Theorem 2. *In the homogeneous parallel contest case, the BNE is: (a) fully based on pure strategies such that $p = 1$ (all agents participate) and $p = 0$ (all agents opt not to participate) when $\frac{c}{M} \leq \frac{1}{k}$ and $\frac{c}{M} \geq \frac{k-1}{k}$, respectively; (b) based on mixed strategies, otherwise, where p is the solution to:*

$$\frac{c}{M} = \frac{1-(1-p)^{k-1}}{kp} \quad (10)$$

Proof. Part (a) directly derives from Proposition 1. In any other case there is necessarily one equilibrium which is based on mixed strategies that use the same p. Equating (8) to (9) results in $\frac{c}{M} = \sum_{j=1}^{k-1} \binom{k-1}{j} \frac{1}{j+1} p^j (1-p)^{k-j-1} + \frac{k-1}{k}(1-p)^{k-1}$. Using the identity $\binom{k}{j+1} = \frac{k}{j+1}\binom{k-1}{j}$ and the binomial theorem we obtain (10). □

The derivative of the right-hand-side of (10) with respect to p is always negative, hence the value of p (and consequently the expected profit of the organizer) increases as participation cost c decreases over the interval $\frac{1}{k} < \frac{c}{M} < \frac{k-1}{k}$.

6.2 Sequential Contest

Theorems 3 and 4 provide a comprehensive characterization of the BNE for the case of sequential contest with homogeneous agents as a function of the problem parameters.

Theorem 3. *The equilibrium solution to the homogeneous case is: (a) have all agents not participate whenever $\frac{c}{M} \geq \frac{k-1}{k}$; (b) have all agents participate whenever $c = 0$; (c) otherwise, have all agents use the same threshold r to determine whether or not to participate, where r satisfies:*

$$\frac{c}{M} = \int_{y=r}^{\infty} f(y)dy = (1 - F(r)) \quad (11)$$

Proof. The proof for the case where $\frac{c}{M} \geq \frac{k-1}{k}$ is a trivial modification of the proof given for case (b) of Proposition 1. The case where $c = 0$ is straightforward—if not incurring a participation cost then participating dominates not participating. Therefore we only need to prove the part where $0 < \frac{c}{M} < \frac{k-1}{k}$. This is proved by induction: The last agent A_k uses r according to (11) based on substituting $F_i^{next}(y) = 1$ (according to (7)) in (6). Now assume that every agent A_j for which $j > i$ uses $r_j = r$ according to (11). We prove that $r_i = r$ as well. Assume otherwise, e.g., $r_i < r$. Now consider any value z such that $r_i < z < r$. Since $z > r_i$, not participating dominates participating if being reached with a value z (i.e., $B_i^P(z) < 0$). However according to the proof's assumption all remaining agents use r and therefore the expected profit of the agent if participating is: $B_i^P(z) = -c + M\Big(\int_{y=r_i}^{r} f(y)(F(y))^{k-i}dy + (1-F(r))\Big)$. Substituting $c = M(1-F(r))$ (according to (11)) in the latter term obtains $B_i^P(z) = M\int_{y=r_i}^{r} f(y)(F(y))^{k-i}dy > 0$ which is a contradiction. Similarly, assume $r_i > r$ and a value z such that $r < z < r_i$. Since $z < r_i$ participating dominates not participating if being reached with a value z (i.e., $B_i^P(z) > 0$). However if participating in the contest the expected profit of the agent is: $B_i^P(z) = -c + M(1 - F(z)) < -c + M(1 - F(r)) = 0$, which leads again to contradiction. □

Theorem 4. *In equilibrium, all agents gain the exact same expected profit regardless of their position in the sequence, depending on the nature of the equilibrium: (a) $\frac{M}{k}$ if the equilibrium strategy is not to participate; (b) $\frac{M}{k} - c$ if they all participate (i.e., when $c = 0$); and (c) otherwise:*

$$B = \frac{M(F(r))^k}{k} \quad (12)$$

Proof. The first two cases are trivial and result from the model definition. As for the third case, the expected profit B_i of agent A_i is given by:

$$B_i = \int_{y=-\infty}^{r} (i-1)f(y)(F(y))^{i-2}\Big(M\int_{z=y}^{r} f(z)(F(z))^{k-i}dz + M\int_{z=r}^{\infty} f(z)dz - c\Big)dy$$

where $(i-1)f(y)(F(y))^{i-2}$ is the probability distribution function of getting to A_i with a performance y and $(F(z))^{k-i}$ is the probability that the remaining $k-i$ agents will fail to perform better than y. Substituting $\int_{y=r}^{\infty} f(y)dy = \frac{c}{M}$ (according to (11)) and further applying several standard mathematical manipulations we obtain (12). □

The implications of Theorems 3 and 4 are quite surprising and counter-intuitive: with homogeneous agents even though each agent has full information concerning the performance of those participating before it, in equilibrium all agents follow the same participation rule and that rule does not depend on the number of participants k (whenever choosing to participate) or the number of remaining contestants. Furthermore, despite the fact that the chance of each agent to participate decreases (as it depends on the chance that preceding agents performed worse than r), along the agents sequence, all agents end up with the same expected profit, regardless of their position in the contest sequence.

Notice that from (11) we obtain that $F(r) = \frac{M-c}{M}$ and substituting it in (12) yields $B = \frac{(M-c)^k}{kM^{k-1}}$, meaning that the expected profit of any of the agents can be calculated even without calculating first the threshold-based strategy. Furthermore, $\frac{(M-c)^k}{kM^{k-1}} < \frac{M}{k}$, meaning that the agents are better off having all of them not participate in the contest. Unfortunately, this solution is not stable, leading as in many other problems to the tragedy of the commons.

If none of the agents participate the organizer's expected profit is v_0. Otherwise, her expected profit is given by (3), where:

$$\bar{F}(y) = \begin{cases} (F(y))^k & y < r \\ (F(r))^k + \frac{1-(F(r))^k}{1-F(r)}(F(y)-F(r)) & y \geq r \end{cases}$$

6.3 Methods Comparison

The analysis provided in the former section unfolds several inherent differences between the parallel and sequential contest in the homogeneous case. The most significant difference is the dependency of the equilibrium strategy on the underlying distribution function $f(x)$ and the number of participants. From Theorem 2 we obtain that the parallel contest equilibrium strategies depend on the number of agents k however do not depend whatsoever on $f(x)$. With the sequential contest we obtain the reversed phenomenon—the probability distribution function plays a key role in shaping the agents' equilibrium strategies while the number of agents has no effect at all (see Theorem 3).

In the homogeneous case, we can identify some conditions for determining the type of contest that will maximize organizer's expected profit. These are captured by Proposition 2.

Proposition 2. *From the organizer's point of view (i.e., B^{org}-wise): (a) when $c = 0$ or $\frac{c}{M} \geq \frac{k-1}{k}$ both the parallel and sequential contests result in the same expected profit; (b) when $0 < \frac{c}{M} \leq \frac{1}{k}$ the parallel contest dominates the sequential one; and (c) for $\frac{1}{k} < \frac{c}{M} < \frac{k-1}{k}$ there exists at least one alternation point where the preference shifts from parallel to sequential contest.*

Proof. In case $c = 0$ all agents participate in the contest, as proved in Theorems 2 and 3. Therefore the expected profit is identical. Similarly, in the case $\frac{c}{M} \geq \frac{k-1}{k}$ all agents opt not to participate in the contest (Theorems 2 and 3) hence the expected profit is zero regardless of the method used. When $0 < \frac{c}{M} \leq \frac{1}{k}$ all agents participate in a parallel contest (Theorem 2), while with a sequential contest it is guaranteed that only the first agent participates, however the participation of the others depend on the performance of their predecessors. Therefore the parallel contest yields a greater expected profit.

For $\frac{1}{k} < \frac{c}{M} < \frac{k-1}{k}$, we already know (proved above) that when $\frac{c}{M} = \frac{1}{k}$ parallel contest dominates sequential contest, hence to prove that there is an alternation point we need to show that for $\frac{c}{M} \to \frac{k-1}{k}$ the sequential contest dominates the parallel. Substituting $\frac{c}{M} \to \frac{k-1}{k}$ in (8) and (9) and equating the two (to obtain the mixed strategy p) obtains:

$$\frac{k-1}{k} = \sum_{j=1}^{k-1} \binom{k-1}{j} \frac{1}{j+1} p^j (1-p)^{k-j-1} + \frac{k-1}{k}(1-p)^{k-1}$$

The above can hold only when $p \to 0$. Therefore, the expected profit is $B^{org} \to 0$. With sequential contest, on the other hand, when $\frac{c}{M} \to \frac{k-1}{k}$ we obtain $r \to F^{-1}(\frac{1}{k})$ (according to (11)), hence there will be at least one participant, leading to an expected profit $B^{org} > 0$. □

Finally, we get to the difference in the effect of the number of agents over the expected profit of the organizer. Surprisingly, with parallel contest it is possible that an increase in the number of agents results in a decrease in the organizer's expected profit. To illustrate, consider a setting of k agents characterized by a participation cost $c = 0.15$, a prize $M = 0.4$, a fallback performance $v_0 = 0$ and a uniform distribution function over the interval $(0, 1)$. Here, with three agents ($k = 3$) and with four agents ($k = 4$) the equilibria are $p = 0.875$ and $p = 0.634$, resulting in expected profit of 0.714 and 0.687, respectively. This cannot happen in the sequential contest (proof is omitted for space considerations).

7 Discussion and Conclusions

As demonstrated numerically, the preference of the model to be used highly varies in the setting parameters, in the general case. This is where the analysis provided in the paper becomes particularly important, as it enables a contest organizer with the appropriate mathematical tools for extracting the equilibrium in both model variants. While the model assumes (much like most existing literature) the prize is always awarded, the transition to a model where the prize is not awarded if none of the agents participates is straightforward and requires only changing the profit in case of not participating ($B^{\neg P}$) to zero. Still, in this latter variant one has to consider a multi-attribute utility function for the organizer, to reflect the tradeoff between an increase in the performance achieved and the prize payments made.

The analysis of the homogeneous case unfolds several interesting properties of the equilibrium in the two models, many of which are highly counter-intuitive (especially in the case of sequential contest). In particular, an important implication of Theorem 4 is that an agent's profit does not depend on its position in the sequence of participants. This has many important social aspects from the contest design point of view, as it completely eliminates the need to compensate agents for their position in the sequence in order to maintain fairness.

We see many directions for future research extending this work. Among these are multi-prize allocation for enhancing participation, sequencing algorithms for maximizing profit in the case of sequential contest with heterogeneous agents and prize-setting algorithms for maximizing social welfare.

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