On the Choice of Obtaining and Disclosing the Common Value in Auctions

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Abstract

This paper introduces a game-theoretic analysis of auction settings where bidders' private values depend on an uncertain common value, and the auctioneer has the option to purchase information that can eliminate that uncertainty. Therefore the auctioneer needs to decide whether to purchase the information, and if so, whether to disclose it to the bidders. Unlike prior work, the model assumes that bidders are aware of the auctioneer's option to purchase the external information but are not necessarily aware of her decision. The modeling of the problem as a Stackelberg game, where the auctioneer is the leader, is complicated by the fact that in cases where the auctioneer decides not to disclose the information, the situation is actually modeled as a version of Stackelberg game where the follower has potentially imperfect information about the leader's actions. Our analysis of the individual expected-benefit-maximizing strategies results in the characterization of the pure-strategy perfect Bayesian Nash equilibrium and proof of its existence for any setting. In addition, we introduce an algorithm for extracting the equilibrium as a function of the information cost, which is of great importance when the information is provided by a strategic information-provider. The analysis is also extended to deal with mixed-strategy perfect Bayesian Nash equilibrium and with noisy information. Overall, the analysis enables the demonstration of various model characteristics, including many non-intuitive properties related to the benefits of competition, the benefits in having the option of the auctioneer to purchase such information and the benefits encapsulated in the bidders' awareness of such an option.

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1. Introduction

Recent advances in information technologies support the emergence of dynamic pricing mechanisms as the successors of fixed pricing in electronic marketplaces. The success of dynamic pricing mechanisms is based on their premise to improve benefit and resource utilization. One important dynamic pricing mechanism, where price emerges from the buyers' (e.g., bidders') willingness to pay, is auctions. Over the past two decades, auctions have become an integral part of electronic commerce, a popular method for transacting business, and a promising field for applying agent and Artificial Intelligence technologies [30, 65, 37, 59, 56].

The key aspect that affects bidding in an auction is the way the bidders value the auctioned item. In this paper we consider an auction model where the auctioned item is characterized by an uncertain common value, on which bidders' private values are based [34, 22, 23, 40]. For example, consider an auction for the lease of an advertising space in a shopping mall. Here, the common value associated with the advertising space is the foot traffic (e.g., number of shoppers that visit the shopping mall in a certain period of time). Each bidder's (i.e., potential advertiser) valuation of her expected benefit from leasing the advertising space depends on the foot traffic. Moreover, since each bidder has a different benefit from having a single shopper see its ad, different bidders will have different valuations for different values of the foot traffic. Another example is the classic oil drilling case [34]. Here, the amount of oil and the depth of its location under the ground are the uncertain common values. However, each bidder's valuation of the benefit from owning the drilling rights depends on the stratum to which she needs to drill, as each bidder can have different equipment and drilling technology. Similar arguments favoring this hybrid-value model can be suggested for other classic auction domains, e.g., the U.S. Federal Communications Commission (FCC) [11]. Even the classic painting example that is often used in the context of private value can be considered as an example for the hybrid model due to the resell factor [22].

One key question in models assuming an uncertain common value concerns the disclosure of information that may eliminate some of the uncertainty associated with the common value, whenever such information is available to the auctioneer. Unlike prior work that considered models combining private and common value aspects [22, 23, 9, 40], bidders in our model are not limited to an additive combination of the two, and the effect of the common value on all bidders' valuation is not necessarily positively correlated. One important implication of this difference is that, in contrast to results obtained in prior work [51], the preferred choice for the auctioneer is not necessarily to always disclose the common value. Instead, the auctioneer needs to selectively disclose the information regarding the common value, based on her beliefs regarding the bids that will be received for any value disclosed and whenever not disclosing any value.

In many real-life situations the auctioneer does not initially possess the information, but rather needs to pay in order to obtain it (either in the form of resources it needs to spend in order to produce it "internally" or in the form of purchasing it from an external information-provider). Taking the examples above, information concerning shoppers' traffic can be obtained either by placing mall employees next to the offered space in order to measure the bypassing traffic, by analyzing data collected from the security cameras, by executing a statistical analysis based on the number of people visiting the shopping mall or by hiring an expert that can produce such an estimate using her own methods. In the oil drilling example the auctioneer can execute an exploratory drill or alternatively pay a specialist for a thorough survey of the field. Our model applies to this case by enabling the option of purchasing information that fully eliminates the uncertainty associated with the common value of the auctioned item. This option is available only to the auctioneer. Bidders in our model have no such option of purchasing the information (for various realistic reasons, e.g., if the production of the information requires access to some data that only the auctioneer has) however become acquainted with the true common value if the auctioneer purchases the information and chooses to disclose the value received. In such a setting, a substantial part of the auctioneer's strategy is deciding whether or not to obtain the external information, and, if so, once obtained, whether or not to disclose it to the bidders. The problem can thus be modeled as a Stackelberg game where the auctioneer is the leader and the followers are the bidders. This modeling is complicated by the fact that in cases where the auctioneer decides not to disclose the information, the situation is actually modeled as a version of a Stackelberg game where the followers have potentially imperfect information about the leader's actions.

The contributions of this paper are threefold: First, the paper formally presents and analyzes an auction model where: (a) bidders' valuations depend on an uncertain common value in a general way; (b) the auctioneer can eliminate the uncertainty associated with the common value through the purchase of information; and (c) bidders are aware of the availability of such an option to the auctioneer. To the best of our knowledge, a model of this type has not been investigated to date. In particular, the incorporation of costly information and bidders' awareness results in several complexities both from the strategy space and the analysis points of view. The analysis given essentially derives from the individual benefit-maximizing strategies of the auctioneer (given the bidders' belief regarding whether or not the auctioneer is planning to purchase the information and which values she will disclose) and the bidders (given the auctioneer's decision of whether to purchase the information and which values she is planning to disclose). This leads to the characterization of stable solutions and a perfect Bayesian Nash equilibrium. As part of the analysis we characterize the influence of the cost of obtaining the information over the equilibrium and the resulting expected benefit of the different players, proving, among other things, that an equilibrium always exists. The analysis leads to an algorithm for calculating the pure-strategy equilibrium for all different possible costs of purchasing the information, which is of much importance for the auctioneer and the social planner. The analysis encompasses various other aspects of the model, such as the expected-benefit-maximizing strategy for a self-interested information-provider and the effect of bidders' homogeneity over the results.

Second, using the equilibrium analysis, we manage to illustrate various properties of the model. The nature of these results is primarily existential (i.e., showing the existence of said solution), and many of them are somehow counter-intuitive. For example, it is demonstrated that, in conflict with classic auction theory, the auctioneer will not necessarily find it beneficial to have more bidders participate in the auction and similarly bidders will not necessarily prefer less competition. Also, bidders' unawareness of the auctioneer's option to purchase the information does not necessarily play into the hands of the auctioneer and, similarly, bidders may sometimes benefit from not knowing that the auctioneer has the option to purchase such information. Furthermore, having the option to purchase the information can be devastating for the auctioneer in some settings, even though she gets to decide whether or not to purchase the information and what portions of it to disclose to the bidders. Similarly, the auctioneer may prefer that the information be offered at a high rather than a low price (and in many cases would even prefer costly information over the option to obtain it for free). Common to all the above results is that they are obtained in situations where, despite switching to what might seem to be a more favorable setting, the solution one would expect to hold in the new setting is found to be unstable and the equilibrium with which the system eventually ends up is worse (in terms of the expected benefit to the different players) than the one held in the less favorable setting.

Finally, the paper shows that in our unique setting, it might be beneficial for the auctioneer to pay the external information-provider in order to change the price she sets for the information. This can either have the form of making the information-provider leave the market (or alternatively publicly increase the price of the information she offers) or making the information-provider announce a decrease in the information price. Similarly, we demonstrate how a social planner can improve social welfare in our model through subsidy and taxation. The taxation aims either to discourage the auctioneer from purchasing the information or drive the system towards a more beneficial equilibrium, whereas the subsidy enables setting a price in which the information is indeed purchased (in cases where it is not purchased otherwise).

The remainder of the paper is organized as follows: In Section 2 we describe the model in detail. The pure-strategy perfect Bayesian Nash equilibrium analysis as well as a procedure for its efficient calculation as a function of the information pricing are given in Section 3. Section 4 extends the analysis to the case where players can use mixed strategies. In Section 5 we illustrate specific properties of the model using synthetic settings. In Section 6 we survey related work, and finally we conclude and outline directions for future work in Section 7.

2. The Model

The model assumes a setting where a single auctioneer offers a single item for sale in a second-price sealed-bid auction (with a random winner selection among highest bids in case of a tie) to n heterogeneous bidders that are interested in the said item.[1] Both the auctioneer and the bidders are assumed to be risk-neutral and fully rational. As is common in auction literature, the auctioned item is assumed to have a characteristic X whose value, x, is a priori unknown both to the auctioneer and the bidders [22, 23]. The only information publicly available with regard to X is the set of possible values it can obtain, denoted $X^* = \{v_1, ..., v_k\}$, and the probability associated with each value, $Pr(X = x)$ ($\sum_{x \in X^*} Pr(X = x) = 1$). In case X obtains values from a set of continuous intervals, we use the probability distribution function $f_x(x)$ instead of $Pr(X = x)$.

Each bidder is assumed to be characterized by a type T, where the set of possible types it can obtain is T^*. Bidders' types are assumed to be independent and identically distributed, such that the a priori probability of any of the bidders being of type $T = t$ is given by $Pr(T = t)$ [22, 23]. For the case where types are continuous we use the probability distribution function $f_t(t)$ instead of $Pr(T = t)$. A bidder's type defines her valuation of the proposed item (i.e., her "private value") for any possible value that X may obtain. We use the function $V_t(x)$ to denote the value bidders of type $T = t$ see in the auctioned item in the case where the characteristic X obtains a value x. The value of X can thus be seen

[1] The use of the second-price auction protocol is due to the fact that it is a very natural, well-known and wide-spread scheme for selling an item [42]. Also, this is the protocol used in most prior literature that deals with selective disclosure of information (e.g., [17, 42, 14]).

as a common value in this context, and the function $V_t(x)$ defines the way that the private value of bidders of type t is affected by it. Unlike prior work that commonly assumed some correlation between the ways that different bidders' valuations are set given the common value (e.g., linear or symmetric dependency on the common value or a symmetric function of the other bidders' signals [23, 40]), the model in this paper does not imply any restriction on the function $V_t(x)$. Each bidder is assumed to know its own type, but not the type of the other bidders that take part in the auction. Similarly, the auctioneer is assumed to be unfamiliar with the specific types of the bidders that take part in the auction.

The model assumes that the auctioneer can disambiguate the common value, i.e., obtain the actual value of X, upon payment of a fee C (e.g., by consuming some of her resources in order to obtain it or by purchasing it from an external information-provider) prior to starting the auction. We refer to this as "obtaining information". If the auctioneer so chooses, the true value x of X becomes available only to her, and she can either disclose it truthfully to all bidders symmetrically, prior to bidding, or keep it to herself. These latter assumptions (truthful and symmetric information disclosure) are common in settings where the auctioneer is regulated or has to consider her reputation, and would not want to lose out on future benefits associated with her privileged knowledge. We assume that bidders cannot independently obtain the value of X (not even for a fee), and the only way that they can become aware of the true common value, before placing their bids, is if the auctioneer obtains and discloses it. The realism for such an assumption may be the fact that the information-provider's services might require direct access to the auctioned item or some private information that cannot be accessed without the auctioneer's permission and cooperation.[2] For example, in the case of auctioning oil and gas mineral rights, only the auctioneer can execute an exploratory drill, as bidders do not have the right to drill in that area.

The model distinguishes between cases where the bidders are aware of the auctioneer's option to purchase the additional information and when they are not. This issue is fundamental whenever no information is disclosed to bidders. In the case where bidders are unaware, the auctioneer's problem becomes an optimization problem as the bidders' bids, whenever no information is being disclosed, will not change based on the the auctioneer's strategy. In the case where bidders are aware, they will act strategically, attempting to distinguish between not receiving the information because the auctioneer intentionally did not purchase the information in the first place, and not receiving because despite purchasing the informa-

[2]The challenges of analyzing the setting where bidders may independently obtain the value are discussed in Section 7.

tion the auctioneer decided not to disclose the specific value obtained. A setting where bidders are unaware of whether the auctioneer can purchase the additional information is common whenever the extraction of the true value requires further complementary information whose availability to the auctioneer is unknown to the bidders. A setting where bidders are aware of the auctioneer's option is common whenever the information is supplied by an external information-provider, whose existence and the price she charges for such information are publicly known.

Both the auctioneer and the bidders are assumed to be self-interested and attempt to maximize their own expected benefit. The auctioneer's expected benefit is defined as the expected benefit from the auction (i.e., the expected second best bid) minus, if choosing to obtain the information, the payment C. A bidder's benefit is her valuation of the item minus her payment to the auctioneer (which is the second highest bid) if she wins the auction and zero otherwise. The model also distinguishes between the case where the external information-provider is a self-interested strategic player that sets the cost of her service, C, in a way that maximizes her benefit and the case where the value of C is exogenously set (e.g., when the information extraction requires some costly exploration or internal data gathering, or when the information-provider uses a fixed price that applies to any customer and is set based on various external considerations).

Finally, the model also assumes the existence of a social planner (e.g., a government or a market/platform owner). The social planner aims to maximize the "social welfare", defined as the sum of the expected benefits of all participants (auctioneer and bidders). Consequently, she can decide to tax or subsidize the purchase of information if this will result in an increase in the social welfare. While taxes can later be re-distributed, we take a strict approach and do not add the social-planner's proceeds from the tax to the social-welfare calculations (however the tax is subtracted from the auctioneer's expected benefit, if the information is indeed purchased). The subsidy (paid to the information-provider), however, is always subtracted from the overall social welfare whenever used.

All players (auctioneer, bidders, information-provider and social planner) are assumed to be acquainted with the general setting parameters: the number of bidders in the auction, n, the cost of purchasing the information, C, the discrete random variables, X and T, their possible values and their discrete probability distributions (or the probability distribution functions when values are continuous).

3. Analysis

We begin by analyzing the bidders' and the auctioneer's individual expected-benefit-maximizing strategies, given the strategies of the other players. Based on these we provide a comprehensive analysis of the pure-strategy perfect Bayesian Nash equilibrium, which, as detailed in the following section, can be extended to deal with the mixed-strategy perfect Bayesian Nash equilibrium in a rather straightforward manner. For exposition purposes the section presents the analysis for the discrete case. The appropriate adjustments required for the continuous case are given in the Appendix.

3.1. Bidders' Side

A bidder's strategy defines its bid given the information disclosed by the auctioneer (or its absence), i.e., the mapping $B(t, x) \to \mathbb{R}$, where t is the bidder's type and x is the information disclosed ($x \in X^*$ or $x = \emptyset$). We use $R^{bidder} \subseteq X^*$ to denote the set of values that the bidder believes the auctioneer will disclose if purchasing the external information. Since the auctioneer is known to be self-interested, the case $R^{bidder} = \emptyset$ necessarily represents the bidder's belief that the auctioneer does not obtain the information in the first place (as not purchasing the information dominates purchasing without disclosing). Since bidders are fully rational and their information regarding the value-distribution of X is identical, they all use the same set R^{bidder} in equilibrium.

Since this is a second-price sealed-bid auction, the bidders' dominating (expected-benefit-maximizing) bids can be calculated according to the following two cases:

- The auctioneer discloses the value x of X - the dominating strategy for each bidder is to bid her private value (which is equivalent to truth telling [60]). Therefore the bid $B(t, x)$ of a bidder of type t in this case is:

  $$B(t, x) = V_t(x) \tag{1}$$

  It is notable that the bidder's bid in this case is affected only by the value x disclosed by the auctioneer and is not affected by the bidder's belief, R^{bidder}, whatsoever. This is because even if the auctioneer discloses a value which the bidder was not expecting to be disclosed (i.e., $x \notin R^{bidder}$), this value "overrides" the set R^{bidder} once it is disclosed, making R^{bidder} irrelevant. The value x dictates the private value of the bidders, therefore whenever the true value is disclosed, the problem maps to a standard second-price sealed bid auction, where every bidder knows her exact private value.

- The auctioneer does not disclose any value - the dominating strategy for each bidder is to bid her expected private value, given that $x \notin R^{bidder}$ [17]. Therefore the bid $B(t, \emptyset)$ of a bidder of type t in this case is:

  $$B(t, \emptyset) = \sum_{x \notin R^{bidder}} V_t(x) \cdot Pr^*(X = x) \tag{2}$$

where $Pr^*(X = x)$ is the posterior probability, given that $x \notin R^{bidder}$. Namely, the probability of having x be the true common value will now be calculated as:

$$Pr^*(X = x) = \begin{cases} 0 & \text{if } x \in R^{bidder} \\ \frac{Pr(X=x)}{\sum\limits_{y \notin R^{bidder}} Pr(X=y)} & \text{if } x \notin R^{bidder} \end{cases} \tag{3}$$

Therefore, bidders' dominating (or "best response") strategies can be compactly represented by the set R^{bidder}, such that for any disclosed value x they bid $B(t, x) = V_t(x)$, and when no value is disclosed they bid $B(t, \emptyset)$ according to (2) where $Pr^*(X = x)$ is calculated according to (3). From this point on we thus use R^{bidder} both as a belief and a strategy interchangeably.

One exception for the above is when the bidder believes that the auctioneer purchases the information and reveals any value obtained (i.e., $R^{bidder} = X^*$). In this case, if the auctioneer does not eventually disclose a value Equation 3 does not hold. In this case, which, as we show in the equilibrium analysis that follows and discuss in more detail, is off the equilibrium path, the bidders will bid according to the common value that minimizes the auctioneer's expected benefit.

The expected benefit of a bidder of type t from participating in the auction, when bidders believe the auctioneer uses R^{bidder} and the auctioneer discloses the set R^{auc}, denoted $u_{bidder}(t)$, is a composition of 4 possible cases:

- In case the value disclosed is $x \in R^{auc}$ and the bidder wins by exclusively placing the highest bid, her expected benefit is: $\sum_{B(t',x)<B(t,x)} \sum_{k=1}^{n-1} \binom{n-1}{k} (Pr(T = t'))^k \Big(\sum_{B(t'',x)<B(t',x)} Pr(T = t'') \Big)^{n-k-1}$ $(V_t(x) - B(t', x))$. The calculation iterates over all of the different values that the second best bid may obtain (i.e., all bids $B(t', x)$ placed by an agent of type $t' \neq t$, such that $B(t', x) < B(t, x)$) and weighs the difference between the agent's valuation of the item and the second best bid, $V_t(x) - B(t', x)$, according to the probability that the second best bid is indeed $B(t', x)$. The probability that the second best bid is $B(t', x)$ is given by $\sum_{k=1}^{n-1} \binom{n-1}{k} (Pr(T = t'))^k \Big(\sum_{B(t'',x)<B(t',x)} Pr(T = t'') \Big)^{n-k-1}$ as it requires having $k \geq 1$ agents of type t' and all of the remaining $n - k - 1$ agents to be of types that bid less than $B(t', x)$ (thus considering all types t'' such that $B(t'', x) < B(t', x)$).

  We note that here and in the following three cases, if other types assign a similar private value to a specific common value x as type t does (i.e., whenever $V_t(x) = V_{t'}(x)$ for two types, t and t', such that $t \neq t'$ for some value x), these types should be united (i.e., their type probabilities should be summed) in the calculation related to the disclosure of that specific value. This applies also for the calculation related to values that are not disclosed (i.e., combining types that assign a similar value to $x = \emptyset$).

- In case the value disclosed is $x \in R^{auc}$ and the bidder wins by placing a bid equal to the highest bid placed by any of the other bidders, her expected benefit is: $\sum_{k=1}^{n-1} \binom{n-1}{k} \frac{1}{k+1} (Pr(T=t))^k \Big(\sum_{B(t',x)<B(t,x)} Pr(T=t') \Big)^{n-k-1} (V_t(x) - B(t,x))$. In this case the value of the second best bid is necessarily $B(t,x)$ (which equals $V_t(x)$ according to (1)). The probability of having k additional bidders with the same bid as the agent's bid is $\binom{n-1}{k}(Pr(T=t))^k$ and in order for this bid to be the winning bid all of the remaining agents must bid below this, i.e., with a probability of $\Big(\sum_{B(t',x)<B(t,x)} Pr(T = t') \Big)^{n-k-1}$. The probability that the agent will be the winner in this case is $1/(k+1)$ as the winner is chosen randomly from the winning bids if there is a tie.

  We note that the difference $V_t(x) - B(t,x)$ in this case is in fact zero. Nonetheless, we have not omitted this case from the formulations given, for completeness. Another important reason for keeping this term is that once it is incorporated into the $u_{bidder}(t)$ calculation (see (4) below), the expression can be easily divided into the expected value of the item when winning (i.e., excluding the payment of the second-best bid) and the expected payment (i.e., the second-best bid when winning) for an agent of type t. The two parts are represented in the four subtractions made, where the subtracted parts are the second-best bid, and the terms from which the subtraction is made are the bidder's expected valuation of the item.

- In case no value is disclosed, and the bidder wins by exclusively placing the highest bid, her expected benefit is: $\sum_{B(t',\emptyset)<B(t,\emptyset)} \sum_{k=1}^{n-1} \binom{n-1}{k} (Pr(T = t'))^k \Big(\sum_{B(t'',\emptyset)<B(t',\emptyset)} Pr(T=t'') \Big)^{n-k-1} \cdot \Big(\frac{\sum_{y \notin R^{auc}} V_t(y) Pr(X=y)}{\sum_{y \notin R^{auc}} Pr(X=y)} - B(t',\emptyset) \Big)$. The principles of the calculation in this case are similar to those used when information is disclosed, while replacing $B(t,x)$, $B(t',x)$ and $B(t'',x)$ with $B(t,\emptyset)$, $B(t',\emptyset)$ and $B(t'',\emptyset)$, respectively. The expression $\frac{\sum_{y \notin R^{auc}} V_t(y) Pr(X=y)}{\sum_{y \notin R^{auc}} Pr(X=y)}$ is the expected private value of an agent of type t when a value is not disclosed (obtained by substituting (3) in (2)).

- In case no value is disclosed and the bidder wins by placing a bid equal to the highest bid placed by any of the other bidders, her expected benefit is: $\sum_{k=1}^{n-1}\binom{n-1}{k}\frac{1}{k+1}(Pr(T=t))^k\Big(\sum_{B(t',\emptyset)<B(t,\emptyset)}Pr(T=t')\Big)^{n-k-1}$ $\cdot\,(\frac{\sum_{y\notin R^{auc}}V_t(y)Pr(X=y)}{\sum_{y\notin R^{auc}}Pr(X=y)}-B(t,\emptyset))$. The principles of the calculation in this case are similar to those used when information is disclosed, while replacing $B(t,x)$ and $B(t',x)$ with $B(t,\emptyset)$ and $B(t',\emptyset)$, respectively. Also, when $R^{auc}=R^{bidder}$, the difference $\frac{\sum_{y\notin R^{auc}}V_t(y)Pr(X=y)}{\sum_{y\notin R^{auc}}Pr(X=y)}-B(t,\emptyset)$ is zero. The choice of including this term in the calculation of $u_{bidder}(t)$ when $R^{auc}=R^{bidder}$ is justified by the same considerations given in the second case above.

Based on these four cases we can calculate the expected benefit of bidders of type t, weighing the expected benefit for each value that X may obtain according to its occurrence probability:

$$
\begin{aligned}
&u_{bidder}(t)=\\
&\sum_{x\in R^{auc}}Pr(X=x)\Big(\sum_{B(t',x)<B(t,x)}\sum_{k=1}^{n-1}\binom{n-1}{k}(Pr(T=t'))^k\Big(\sum_{B(t'',x)<B(t',x)}Pr(T=t'')\Big)^{n-k-1}(V_t(x)-B(t',x))\\
&+\sum_{k=1}^{n-1}\binom{n-1}{k}\frac{1}{k+1}(Pr(T=t))^k\Big(\sum_{B(t',x)<B(t,x)}Pr(T=t')\Big)^{n-k-1}(V_t(x)-B(t,x))\Big)\\
&+\sum_{x\notin R^{auc}}Pr(X=x)\Big(\sum_{B(t',\emptyset)<B(t,\emptyset)}\sum_{k=1}^{n-1}\binom{n-1}{k}(Pr(T=t'))^k\Big(\sum_{B(t'',\emptyset)<B(t',\emptyset)}Pr(T=t'')\Big)^{n-k-1}\\
&\cdot\,(\frac{\sum_{y\notin R^{auc}}V_t(y)Pr(X=y)}{\sum_{y\notin R^{auc}}Pr(X=y)}-B(t',\emptyset))\\
&+\sum_{k=1}^{n-1}\binom{n-1}{k}\frac{1}{k+1}(Pr(T=t))^k\Big(\sum_{B(t',\emptyset)<B(t,\emptyset)}Pr(T=t')\Big)^{n-k-1}\\
&\cdot\,(\frac{\sum_{y\notin R^{auc}}V_t(y)Pr(X=y)}{\sum_{y\notin R^{auc}}Pr(X=y)}-B(t,\emptyset))\Big)
\end{aligned}
\tag{4}
$$

In general, the influence of R^{bidder} over the calculation taking place in Equation (4) is through the calculation of the bids $B(t,\emptyset)$, $B(t',\emptyset)$ and $B(t'',\emptyset)$, as given in (2) and (3).

Using (4) we can now calculate the expected benefit of a random bidder, denoted $u_{bidders}$. $u_{bidders}$ is different from $u_{bidder}(t)$ in the sense that while the latter is the expected benefit of a bidder of a specific type t, the first is the expected benefit of a bidder with no a priori information regarding her type, i.e., calculated as: $u_{bidders} = \sum_{t \in T} Pr(T = t) u_{bidder}(t)$. When separating the calculation into its two parts, as explained above, and multiplying by the number of bidders we obtain: (1) the expected valuation of the winning bidder, which also captures the efficiency of the allocation made and consequently the social welfare (as will be explained in 3.3); and (2) the expected second-best bid, which is actually the auctioneer's expected benefit.

3.2. Auctioneer's Side

We now turn to the analysis of the auctioneer's expected benefit, given her information purchasing and disclosure strategy and the bidders' strategies (captured by their set R^{bidder} according to which they update the probability distribution $Pr^*(X = x)$). Indeed from 3.1 we can calculate the auctioneer's expected benefit, given any set of strategies (R^{auc}, R^{bidder}), due to the ability to decompose the bidders' expected benefit into the payment to the auctioneer and the expected valuation of the winning bidder. Still, that calculation method is bidder-type oriented, i.e., requires calculating the two elements for each bidder type. In the following paragraph we supply a disclosed-value-oriented calculation for the auctioneer's expected benefit. The alternative calculation method is both more intuitive and enables a direct calculation of the expected benefit from disclosing any specific disclosed value or when no value is being disclosed.

As discussed earlier, since all bidders have the same information regarding the market structure, they all use the same belief R^{bidder}. In order to formalize the expected second-best bid if disclosing a value x, we first define two probability functions. The first is the probability that given that the value disclosed by the auctioneer is x, the bid placed by a random bidder equals w, denoted $g(w, x)$, and the second is the probability that the bid placed by a random bidder equals w or below, denoted $G(w, x)$. The functions $g(w, x)$ and $G(w, x)$ are given by:

$$G(w, x) = \sum_{B(t,x) \leq w} Pr(T = t) \quad ; \quad g(w, x) = \sum_{B(t,x) = w} Pr(T = t) \tag{5}$$

The expected benefit of the auctioneer when disclosing the information $X = x$, denoted $u_{auc}(X = x)$, equals the expected second-best bid when the bidders are given x, formally calculated as:

$$u_{auc}(X = x) = \sum_{w \in \{B(t,x)|t \in T\}} w(\sum_{k=1}^{n-1} n \binom{n-1}{k}$$
$$(1 - G(w,x))(g(w,x))^k (G(w,x) - g(w,x))^{n-k-1} \tag{6}$$
$$+ \sum_{k=2}^{n} \binom{n}{k} (g(w,x))^k (G(w,x) - g(w,x))^{n-k})$$

The calculation iterates over all of the possible second-best bid values, assigning for each its probability of being the second-best bid. As we consider discrete probability functions, it is possible to have two bidders placing the same highest bid (in which case it is also the second-best bid). For any given bid value, w, we therefore consider the probability of having either: (i) one bidder bidding more than w, $k \in 1, ..., (n-1)$ bidders bidding exactly w and all of the other bidders bidding less than w; or (ii) $k \in 2, ..., n$ bidders bidding exactly w and all of the others bidding less than w. Notice that (6) also holds for the case where $x = \emptyset$ (in which case bidders use $B(t, \emptyset)$ according to (2)). Consequently, if the strategy of the auctioneer is R^{auc}, and the bidders' strategy is R^{bidder}, then the auctioneer's expected benefit from the auction itself (i.e., excluding the payment C), denoted by $u(R^{auc}, R^{bidder})$, is:

$$u(R^{auc}, R^{bidder}) = \sum_{x \in R^{auc}} Pr(X = x) \cdot u_{auc}(x)$$
$$+ \sum_{x \notin R^{auc}} Pr(X = x) \cdot u_{auc}(\emptyset) \tag{7}$$

We note that the influence of R^{bidder} over the calculation given in (7) is through the calculation of $B(t, x)$ (according to (1) and (2)) which is part of the calculation of $G(w, x)$ and $g(w, x)$ according to (5).

The overall expected benefit of the auctioneer, denoted $U(R^{auc}, R^{bidder})$, is given by:

$$U(R^{auc}, R^{bidder}) = \begin{cases} u(R^{auc}, R^{bidder}) & \text{if } R^{auc} = \emptyset \\ u(R^{auc}, R^{bidder}) - C & \text{otherwise} \end{cases} \tag{8}$$

Given the bidders' belief R^{bidder} and the cost of obtaining the information C, the auctioneer's expected-benefit-maximizing strategy can be obtained by iterating over the different values in X^* and determining for each value x whether it should or should not be disclosed, based on the difference $u_{auc}(x) - u_{auc}(\emptyset)$, resulting in the best R^{auc} subset. Then, if $U(R^{auc}, R^{bidder}) > U(\emptyset, R^{bidder})$, the information should be purchased, and the values to be disclosed are those in the set R^{auc}. This solution, however, will hold only if the bidders are not acting strategically and stick with their belief R^{bidder}. One important case where the bidders are not acting strategically is when they are not aware of the auctioneer's option to purchase the information (in which case $R^{bidder} = \emptyset$). For example, if the information-provider does not publish her services but rather contacts the auctioneer directly, or if the information provider's assessment requires some complementary information that needs to be supplied by the auctioneer and the bidders mistakenly believe such complementary information is not available to the auctioneer. We defer the analysis and discussion of this case to Section 3.7.

3.3. Social Welfare

The social welfare measure sums the auctioneer's and bidders' expected benefits. Since the auctioneer's expected benefit is actually the second best bid paid by the winner minus the cost of purchasing the information, the social welfare essentially measures the true valuation of the item in the eyes of the winner. This represents the efficiency of the allocation made and aligns with prior work ([33] p.75-76). Alternatively, one may choose to measure the social welfare as the sum of the auctioneer's, the bidders' and the information provider's benefits. While we rely on the first measure in this paper, none of the qualitative results that are given in Section 5, excluding one example, changes when the latter measure is used for social welfare.

While the second-price sealed-bid auction when the common value is certain maximizes social welfare, this property does not hold when switching to an uncertain common value model. In the latter case, it is common that the winner is not necessarily the one that values the item most based on her true common value. Interestingly, with the option to purchase the external information, the social welfare in many cases actually improves (compared to cases without this option). This is illustrated in Section 5. Furthermore, as we later demonstrate, the use of subsidy and taxation in our model can also improve social welfare if used wisely by a social planner.

3.4. Equilibrium Dynamics

We now analyze the case where bidders are aware of the auctioneer's option to obtain the information at a cost C. This case can be considered a variant of a Stackelberg game [19] where the leader is the auctioneer and the followers are the bidders. In such scenarios the leader first commits to a strategy and then the followers selfishly optimize their own best strategy. As explained earlier, when the auctioneer does not disclose the true outcome the followers, i.e., the bidders, have imperfect information about the leader's actions. This is illustrated in Figure 1, which provides the extensive form representation of the model. First, the common value is determined by nature (node "0"). Then, the auctioneer needs to decide whether to purchase or not purchase the information (upper nodes "1"), and if purchasing then whether to disclose or hide the value obtained (lower nodes "1"). For the first decision, the auctioneer cannot distinguish between the different world states defined by the true value assigned to the common value. For the second (i.e., after purchasing the information) the value of X is known to the auctioneer. Then, bidders need to decide on their bid. At this stage the bidders can distinguish their state only if the information was purchased and disclosed by the auctioneer. In any other case, bidders are not only unaware of the true common value but also do not know if it was not received because the information was originally not purchased or it was purchased but not disclosed.

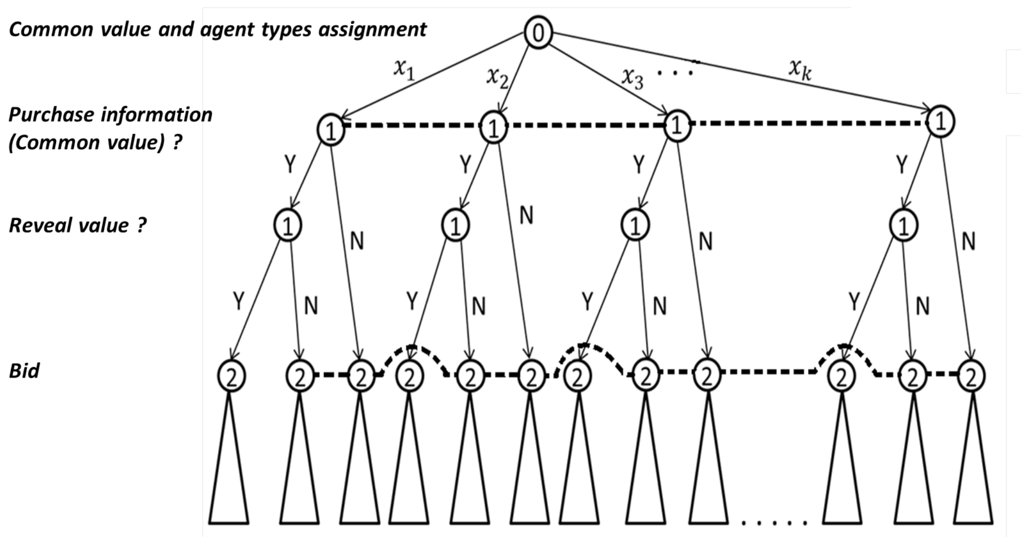


Figure 1: Extensive form representation of the game.

We note that both the auctioneer and the bidders do not know the type of the specific bidders taking part in the auction (other than a bidder's own type) - only the distribution of the types is known. Similarly, the players also do not know the true common value of the auctioned item. Therefore, our analysis considers the principles of perfect Bayesian Nash equilibria. In the remaining of the paper, unless stated otherwise, we use the term equilibrium to refer to a perfect Bayesian Nash equilibrium. An equilibrium profile of strategies in this case can be represented by the pair $(R^{auc}, \{B(t,x)|x \in X^*$ or $x = \emptyset)\})$, from which neither the auctioneer nor the bidders have an incentive to deviate individually. This pair maps to whether or not the auctioneer purchases the information (purchases if $R^{auc} \neq \emptyset$), what values she discloses (any value $x \in R^{auc}$), what values she does not disclose (any value $x \notin R^{auc}$) and the bidders' bids, depending on the value they receive. Since bidding according to (1) and (2), given R^{bidder}, dominates any other bidding strategy, we can use a compact representation of strategy profiles (or "solutions") in the form (R^{auc}, R^{bidder}), where bidders' bids are calculated according to (1) and (2), based on R^{bidder}.[3]

We first show that the pair (R^{auc}, R^{bidder}) where $R^{auc} \neq R^{bidder}$ cannot be stable, hence the potential pure equilibrium solutions are necessarily of the form $R^{auc} = R^{bidder} = R^*$. If indeed the solution is of the form (R^{auc}, R^{bidder}), where $R^{auc} \neq R^{bidder}$, then the bidders can benefit from deviating to (i.e., calculating $Pr^*(X = x)$ according to) $R^{bidder'} = R^{auc}$, since this will result in a bid that equals the bidder's expected valuation of the item in cases where no information is disclosed (whereas when R^{bidder} is used the bid is either greater or lesser than the bidder's expected valuation of the item).[4] In other words, since the bidders' best response is to bid according to R^{auc} then a solution (R^{auc}, R^{bidder}) of the form $R^{auc} \neq R^{bidder}$ is never stable.[5] The stability of solutions of the form $R^{auc} = R^{bidder} = R^*$, however, depends on the auctioneer's considerations. Being the leader, the auctioneer may find it beneficial to deviate to a strategy $R^{auc'} \neq R^{auc}$, given the bidders' belief $R^{bidder} = R^{auc}$. A deviation from a strategy $R^{auc} = \emptyset$ means purchasing the information and using a set $R^{auc'} \neq \emptyset$. Otherwise, if $R^{auc} \neq \emptyset$, the auctioneer may deviate to not purchasing the information (thus necessarily using $R^{auc'} = \emptyset$) or using $R^{auc'} \neq R^{auc}$ such that $R^{auc'} \neq \emptyset$.

To find equilibria of the form $R^{auc} = R^{bidder} = R^* \neq X^*$, one needs to iterate over all possible sets and calculate for any set $R' \subseteq X^*$ the auctioneer's expected benefit from using that set, $U(R', R')$, and her expected benefit $U(R^{auc}, R')$ when deviating to any other possible set $R^{auc} \neq R'$, assuming all bidders are using R' according to (3) to construct their bids when the true value

[3]For completeness, in the case of $R^{bidder} = X^*$ this representation should also specify bidders' bids when no value is disclosed.

[4]In the case where there is a solution $(R^{auc'}, R^{bidder'})$, where $R^{auc'} \neq R^{bidder'}$, which results in a benefit to the auctioneer that is identical to the one obtained with $R^{bidder''} = R^{auc'}$ the latter will be chosen as the bidders' strategy.

[5]The optimality of bidding the true valuation of the item ("strategy proofness") is a known characteristic of the second-price sealed-bid auction for the case of "deterministic" single item auction [60, 31, 40, 33]. Similarly, bidding the expected valuation is optimal in such auctions when the item's value is probabilistic as in our case [17].

is not received from the auctioneer. If the auctioneer benefits by deviating from R' to R^{auc}, then (R', R') is necessarily not in equilibrium, since it is not stable. Formally, a given solution (R', R') is a pure Perfect Bayesian Nash Equilibrium if $U(R^{auc}, R') \leq U(R', R'), \forall R^{auc} \neq R'$, where $U(R^{auc}, R')$ is the expected benefit of the auctioneer from using strategy R^{auc} while the bidders believe that she is using the strategy R' (and bid accordingly), calculated according to (8). It is notable that the set $\{R^{auc} | R^{auc} \neq R'\}$ is of size $2^k - 1$, thus the number of calculations required is exponential in k. While this computational aspect is beyond the scope of the current paper, we emphasize that in many domains the number of possible outcomes is moderate thus this is of less concern.[6]

As for the case where $R^{auc} = R^{bidder} = X^*$, a Perfect Bayesian Nash Equilibrium of this form requires some off-path beliefs that would result in optimal bids that are undesirable for the auctioneer, hence pushing her to full disclosure of values. A possible solution in this case is to use the off-path bidder beliefs that, with probability 1, the reached node is the one encoding the history in which the auctioneer purchased the information and found the common value to be $x_{min} = \text{argmin}_x \{u_{auc}(X = x)\}$. That is, all bidders think that, with probability 1, the undisclosed value is the one that minimizes the auctioneer's expected benefit upon disclosure. A bidder with type t should then bid $V_t(x_{min})$. Such off-path actions would not result in $R^{auc} = R^{bidder} = X^*$ always being a Perfect Bayesian Nash Equilibrium, as it does not guarantee that the auctioneer has no incentive to deviate to a strategy $R^{auc} = \emptyset$. Still, later on we prove that if there is no stable solution of the form (R', R'), where $R' \neq X^*$, then (X^*, X^*) with the above off-path bidder beliefs is necessarily a Perfect Bayesian Nash Equilibrium.

Table 1 illustrates the general solution space in the form of a bi-dimensional matrix, where the rows are the auctioneer's possible strategies and the columns are the bidders' beliefs. Each cell in the matrix contains the expected benefit of the auctioneer given her strategy (determined by the row) and the bidders' beliefs (determined by the column). The matrix is of size $2^k * 2^k$ since any possible value of X can be either disclosed or not disclosed. The first row, in which $R^{auc} = \emptyset$, is the only row where the cost C does not need to be subtracted from the auctioneer's expected benefit from the auction itself, because, as discussed earlier, if she chooses not to reveal any information, the dominating strategy for her is not to purchase the information in the first place (since this has the same outcome as

[6]For example, in oil drilling surveys, geologists usually specify 3-4 possible ranges for the amount of oil or gas that is likely to be found in a given area. Similarly, when requesting an estimate of the amount of traffic next to an advertising space, the answer would usually be in the form of ranges rather than exact numbers.

purchasing and not revealing any value, however the cost C is saved).

Bidders' Strategy → / Auc. Strategy ↓	$R_1^{bidder} = \emptyset$	...	...	$R_{2k}^{bidder} = \{v_1, v_2, ..., v_k\} = X^*$
$R_1^{auc} = \emptyset$	$u(R_1^{auc}, R_1^{bidder})$ ...	...	...	$u(R_1^{auc}, R_{2k}^{bidder})$
$R_2^{auc} = \{v_1\}$	$u(R_2^{auc}, R_1^{bidder}) - C$	...	...	$u(R_2^{auc}, R_{2k}^{bidder}) - C$
...	...	...	...	...
$R_{2k}^{auc} = \{v_1, v_2, ..., v_k\} = X^*$	$u(R_{2k}^{auc}, R_1^{bidder}) - C$	...	...	$u(X^*, X^*) - C$

Table 1: The model's solution space (rows represent the auctioneer's strategy and columns represent bidders' strategies, captured by their belief R^{bidder} as explained above). The term in each cell is the auctioneer's expected benefit.

The last row represents the case where the auctioneer always reveals the true outcome of X ($R^{auc} = X^*$). In this case, the auctioneer's expected benefit is the same regardless of the bidders' belief (i.e., $u(X^*, R_i) = u(X^*, R_j)\ \forall R_i, R_j$). This is because in this case the bidders will not be required to perform any probabilistic update and will always bid based on the value revealed, using (1). A solution (R^*, R^*) that yields the auctioneer an expected benefit greater than or equal to the expected benefit if switching to any other strategy R^{auc} is necessarily an equilibrium. The characterization of such a solution within the context of the matrix in Table 1 is a value on the diagonal that is greater than or equal to any other value in its column.

We note that it is possible to have more than a single equilibrium in our model. The research on multiple non-dominating equilibrium strategies in game and agents theory is quite rich and is beyond the scope of the current paper. Nonetheless, when considering the problem in its extensive form, each set R^{auc} defines a subgame whose equilibrium solution is associated with some expected benefit to the auctioneer. Therefore, since the auctioneer is the first to act in this game, the equilibrium that will be used is the one associated with the maximum expected benefit for the auctioneer.[7] For exposition purposes, in the remainder of the paper we will refer to all the equilibria as "stable solutions" and use the term "equilibrium" to refer only to the one which is preferred by the auctioneer.

Theorem 1. *If the solution (X^*, X^*), where bidders' off-path belief is that the common value is $x_{min} = \text{argmin}_x\{u_{auc}(X = x)\}$, is not stable, then the solution $(\emptyset, \emptyset)$ is necessarily stale.*

[7] And if there are several equilibria yielding the same highest expected benefit for the auctioneer then the one represented first (as a row) in Table 1 among them is used.

Proof. If the solution (X^*, X^*), in which bidders' off-path belief is that the common value is $x_{min} = \text{argmin}_x\{u_{auc}(X = x)\}$, is not stable then the auctioneer must have an incentive to deviate to some $R^{auc'}$. Deviating from $R^{auc} = X^*$ to $R^{auc'} \notin \{X^*, \emptyset\}$ cannot be a dominating strategy for the auctioneer in this case. This is because the expected benefit when disclosing any value $x_i \in R^{auc'}$ is the same as when disclosing it with $R^{auc} = X^*$, and for any value x_i not disclosed ($x_i \notin R^{auc'}$) the expected benefit is $\min\{u_{auc}(X = x)\} \leq u_{auc}(X = x_i)$. Therefore, the only plausible deviation is to $R^{auc} = \emptyset$ (for the benefit of saving the cost C). However if deviating to $R^{auc} = \emptyset$ is beneficial then the marginal benefit from purchasing the information compared to when not purchasing the information, given that the bidders bid according to $x_{min} = \text{argmin}_x\{u_{auc}(X = x)\}$ whenever no value is disclosed, is lesser than the cost C of purchasing the information.The expected benefit is given by $\sum_x u_{auc}(X = x)Pr(X = x)$, hence the marginal benefit from deviating to $R^{auc} = \emptyset$ is thus given by $\sum_x(u_{auc}(X = x) - \min_y\{u_{auc}(X = y)\})Pr(X = x)$, and according to the above must satisfy $\sum_x(u_{auc}(X = x) - \min_y\{u_{auc}(X = y)\})Pr(X = x) < C$. We now show that if the latter inequality holds then $(\emptyset, \emptyset)$ is necessarily a stable solution. When bidders use $R^{bidder} = \emptyset$ then the benefit for the auctioneer from deviating to any strategy $R^{auc} = X' \neq \emptyset$ is given by $\sum_{x \in X'}(u_{auc}(X = x) - u_{auc}(X = \emptyset))Pr(X = x) < \sum_x(u_{auc}(X = x) - \min_y\{u_{auc}(X = y)\})Pr(X = x) < C$. Therefore there is no incentive for the auctioneer to deviate from $(\emptyset, \emptyset)$.

□

The above theorem establishes the following corollary.

Corollary 1. *There is always a Perfect Bayesian Nash Equilibrium of the form* (R, R) *to the problem.*

Among the potential solutions to the problem, the solution $(\emptyset, \emptyset)$ is unique in the sense that if it is stable, then it is necessarily the equilibrium solution, i.e., there is no other stable solution (R, R) associated with a greater expected benefit to the auctioneer. This is proved in Theorem 2.

Theorem 2. *If the solution* $(\emptyset, \emptyset)$ *is stable then it is necessarily the equilibrium.*

Proof. The stability of the solution according to which the auctioneer does not purchase the information is achieved only when $u(\emptyset, \emptyset)$ is greater than or equal to any other element in the first column of the solution space matrix (see Table 1). We show that when this condition is satisfied, any other potential solution of the form (R, R) is either an unstable solution or offers an expected benefit lower than $u(\emptyset, \emptyset)$ to the auctioneer.

Consider a potential solution (R, R), where $R \neq \emptyset$. We distinguish between two cases. The first is $u(\emptyset, R) \leq u(\emptyset, \emptyset)$, i.e., when the second-best bid, when not disclosing any information, given the bidders' belief R, is less than or equal to the second best bid when no information is being disclosed and the bidders' belief is $R^{bidder} = \emptyset$. In this case, for any value $x \in R$ the second-best bid is the same in both cases (i.e., both when $R^{bidder} = R$ and when $R^{bidder} = \emptyset$) and for any value $x \notin R$ the auctioneer obtains $u(\emptyset, R)$ (if bidders use R) and $u(\emptyset, \emptyset)$ (if bidders use $\emptyset$).[8] Therefore, given that $u(\emptyset, R) \leq u(\emptyset, \emptyset)$, $u(R, \emptyset) \geq u(R, R)$ (because for any value $x \in R$ the auctioneer's expected benefit is the same for $(R, \emptyset)$ and (R, R), and for the value $x \notin R$ her expected benefit is greater for $(R, \emptyset)$ than for (R, R) because $u(\emptyset, R) \leq u(\emptyset, \emptyset)$). Consequently $U(R, \emptyset) = u(R, \emptyset) - C \geq u(R, R) - C = U(R, R)$. Since $(\emptyset, \emptyset)$ is stable, $U(\emptyset, \emptyset) \geq U(R, \emptyset) \geq U(R, R)$, and this solution is the equilibrium.

The second case is $u(\emptyset, R) > u(\emptyset, \emptyset)$, i.e., when the second-best bid when not disclosing any information, given the bidders' belief R, is greater than the second-best bid when no information is being disclosed and the bidders' belief is $R^{bidder} = \emptyset$. We note that in general for any bidders' belief R', the difference in the auctioneer's expected revenue from the auction (excluding the payment C) when deviating from $\emptyset$ to R'' is given by $u(R'', R') - u(\emptyset, R') = \sum_{x \in R''}(u_{auc}(x) - u_{auc}(\emptyset, R'))Pr(X = x)$ (as for any value $x \notin R''$ the bids in both cases are similar). Given that $u(\emptyset, R) > u(\emptyset, \emptyset)$, we obtain $u(R, \emptyset) - u(\emptyset, \emptyset) = \sum_{x \in R}(u_{auc}(x) - u_{auc}(\emptyset, \emptyset))Pr(X = x) > \sum_{x \in R}(u_{auc}(x) - u_{auc}(\emptyset, R))Pr(X = x) = u(R, R) - u(\emptyset, R)$ (because of subtracting a smaller value). Furthermore since $(\emptyset, \emptyset)$ is stable, we know that $u(R, \emptyset) - C \leq u(\emptyset, \emptyset)$. Therefore $u(R, R) - C < u(\emptyset, R)$, hence the solution (R, R) is not stable. $\square$

The implications of the latter proof are important for three reasons. First, this facilitates the study of the change in the equilibrium structure as a function of the cost C as discussed in the following paragraphs. Second, it enables immediate determination of whether or not the external information is purchased by the auctioneer for any given setting — the information is necessarily purchased if there exists R such that $U(R, \emptyset) > u(\emptyset, \emptyset)$, and not purchased otherwise. This is because there is always an equilibrium (according to Corollary 1) and unless $(\emptyset, \emptyset)$ is stable, the equilibrium is different than $(\emptyset, \emptyset)$. Finally, it suggests that at the point of switching from purchasing to not purchasing the information, the auctioneer's

[8] This is because given bidders' belief R^{bidder} the second-best bid when not disclosing the information is similar, regardless of the strategy used by the auctioneer. In particular, the second-best bid when not disclosing the information equals $u(\emptyset, R^{bidder})$ since in that case the auctioneer never discloses the information thus bidders always bid $B(t, \emptyset)$ according to R^{bidder}.

expected benefit necessarily does not decrease by switching to not purchasing the information (which is equivalent to not having the information provider in the market in the first place).

3.5. The Effect of the Information Cost

Based on the equilibrium analysis that was introduced in the former section, we now analyze the effect of the cost of obtaining the information (C) on the equilibrium structure. We begin with Proposition 1 that states that once a solution becomes unstable due to the cost of obtaining the information, it will remain unstable for any greater cost C.

Proposition 1. *If a solution (R, R), where $R \neq \emptyset$, is not stable for a cost C then it is necessarily not stable for $C' > C$.*

Proof. The solution (R, R) can become stable only if its value to the auctioneer becomes greater than or equal to the value from any other solution (R', R), i.e., if $U(R, R) \geq U(R', R)$ for any R'. However, according to (8) an increase in the value of C does not affect the value of the solution $(\emptyset, R)$, however it decreases the value of any other solution (R', R) equally. Therefore, if (R, R) was initially unstable, there was at least one solution in its column associated with a greater expected benefit to the auctioneer, and the increase in C will necessarily keep or even strengthen (in the case of $(\emptyset, R)$) the dominance of these other solution/s over (R, R) from the auctioneer's point of view. □

The immediate implication of Proposition 1 is that there is at most one cost C for which the equilibrium switches from (R, R) to (R', R') (where $R \neq R'$ and $R, R' \subseteq X^*$). Proposition 2 characterizes the interval of costs for which the equilibrium is $(\emptyset, \emptyset)$.

Proposition 2. *For any setting the solution $(\emptyset, \emptyset)$ is the equilibrium if and only if $C > C_{\emptyset} = max(u(R, \emptyset)|R \subseteq X^*) - u(\emptyset, \emptyset)$.*

Proof. This is straightforward from Theorem 2 — the solution $(\emptyset, \emptyset)$ is stable if its expected benefit to the auctioneer is greater than the one obtained from any other solution $(R, \emptyset)$, i.e., if $U(R, \emptyset) = u(R, \emptyset) - C \leq u(\emptyset, \emptyset)$ which is equivalent to the condition given in the proposition. Once the solution $(\emptyset, \emptyset)$ is stable, it is also the equilibrium according to Theorem 2. □

Based on Propositions 1-2 we can summarize the dependency of the equilibrium solution (and consequently the expected benefit of the auctioneer) in the cost of obtaining the information C. Since, as shown above, an equilibrium solution always exists, then for $C = 0$ the equilibrium is either:

- $(\emptyset, \emptyset)$ - in this case it remains the equilibrium for any $C' > 0$ according to Proposition 2.

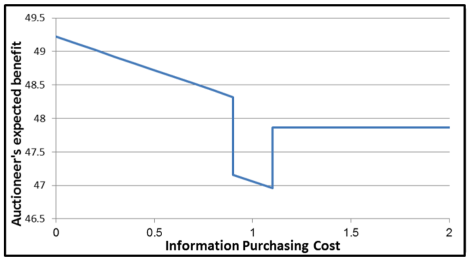
- (X^*, X^*) - in this case, as C increases, none of the solutions $R \notin \{\emptyset, X^*\}$ will become the equilibrium, and (X^*, X^*) will remain the equilibrium for any cost $C < C_\emptyset$ and change to $(\emptyset, \emptyset)$ for any cost $C > C_\emptyset$ (Theorem 2).

- (R, R) where $R \notin \{\emptyset, X^*\}$ - in this case, as the value of C increases the equilibrium may change to (R', R'), $R' \subset X^*$ $(R' \neq \emptyset)$, (and possibly again to other equilibria of that form, however will never return to an equilibrium that already ceased to exist for some lower C value, according to Proposition 1). Regardless of whether changes of the latter type occur, for some $C' > 0$ the equilibrium will become either: (a) $(\emptyset, \emptyset)$, in which case it remains the equilibrium for any $C'' > C'$ according to Proposition 2; or (b) (X^*, X^*), in which case it will remain the equilibrium for any cost $C < C_\emptyset$ and change to $(\emptyset, \emptyset)$ for any cost $C > C_\emptyset$, according to Proposition 2.

Proposition 3 characterizes the nature of the auctioneer's expected benefit as a function of C. It suggests that for the interval $(0, C_\emptyset)$, i.e., as long as $(\emptyset, \emptyset)$ is not the equilibrium solution, the auctioneer's expected benefit necessarily decreases as C increases. Then, at cost $C_\emptyset$ it increases by some constant $C' \geq 0$ (in a "phase transition"-like pattern) and remains at that value for any $C > C_\emptyset$.

Proposition 3. *If (R, R) is the equilibrium associated with a cost C, and (R', R') is the equilibrium associated with $C' > C$ then: (a) if $R = R' \neq \emptyset$ then $U(R, R) = U(R', R') + C' - C > U(R', R')$; (b) if $R \neq \emptyset$ and $R' \notin \{R, \emptyset\}$ then $U(R, R) > U(R', R')$; (c) if $C' = C_\emptyset$ and $C = C_\emptyset - \epsilon$ (where $\epsilon \to 0$) then $U(R', R') \geq U(R, R)$; and (d) if $R = R' = \emptyset$ then $U(R, R) = U(R', R')$.*

Proof. If $R = R' \neq \emptyset$ then according to (8) the only change in the expected benefit is the difference $C' - C$, thus proving part (a). In order to prove part (b) we show that any transition from equilibrium (R, R) to (R', R') where $R, R' \neq \emptyset$ can never result in an increase in the auctioneer's expected benefit. The transition from (R, R) to (R', R') can occur only if (R, R) is no longer stable (i.e., when $U(\emptyset, R) > U(R, R)$), in which case we necessarily switch to another equilibrium that is associated with an equal or lower expected benefit (as otherwise it would have been the equilibrium for lower C values). The proof for part (c) immediately derives from Theorem 2 (see discussion of the implications given right after that proof). Finally, part (d) holds since $U(\emptyset, \emptyset)$ does not depend on C. $\square$



n = 3

		private values			
		x1	x2	x3	x4
Type	p(type) \ p(value)	0.626	0.159	0.143	0.072
1	0.252	54	25	84	10
2	0.33	79	56	0	53
3	0.386	21	59	46	69
4	0.032	25	94	69	93

Figure 2: An example for a case where there are three different equilibria, each for a different interval of the information purchasing cost. The intervals are $[0, 0.93]$, $[0.93, 1.15]$ and $[1.15, \infty]$.

	{}	{1}	{2}	{3}	{4}	{1,2}	{1,3}	{1,4}	{2,3}	{2,4}	{3,4}	{1,2,3}	{1,2,4}	{1,3,4}	{2,3,4}	{1,2,3,4}
{}	47.86	46.32	47.95	47.62	48.13	45.7	52.32	47.01	47.67	48.29	47.91	52.31	41.19	52.33	48.05	41.17
{1}	47.98	47.4	48.01	47.89	48.08	47.17	49.65	47.66	47.91	48.14	48	49.65	45.48	49.65	48.05	45.49
{2}	48.57	47.27	48.65	48.37	48.8	46.75	52.32	47.86	48.41	48.93	48.62	52.32	42.97	52.33	48.73	42.94
{3}	46.91	45.58	46.98	46.7	47.14	45.05	50.73	46.18	46.75	47.27	46.95	50.72	41.19	50.74	47.07	41.17
{4}	48.18	46.75	48.27	47.96	48.43	46.17	52.32	47.4	48.01	48.58	48.23	52.31	41.99	52.33	48.36	41.97
{1,2}	48.69	48.36	48.71	48.64	48.75	48.23	49.65	48.51	48.65	48.78	48.7	49.65	47.26	49.65	48.73	47.27
{1,3}	47.03	46.67	47.05	46.97	47.09	46.53	48.06	46.83	46.98	47.12	47.04	48.06	45.48	48.06	47.07	45.49
{1,4}	48.3	47.83	48.33	48.23	48.38	47.65	49.65	48.04	48.24	48.43	48.32	49.65	46.28	49.65	48.36	46.3
{2,3}	47.62	46.54	47.68	47.45	47.81	46.11	50.73	47.03	47.49	47.92	47.66	50.73	42.97	50.74	47.75	42.94
{2,4}	48.89	47.71	48.96	48.71	49.1	47.23	52.32	48.24	48.75	49.22	48.94	52.32	43.77	52.33	49.04	43.74
{3,4}	47.23	46.02	47.3	47.04	47.44	45.53	50.73	46.56	47.08	47.56	47.27	50.72	41.99	50.73	47.38	41.97
{1,2,3}	47.74	47.63	47.74	47.72	47.76	47.58	48.06	47.68	47.72	47.77	47.74	48.06	47.26	48.06	47.75	47.27
{1,2,4}	49.01	48.79	49.03	48.98	49.05	48.7	49.65	48.9	48.99	49.07	49.02	49.65	48.06	49.65	49.04	48.06
{1,3,4}	47.35	47.1	47.36	47.31	47.39	47	48.06	47.21	47.32	47.41	47.36	48.06	46.28	48.06	47.38	46.3
{2,3,4}	47.94	46.97	47.99	47.79	48.11	46.59	50.73	47.4	47.82	48.21	47.97	50.73	43.77	50.73	48.06	43.74
{1,2,3,4}	48.06	48.06	48.06	48.06	48.06	48.06	48.06	48.06	48.06	48.06	48.06	48.06	48.06	48.06	48.06	48.06

Table 2: The auctioneer's expected benefit table for $C = 0$, using the setting of Figure 2.

Figure 2 depicts the typical behavior of the auctioneer's expected benefit as a function of the cost C as outlined above. The setting considers three agents of four possible types, and is fully described in the table to the right of the figure. The first column of the table depicts the different bidder types and the second column gives their probability. Similarly, the second and third rows depict the different possible values of X (denoted x_1,x_2,x_3 and x_4) and their probabilities. The remaining values are the values that bidders of different types assign different possible values of the parameter X. For example, if a bidder is of type 1, then her valuation of x_1 is 54. The auctioneer's expected benefit table for the different strategy combinations for the case $C = 0$ is given in Table 2. Here the notation $\{i, j, k\}$ is used to denote a strategy according to which the auctioneer discloses $\{x_i, x_j, x_k\}$. According to the table there are three stable solutions: $(\{2, 4\}, \{2, 4\})$, $(\{1, 2, 4\}, \{1, 2, 4\})$ and $(\{1, 2, 3, 4\}, \{1, 2, 3, 4\})$, where the one associated with the higher expected benefit to the auctioneer is $(\{2,4\},\{2,4\})$. Therefore the latter solution becomes the equilibrium and remains the equilibrium for any $C < 0.93$. For $C = 0.93$ that solution is no longer stable, as the auctioneer has an incentive to deviate to $(\emptyset, \{2,4\})$ which offers her a greater expected benefit. The two other solutions are, however, stable for $C = 0.93$ and thus one of them becomes the equilibrium at that point, until $C = 1.15$. When $C = 1.15$, these solutions are still stable, however the solution $(\emptyset, \emptyset)$ also becomes stable, and since the latter offers a greater expected benefit to the auctioneer it becomes the equilibrium. From this point on, the solution $(\emptyset, \emptyset)$ remains the equilibrium according to Proposition 2.

Since any equilibrium that holds in our model holds for a continuous interval of C values, the bidder's expected benefit as a function of C can be described as a step function, according to which the bidders' expected benefit changes only in C values for which the equilibrium changes. The expected social welfare, which is the sum of the auctioneer's and bidders' expected benefit, exhibits a behavior similar to the auctioneer's expected benefit except that with any equilibrium transition the overall sum can either increase or decrease by some constant C', due to the change in the bidders' expected benefit.

Algorithm 1 Calculating Equilibrium as a Function of the Cost C.

Input: M - Matrix of the auctioneer's expected benefit as in Table 1 for $C = 0$;
Output: $Solution$ - The set of equilibria and the interval of C values for which each equilibrium solution holds.

Set $C' = 0$; $Solution = \{\emptyset\}$;
Set $S = [R'_1, ..., R'_{n'}]$ such that: (a) $R'_i \notin \{\emptyset, X^*\}$; (b) $M[R'_i, R'_i] \geq M[R'_j, R'_i]$ $\forall R'_j \subseteq X^*$; (c) $M[R'_i, R'_i] \geq M[R'_j, R'_j]$ $\forall i < j$;
Set $C_\emptyset = max\{M[R, \emptyset] - M[\emptyset, \emptyset] | R \subseteq X^*\}$
while $S \neq \emptyset$ **do**
  Set $C^* = M[S[1], S[1]] - M[\emptyset, S[1]]$
  if $C^* > C_\emptyset$ **then**
    Add tuple $(C', C_\emptyset, S[1])$ to $Solution$;
    Add tuple $(C_\emptyset, \infty, \emptyset)$ to $Solution$;
    return $Solution$;
  end if
  Add tuple $(C', C^*, S[1])$ to $Solution$;
  Remove from S any element i for which $M[S[i], S[i]] - C^* < M[\emptyset, S[i]]$;
  Set $C' = C^*$;
end while
Add tuple $(C', C_\emptyset, X^*)$ to $Solution$;
Add $(C_\emptyset, \infty, \emptyset)$ to $Solution$;
return $Solution$

Based on the above analysis we introduce Algorithm 1, which enables efficient calculation of the pure equilibrium for all different possible C values. Algorithm 1 receives as an input the matrix M which is equivalent to Table 1 for $C = 0$ and outputs a set of intervals and their corresponding equilibrium strategy, stored as tuples in the vector $Solution$. Throughout the algorithm's execution, we use the vector S to store the set of solutions (other than $R = \emptyset$ and $R = X^*$) that are stable for a cost C'. The solutions in S are sorted according to their value to the auctioneer, therefore the first element in S is necessarily the equilibrium solution for $C = C'$. The initialization of S is given in Step 2 (where condition (a) is for precluding the solutions $R = \emptyset$ and $R = X^*$, condition (b) is for validating that the solution is stable (i.e., requires comparing each matrix element on the diagonal with its column members) and condition (c) is for the sorting of the solutions according to their expected benefit to the auctioneer).

Next, the algorithm executes its main loop (Steps 4-14). On each iteration, the algorithm finds the value C^* for which the first solution in S (which is the equilibrium for C') is no longer stable (Step 5). This can happen only if $M[\emptyset, S[1]]$ becomes greater than $M[S[1], S[1]]$ (as all other elements of the column $S[1]$ decrease similar to $M[S[1], S[1]]$ due to the change in the cost, except for $M[\emptyset, S[1]]$ which does not depend on C). The value of C^* is thus $M[S[1], S[1]] - M[\emptyset, S[1]]$. Yet, the fact that $S[1]$ becomes non-stable at cost C^* does not necessarily mean that this is the lowest cost for which it is no longer the equilibrium. It is possible that for some $C < C^*$ the solution $(\emptyset, \emptyset)$ is stable and in such case, according to Theorem 2, it offers an expected benefit to the auctioneer greater than $(S[1], S[1])$ for that cost. This latter case is handled in Steps 6-10. Here, if indeed $(\emptyset, \emptyset)$ becomes stable at a cost $C_\emptyset$ (calculated in Step 3, according to Proposition 2) lower than C^*, then according to Theorem 2 the solution $(\emptyset, \emptyset)$ becomes the equilibrium. Therefore there is no need to check the remaining elements of S. In that case the algorithm adds to $Solution$ the appropriate intervals for the equilibria $(S[1], S[1])$ and $(\emptyset, \emptyset)$ (Steps 7 and 8), terminating and returning $Solution$ right after (Step 9). Otherwise, if $C^* \leq C_\emptyset$ in the check made in Step 6, the interval (C', C^*) is added to $Solution$, with $S[1]$ as the equilibrium (Step 11), and all other solutions in S are re-checked for stability given the new C^* value that was found (Step 12). Then, at Step 13, the value of C' is set to C^* as it is now the beginning of the next interval to be added to $Solution$ (according to Proposition 3). The iterations in the main loop continue until either identifying the solution $(\emptyset, \emptyset)$ as a stable solution and terminating (Step 9), or when there are no further stable solutions for the current C' (i.e., S is empty). In the latter case, the solution $(\emptyset, \emptyset)$ is necessarily not stable for the current C' value (as otherwise the algorithm would have terminated in Step 9). Therefore, according to Corollary 1, in the interval $(C', C_\emptyset)$ the equilibrium is necessarily (X^*, X^*), and from $C_\emptyset$ and on $(\emptyset, \emptyset)$ is the equilibrium solution (according to Proposition 1). Therefore the algorithm adds these two intervals to $Solution$ (Steps 15-16) and outputs the vector $Solution$ (Step 17).

One important thing about Algorithm 1 is that it does not require any knowledge of the problem setting other than the matrix M.[9] Therefore, in order to provide a complete analysis of the equilibrium solutions and the resulting benefit of any of the players as a function of the cost C, one only needs to calculate once the expected benefit of the auctioneer for the $2^k \cdot 2^k$ possible strategy combinations. This capability provided by Algorithm 1 becomes of great importance whenever the auctioneer attempts to influence her expected benefit by paying the information provider to change the price charged for the information (e.g., see Section 5.3) and when the social planner considers the option for using taxation or subsidy (e.g., see Section 5.4).

3.6. Self-Interested Information-Provider

The analysis up to this point assumed that the cost of obtaining the information is exogenously pre-set by the information-provider. In the following paragraphs we discuss the implications of a self-interested information-provider. If the information-provider is self-interested then this becomes a three-way Stackleberg game: the information-provider first sets its price, then the self-interested auctioneer decides whether to buy the information, and if so, what to disclose, and finally the bidders respond. In this case the information-provider will set the highest cost C for which information is purchased.

Finding the highest cost C for which the information is still purchased is straightforward, based on the analysis given in 3.5 — the information-provider merely needs to set its cost as $C = C_\emptyset - \epsilon$, where $\epsilon \to 0$. Since there is always an equilibrium solution to the problem, then given Proposition 2, it is guaranteed that, according to the equilibrium, the information is purchased for cost $C_\emptyset - \epsilon$, and that for any cost $C \geq C_\emptyset$ the information is not purchased.

In some cases the self-interested information provider may choose to set a price different from $C_\emptyset - \epsilon$. For example, in cases where a fixed price needs to be set in a market where different auctions with varying settings continuously take place, or in cases where there is some uncertainty associated with the setting used (e.g., the number of bidders participating in the auction is unknown to the information-provider or is associated with some probability distribution). In these

[9]Indeed the calculation of M for $C = 0$ is setting-dependent, however once M is calculated and given to the algorithm, all calculations are setting-independent and apply to the matrix M in general.

cases the self-interested information-provider will need to calculate for each cost C the number of auctions in which her services will be purchased and choose the price associated with the maximum product.

3.7. Bidder's Unawareness of the Auctioneer's Option to Purchase Information

The analysis of the case where bidders are unaware of the auctioneer's option to purchase the information is an instance of the case where the auctioneer needs to calculate her expected-benefit-maximizing strategy given the bidders' belief R^{bidder}, as given in 3.2, for $R^{bidder} = \emptyset$. In this case, the auctioneer will simply choose the subset R^{auc} associated with the maximum value according to the first column in Table 1. Since all elements in the first column, except for $(\emptyset, \emptyset)$, decrease equally due to the increase in the value C, the auctioneer will keep on using this strategy for any setting with greater C values until her expected benefit is less than $u(\emptyset, \emptyset)$.

As for the dependency of the auctioneer's expected benefit in C for the case in which the bidders are unaware of the auctioneer's option of purchasing the information, here the only possible switch is from $R^{auc} \neq \emptyset$ to $R^{auc} = \emptyset$ (though this derives only from the auctioneer's optimization considerations rather than stability considerations). This is because once the auctioneer determines the strategy R^{auc} that yields her the maximum expected benefit, the expected benefit from this strategy and from any other strategy $R \neq \emptyset$ decreases equally due to an increase in the value of C. Therefore, the only way that the choice of a preferred subset-to-be-disclosed can change is when $R^{auc} = \emptyset$ becomes associated with the highest value in its column in Table 1. Furthermore, the expected benefit of the auctioneer at the cost of transition does not change (since otherwise the auctioneer would have kept the $R^{auc} \neq \emptyset$ strategy), and remains constant as C further increases (as the expected benefit with $R^{auc} = \emptyset$ does not depend on C).

Interestingly, despite what might seem like an advantage for the auctioneer in having the bidders unaware of her option to purchase the information, the cost for which the information is no longer purchased is the same both with and without the bidders' awareness.

Proposition 4. *The cost C for which the auctioneer switches to not purchasing the information in the model variant where bidders are aware of the option to purchase the information is equal to the cost for which she switches to not purchasing the information in the model variant where bidders are unaware of that option.*

Proof. The stability of the solution according to which the auctioneer does not purchase the information in the model with aware bidders is achieved only when $u(\emptyset, \emptyset)$ is greater than or equal to any other element in the first column of the solution space matrix (see Table 1). This is also, however, the condition for switching to not purchasing the information in the model with unaware bidders. According to Theorem 2, if $(\emptyset, \emptyset)$ is a stable solution then it is necessarily also the equilibrium. $\square$

3.8. Homogeneous Bidders

The benefit for the auctioneer in selectively disclosing information to the bidders becomes mostly apparent when bidders are heterogeneous. This is because when bidders are heterogeneous the auctioneer can take advantage of the differences in their valuations of the information she discloses. Yet, in some cases bidders can be quite homogeneous. Lemma 1 relates to the case of fully homogeneous bidders, revealing an interesting characterization of the solution space illustrated in Table 1 that concerns this case.

Lemma 1. *If the bidders are all of the same type (i.e., homogeneous in their private valuations) then all solutions of type (R^{auc}, R^{auc}) where $R^{auc} \neq \emptyset$ will yield the same expected benefit to the auctioneer, regardless of the values being disclosed in each of them.*

Proof. Since the agents are of the same type t, they all place the same bid, which is necessarily also the second-best bid. Therefore the expected benefit of the auctioneer in equilibrium when the set R is used is given by $U(R,R) = \sum_{x \in R} Pr(X=x) \cdot V_t(x) + B(t,\emptyset) \cdot \sum_{x \notin R} Pr(X=x) - C$. Substituting (3) in (2) obtains: $B(t,\emptyset) = \sum_{x \notin R} V_t(x) \cdot \frac{Pr(X=x)}{\sum_{y \notin R} Pr(X=y)}$. Substituting the latter term in $U(R,R)$ above obtains: $U(R,R) = \sum_{x \in R} Pr(X=x) \cdot V_t(x) + \sum_{x \notin R} Pr(X=x) \cdot V_t(x) - C = \sum Pr(X=x) \cdot V_t(x) - C$, which means that regardless of the equilibrium that will hold (other than $(\emptyset, \emptyset)$) the expected benefit for the auctioneer is similar. $\square$

Furthermore, Proposition 5 reveals an interesting property of the model when bidders are homogeneous, according to which the auctioneer necessarily loses due to the existence of a self-interested information-provider.

Proposition 5. *When bidders are homogeneous in their private valuations, then if there is a price C for which the equilibrium is (R,R), where $R \neq \emptyset$, then the auctioneer necessarily loses from having a self-interested information-provider in the market.*

Proof. The proof is based on Lemma 1: For $C = 0$ all potential equilibrium solutions yield the same expected benefit to the auctioneer as $R = \emptyset$ and less for any $C > 0$. Therefore, if there is a value C' for which $R \neq \emptyset$ is the equilibrium, then the self-interested information-provider necessarily charges $C' > 0$ such that the equilibrium is $R \neq \emptyset$ and the auctioneer's expected benefit is worse than

$u(\emptyset, \emptyset)$ (which is equivalent to not having the information-provider's option in the first place). □

3.9. The Case of a Noisy Information Provider

In some cases, the information provider is incapable of providing the exact value x, but instead can provide a more accurate distribution of outcomes. While the bidding strategy of bidders in this case slightly changes, as detailed in the following paragraph, the remaining analysis, as given in this section, holds.

Assume that instead of providing the exact value x, the information provider provides some distribution D_j such that $Pr_{D_j}(X = x)$ is the revised probability of x ($\sum Pr_{D_j}(X = x) = 1$). We assume that the information provider's distribution is more accurate than the a priori probability $Pr(X = x)$, hence overriding it if obtained as the information provider's output. We use $D = \{D_1, ..., D_{|D|}\}$ to denote the set of possible revised distributions the information provider will possibly provide, and $Pr(D = D_j)$ to denote the probability of each distribution D_j being the one returned by the information provider ($\sum Pr(D = D_j) = 1$). Both the set D and the probabilities assigned to outputting its different elements derive from the a priori probability $Pr(X = x)$ and the information provider's capabilities to distinguish between the different values the common value may obtain (capabilities which are assumed to be known to the auctioneer and bidders). Since information disclosure is assumed to be truthful, then if disclosing the information the auctioneer will disclose the distribution D_j obtained from the information provider as-is. The auctioneer's strategy R^{auc} and the set R^{bidder} used for the bidders' strategy thus include a subset of D rather than discrete values.

In this case, if a new distribution D_j is disclosed then, for the same analysis principles given above, regardless of the bidders' belief, a type t bidder's bid will be $B(t, D_j) = \sum V_t(x) \cdot Pr_{D_j}(X = x)$. Similarly, if no information is disclosed then: (a) if $R^{bidder} = \emptyset$ then a type t bidder's bid will be $B(t, \emptyset) = \sum V_t(x) \cdot Pr(X = x)$; and (b) Otherwise, i.e., if $R^{bidder} \neq \emptyset$, then $B(t, \emptyset) = \sum_{x \notin R^{bidder}} V_t(x) \cdot Pr^*(X = x)$, where:

$$Pr^*(X = x) = \frac{\sum\limits_{D_j \notin R^{bidder}} Pr(D = D_j) Pr_{D_j}(X = x)}{\sum\limits_{D_j \notin R^{bidder}} Pr(D = D_j)} \tag{9}$$

The rest of the analysis remains unchanged, however the bi-dimensional matrix that is used for the analysis is of size $2^{|D|}$ rather than 2^k as before, i.e., each row/column relates to a different subset of D.

4. Using Mixed Strategies

In this section we extend the equilibrium analysis to the case of using mixed strategies. The nature of the mixed-strategy perfect Bayesian Nash equilibrium in our model is necessarily such that while the auctioneer randomizes between the different pure strategies the bidders keep using a pure strategy. This is due to the fact that if the auctioneer discloses the value x of X then the dominating strategy for each bidder is to bid her expected private value, as before, i.e.,

$$B(t, x) = V_t(x) \tag{10}$$

Similarly, when a value is not disclosed, the dominating strategy for the bidders is to bid their expected individual private value, based on their belief of the strategy used by the auctioneer (and consequently the probability they assign to each value $x \in X$ when information is not disclosed). Therefore, an auctioneer's strategy can be represented as $R^{auc} = (p, p_1, ..., p_k)$, where p is the probability that the information indeed will be purchased and p_i $(1 \leq i \leq k)$ is the probability that the value x_i will be disclosed if indeed the information is purchased and this value turns out to be the true common value.

In this case, if the auctioneer does not disclose any value, the probabilistic update, according to which the new posterior probability $Pr^*(X = x)$ is calculated for any value x_i being the true common value, given the bidder's belief $R^{bidder} = (p, p_1, ..., p_k)$, is given by:

$$Pr^*(X = x_i) = \frac{Pr(X = x_i)(p(1 - p_i) + (1 - p))}{(1 - p) + p \sum (1 - p_i) Pr(X = x_i)} \tag{11}$$

The term in the numerator is the probability that x_i indeed will be the true value and will not be disclosed. If indeed x_i is the true value (i.e., with a probability of $Pr(X = x_i)$) then it will not be disclosed either if the information is not purchased (i.e., with a probability of $(1 - p)$) or if purchased but not disclosed (i.e., with a probability of $p(1 - p_i)$). The term in the denominator is the overall probability that the information will not be disclosed. This can happen either if the information will not be purchased (i.e., with a probability of $(1 - p)$) or when the information will be purchased however the value will not be disclosed (i.e., with probability of $p \sum (1 - p_i) Pr(X = x_i)$).

The bid placed by a bidder of type t in this case, $B(t, \emptyset)$, equals her expected private value as before, i.e.:

$$B(t, \emptyset) = \sum V_t(x) \cdot Pr^*(X = x_i) \tag{12}$$

Consequently, if the strategy of the auctioneer is R^{auc}, and the bidders' strategy is to bid according to R^{bidder}, then the auctioneer's expected benefit from the auction itself (i.e., excluding the payment C), $u(R^{auc}, R^{bidder})$, is:

$$\begin{aligned} u(R^{auc}, R^{bidder}) &= p \sum Pr(X = x_i) p_i \cdot u_{auc}(x) \\ &+ ((1-p) + p \sum (1-p_i) Pr(X = x_i)) \cdot u_{auc}(\emptyset) \end{aligned} \tag{13}$$

where $u_{auc}(x)$ is calculated according to (6). Consequently $U(R^{auc}, R^{bidder}) = u(R^{auc}, R^{bidder}) - p * C$.

A stable solution in this case is in the form of $R^{auc} = R^{bidder} = (p, p_1, ..., p_k)$, as bidding according to $R^{bidder} = c$ dominates bidding according to any other $R^{bidder'}$ if the auctioneer uses R^{auc} (as shown in 3.4 for the pure strategies equilibrium). Similarly, the stability conditions for a solution $(p, p_1, ..., p_k)$ require equal expected benefit for the auctioneer if disclosing or not disclosing certain information (or purchasing or not purchasing the information in the first place), whenever a mixed strategy is used, or greater expected benefit for a pure strategy used, i.e.: (a) for any $0 < p_i < 1$ (or $0 < p < 1$): $u_{auc}(\emptyset, R^{bidder}) = u_{auc}(x_i)$ (or $u_{auc}(\emptyset, R^{bidder}) = u_{auc}((1, p_1, ..., p_k), R^{bidder})$); (b) for any $p_i = 0$ (or $p = 0$): $u_{auc}(\emptyset, R^{bidder}) \geq u_{auc}(x_i)$ (or $u_{auc}(\emptyset, R^{bidder}) \geq u_{auc}((1, p_1, ..., p_k), R^{bidder})$; and (c) for any $p_i = 1$ (or $p = 1$): $u_{auc}(\emptyset, R^{bidder}) \leq u_{auc}(x_i)$ (or $u_{auc}(\emptyset, R^{bidder}) \leq u_{auc}((1, p_1, ..., p_k), R^{bidder})$. Therefore, when considering also mixed equilibria, one needs to evaluate all the possible solutions of the form $(p, p_1, ..., p_k)$ that may hold (where each probability is either assigned 1, 0 or a value in-between). Each mixed solution of these $2 \cdot 3^k$ combinations (as only one solution where $p = 0$ is applicable) should be first solved for the appropriate probabilities according to the above stability conditions, and included in the matrix (Table 1) only if such a solution indeed exists.

It is notable that an equilibrium of the form $(p, p_1, ..., p_k)$ where more than one of the probabilities in the set obtains a value that is not 1 or 0, is very rare, as it requires that there are at least two values x_i and x_j for which $u_{auc}(x_i) = u_{auc}(x_j) = u_{auc}(\emptyset, R^{bidder})$. Namely, there are at least two values which the common value may obtain, for which the expected second best bid, if these two values are disclosed, is equal.[10] Therefore, a mixed equilibrium in our case will typically be one where the auctioneer randomizes either between purchasing or not purchasing the information in the first place (while using a pure strategy for disclosing

[10] And similarly, for $0 < p < 1$ and $0 < p_i < 1$ we require $u_{auc}(x_i) = u_{auc}((1, p_1, ..., p_k), R^{bidder}) = u_{auc}(\emptyset, R^{bidder})$.

values if information is purchased) or one where information is purchased and the auctioneer randomizes between disclosing and not disclosing one of the values.

An example where a mixed equilibrium holds is illustrated in Figure 3. The figure depicts the auctioneer's expected benefit as a function of the information cost, for the setting given in the following table:

	$n = 3$	Private Values					
		x_1	x_2	x_3	x_4	x_5	x_6
	Value's Prob / Type's Prob	0.166	0.166	0.166	0.166	0.166	0.167
Type 1	0.333	100	100	0	100	0	0
Type 2	0.333	100	0	100	0	100	0
Type 3	0.334	0	100	100	0	0	100

A mixed equilibrium in this case holds for all information purchasing costs for which information is not purchased according to the pure equilibrium (up to a certain limit). The nature of this mixed equilibrium is to have $0 < p < 1$ and $p_i \in \{0, 1\}$ for any i, i.e., when the auctioneer randomizes between purchasing and not purchasing the information, however if purchasing, uses a pure strategy. For example, when the information purchasing cost is $C = 14$, the mixed equilibrium is to purchase the information with a probability of $p = 0.38$ and reveal the value if it turns out to be x_1, x_2 and x_3, i.e., $R^{auc} = R^{bidder} = (0.38, 1, 1, 1, 0, 0, 0)$. The auctioneer's expected benefit from this solution is 46.07, compared to 50 with the pure equilibrium solution. Overall, the figure shows that the mixed equilibrium, in this case, is dominated by the pure equilibrium of not purchasing the information for all values of C where a mixed equilibrium holds.

We note that other than extending the number of strategies that should be considered in the matrix, as detailed above, the entire analysis given for the properties of the equilibrium given some cost C still hold (e.g., Theorem 1 and Theorem 2). The only parts of the analysis that need to be further extended when considering also mixed equilibria are those related to Algorithm 1 (i.e., Proposition 1, Proposition 2, and Proposition 3). This is mainly because the algorithm relies on the property whereby the expected benefit of the auctioneer decreases by the same amount that C increases as long as the same equilibrium holds. With a mixed equilibrium, the probability the information is purchased changes as the cost of purchasing the information changes. Consequently, in the examples given in the following section, we had to calculate the mixed equilibria for any value C, rather than use Algorithm 1.

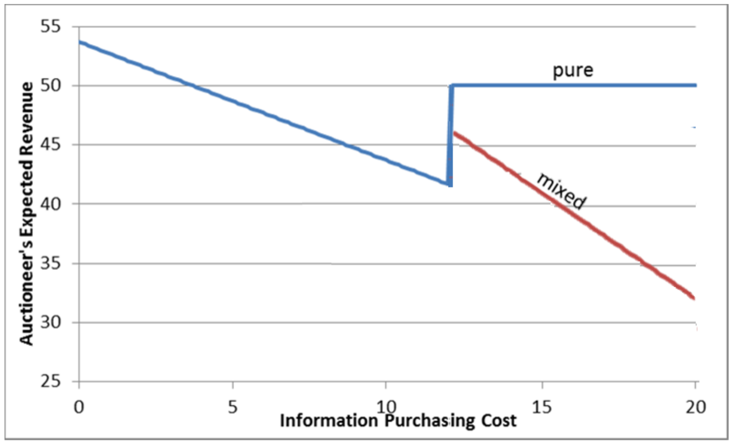


Figure 3: The auctioneer's expected benefit as a function of the information purchasing cost, considering both mixed and pure solutions.

5. Numerical Illustrations and Market Design Implications

The analysis given in the previous sections enables the demonstration of the effect of different model parameters (the number of bidders, the cost of purchasing the information and the bidders' private values) over the resulting equilibrium in terms of the equilibrium structure, the different players' expected benefit and the social welfare in our hybrid setting. In addition, we demonstrate the possible effects of having the option to purchase the information externally and of the bidders' awareness of that option over the auctioneer's and bidders' expected benefit and the social welfare. The results obtained in some of the cases differ from the ones known for traditional auction settings that do not consider the option for purchasing the information and selectively disclosing it. Finally, we demonstrate the effectiveness of market intervention in the form of taxation and subsidy. Since the goal of this section is primarily illustrative, it uses abstract synthetic settings where different bidder types are arbitrarily assigned their private value for any common value. In general, and unless otherwise stated, all of the examples used in this section are based on three types of bidders, with an equal probability and three different possible common values (to which each bidder type assigns a different private value). It is notable that the fact that there are three possible bidder types does not constrain the number of bidders in the tested settings in any way, and this latter parameter can obtain any (integer) value greater than 1.

5.1. The Effect of the Different Model Parameters

The three main parameters that define the problem setting in our model are the number of bidders, the cost of obtaining the information and bidders' valuations based on the values the common value obtains. Figure 4 illustrates that any variation in these parameters can result in a different equilibrium. It depicts the auctioneer's expected utility as a function of her information disclosing strategy (R^{auc}) and the bidders' belief (R^{bidder}) by which they place their bids. The archetypal setting used for this figure is described in the following table:

		Private Values	
		x_1	x_2
	Value's Prob / Type's Prob	0.5	0.5
Type 1	0.5	3^{α}	4^{α}
Type 2	0.5	$3^{2\alpha}$	$4^{2\alpha}$

The table is similar in its structure to the one used in Figure 2. In this example there are two possible types of bidders and each agent can be of each type with an equal probability. Similarly, there are two different possible values of X and their probabilities are equal. The remaining values in the table are the private values that agents of different types assign to the different possible values of the parameter X (i.e., the common value). For example, if a bidder is of type 1, then her valuation of having the common value be x_1 is 3^{α}. The parameter α is thus used to control bidders' valuations in order to generate a wide range of setting variants.

Each of the upper small tables in Figure 4 represents a different setting, differing from the other settings in its column by either C or n (or both) and from the settings in its row by the bidders' private valuations of the different possible outcomes. The stable solutions in each setting are presented with a different background color. As depicted in the figure, for some settings there is more than a single stable solution, and the equilibrium is the one that yields the highest benefit for the auctioneer. The tables provide only pure-strategy solutions since in all settings, except one, no mixed stable solution was found. The only setting where a mixed stable solution was found is where $\alpha = 1$, $n = 3$, and $C = 1.75$. The mixed stable solution in this case is $R^{auc} = R^{bidder} = (0.86, 0, 1)$ and its expected benefit for the auctioneer is 6.5 (i.e., dominated by the pure stable solution). The table at the bottom of the figure depicts the auctioneer's expected benefit, the bidders' expected individual benefit and the social welfare in the pure-strategy stable solutions found for each of the settings (other than the (X^*, X^*) stable solutions, as the auctioneer and bidders' gain (and consequently the social welfare) in this

n=3 , C=1.75, α = 1.3

Auctioneer \ Bidders	{ }	{1}	{2}	{1,2}
{ }	16.1	21.4	10.8	10.8
{1}	11.7	14.3	9	9.03
{2}	17	19.7	14.3	14
{1,2}	14.3	14.3	14.3	14.3

n=3 , C=1.75, α = 1

Auctioneer \ Bidders	{ }	{1}	{2}	{1,2}
{ }	8	10	6	6
{1}	5.3	6.3	4.3	4.25
{2}	7.3	8.3	6.3	5.9
{1,2}	6.3	6.3	6.3	6.3

n=3 , C=1.75, α = 0.1

Auctioneer \ Bidders	{ }	{1}	{2}	{1,2}
{ }	1.2	1.2	1.1	1.1
{1}	-0.55	-0.54	-0.56	-0.56
{2}	-0.52	-0.51	-0.54	-0.57
{1,2}	-0.54	-0.54	-0.54	-0.54

n=3 , C = 0.6, α = 1.3

Auctioneer \ Bidders	{ }	{1}	{2}	{1,2}
{ }	16.1	21.4	10.8	10.8
{1}	12.8	15.5	10.2	10.2
{2}	18.2	20.8	15.5	15
{1,2}	15.5	15.5	15.5	15.5

n=3 , C = 0.6, α = 1

Auctioneer \ Bidders	{ }	{1}	{2}	{1,2}
{ }	8	10	6	6
{1}	6.4	7.4	5.4	5.4
{2}	8.4	9.4	7.4	7.1
{1,2}	7.4	7.4	7.4	7.4

n=3 , C = 0.6, α = 0.1

Auctioneer \ Bidders	{ }	{1}	{2}	{1,2}
{ }	1.2	1.2	1.2	1.18
{1}	0.59	0.6	0.58	0.58
{2}	0.62	0.63	0.61	0.6
{1,2}	0.61	0.61	0.61	0.61

n=30 , C =1.75, α = 1.3

Auctioneer \ Bidders	{ }	{1}	{2}	{1,2}
{ }	27.1	36.8	17.4	17.4
{1}	20.5	25.3	15.6	15.6
{2}	30.2	35	25.3	23.3
{1,2}	25.3	25.3	25.3	25.3

n=30 , C =1.75, α = 1

Auctioneer \ Bidders	{ }	{1}	{2}	{1,2}
{ }	12.5	16	9	9
{1}	9	10.8	7.3	7.25
{2}	12.5	14.3	10.8	10
{1,2}	10.8	10.8	10.8	10.8

n=30 , C =1.75, α = 0.1

Auctioneer \ Bidders	{ }	{1}	{2}	{1,2}
{ }	1.3	1.3	1.2	1.2
{1}	-0.5	-0.47	-0.5	-0.53
{2}	-0.45	-0.43	-0.47	-0.6
{1,2}	-0.47	-0.47	-0.47	-0.47

n=30 , C = 0.6, α = 1.3

Auctioneer \ Bidders	{ }	{1}	{2}	{1,2}
{ }	27.1	36.8	17.4	17.4
{1}	21.6	26.5	16.8	16.8
{2}	31.3	36.2	26.5	24.6
{1,2}	26.5	26.5	26.5	26.5

n=30 , C = 0.6, α = 1

Auctioneer \ Bidders	{ }	{1}	{2}	{1,2}
{ }	12.5	16	9	9
{1}	10.1	11.9	8.4	8.4
{2}	13.6	15.4	11.9	11.6
{1,2}	11.9	11.9	11.9	11.9

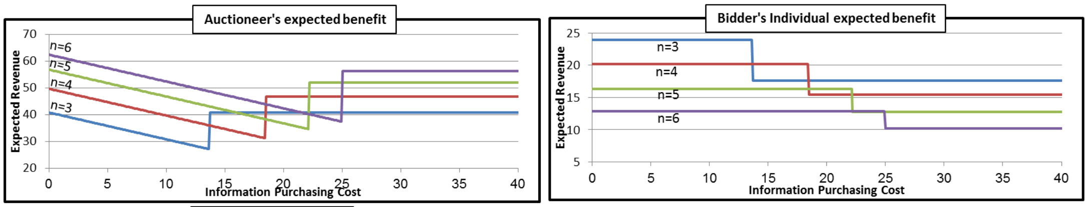
n=30 , C = 0.6, α = 0.1

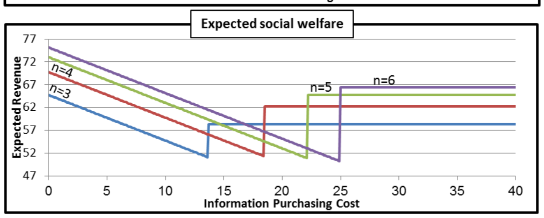
Auctioneer \ Bidders	{ }	{1}	{2}	{1,2}
{ }	1.28	1.32	1.25	1.25
{1}	0.66	0.68	0.65	0.65
{2}	0.7	0.72	0.68	0.68
{1,2}	0.68	0.68	0.68	0.68

n	C	α	Stable Strategy	Auc Gain	Overall Bidders Gain	Social Welfare
3	1.75	0.1	{ } (equilibrium)	1.2	0.056	1.26
3	1.75	1	{ } (equilibrium)	8	3.4	11.4
			{2}	6.3	3.4	9.6
3	1.75	1.3	{2} (equilibrium)	14.3	8.2	22.6
3	0.6	0.1	{ } (equilibrium)	1.2	0.056	1.26
3	0.6	1	{2} (equilibrium)	7.4	3.4	10.8
3	0.6	1.3	{2} (equilibrium)	15.5	8.2	23.7
30	1.75	0.1	{ } (equilibrium)	1.3	0	1.3
30	1.75	1	{ } (equilibrium)	12.5	0	12.5
			{2}	10.8	0	10.8
30	1.75	1.3	{2} (equilibrium)	25.3	$1*10^{(-6)}$	25.3
30	0.6	0.1	{ } (equilibrium)	1.28	0	1.28
30	0.6	1	{2} (equilibrium)	11.9	0	11.9
30	0.6	1.3	{2} (equilibrium)	26.5	$1*10^{(-6)}$	26.5

Figure 4: The auctioneer's expected benefit for each set of strategies (R^{auc}, R^{bidder}) (in the upper small tables where the stable solutions are presented with a different background color) and the expected benefit of the different players, and social welfare, in stable solutions (bottom table) for different values of the model parameters (number of agents, cost of obtaining the information and valuation of outcomes). The setting includes two agent types (of equal probability) and two possible outcomes (with equal probability), where α controls bidders' valuations as described in the text.

example is the same for $\{1, 2\}$ as for $\{2\}$). As illustrated in the figure, both the equilibrium strategies and the different players' individual performance in equilibrium are highly affected by the choice of the setting used.

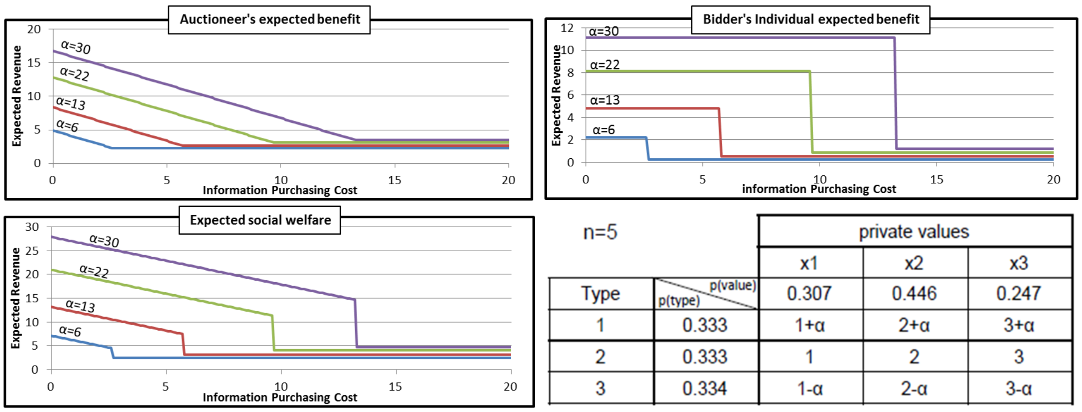




		private values		
		x1	x2	x3
Type	p(value) / p(type)	0.438	0.246	0.316
1	0.333	8	68	35
2	0.333	16	86	3
3	0.334	6	65	162

Figure 5: Auctioneer's and bidders' expected benefit and the social welfare as a function of the information cost, for different numbers of bidders.

To further illustrate the effect of the three parameters over the different players' expected benefit we introduce Figures 5 and 6. Figure 5 depicts the equilibrium's expected benefit for the auctioneer and bidders and the expected social welfare as a function of the cost of the external information, for different numbers of bidders. The setting used for this figure is given in the right bottom part of the figure. The behaviors exhibited in the figure are correlated with Proposition 3 and the analysis given in 3.5 in general. Taking the case where $N = 5$ as an example, the equilibrium strategy is to have the auctioneer purchase the external information if its cost is lower than 22.5 and disclose the true outcome only if it belongs to the set $\{x_2, x_3\}$ (which does not change, based on Proposition 1). Since the expected second-best bid if the set $\{x_2, x_3\}$ is revealed does not depend on the amount paid for purchasing the information, the decrease in the auctioneer's expected benefit curve equals the change in C. For any cost greater than $C = 22.5$, the auctioneer will avoid purchasing the information, hence her expected benefit is fixed. The information-provider, if self-interested in this case, will set the cost of information to be the maximum value in which the information is still purchased by the auctioneer (22.5 for $N = 5$). The bidder's expected benefit curves are fixed as long as the same equilibrium is used, as bidders are not the ones who pay for the information, and change at the points of transition between equilibria. The social welfare curves are the composition of the auctioneer's and bidders' expected benefit and therefore exhibit a decrease in value that is equal to the increase in C, with sharp changes in the value whenever shifting between equilibria. Alternatively, if one chooses to include the information provider's benefits in the social welfare measure, then the social welfare curve would keep the same value obtained at the transition cost as long as the same equilibrium holds. We emphasize that all the qualitative results related to auctioneer's and bidders' preferences given in this section are not affected by the choice of the social welfare measure.



n=5		private values		
		x1	x2	x3
Type	p(type) \ p(value)	0.307	0.446	0.247
1	0.333	1+α	2+α	3+α
2	0.333	1	2	3
3	0.334	1-α	2-α	3-α

Figure 6: Auctioneer's and bidders' expected benefit and the social welfare as a function of the information cost, for different levels of the bidders' heterogeneity.

It is notable that the cost threshold whereby the auctioneer no longer purchases the information in equilibrium increases as the number of bidders increases. The intuitive explanation for this phenomenon is as follows: the benefit of disclosing the information is in having the bidders bid their true private value. When the number of bidders is relatively small, however, even if a bidder of the type that most values the auctioned item bids its true value, based on the disclosed common value, it is possible that the second-best bid will be relatively low and consequently the auctioneer will not benefit enough from revealing the information. As the number of agents increases, it is more probable that at least two agents of types that assign a relatively high private value to the disclosed common value will take part in the auction, and therefore the cost that the auctioneer will be willing to pay for the information will increase. Overall, we observe that the increase in the number of bidders results in an increase in the auctioneer's expected benefit and a decrease in the bidders' expected benefit (for any number of agents $n_1 > n_2$ and cost C for which the same decision of whether or not to purchase the information is made). This is explained, once again, by the effect of the increase in the number of bidders on the probability that at least two agents of types that assign a relatively high private value to the disclosed common value will take part in the auction. A similar effect is observed with the expected social welfare, as this measure represents the efficiency of the item's allocation — the more bidders available in the auction, the greater the winner's valuation of the item is likely to be.

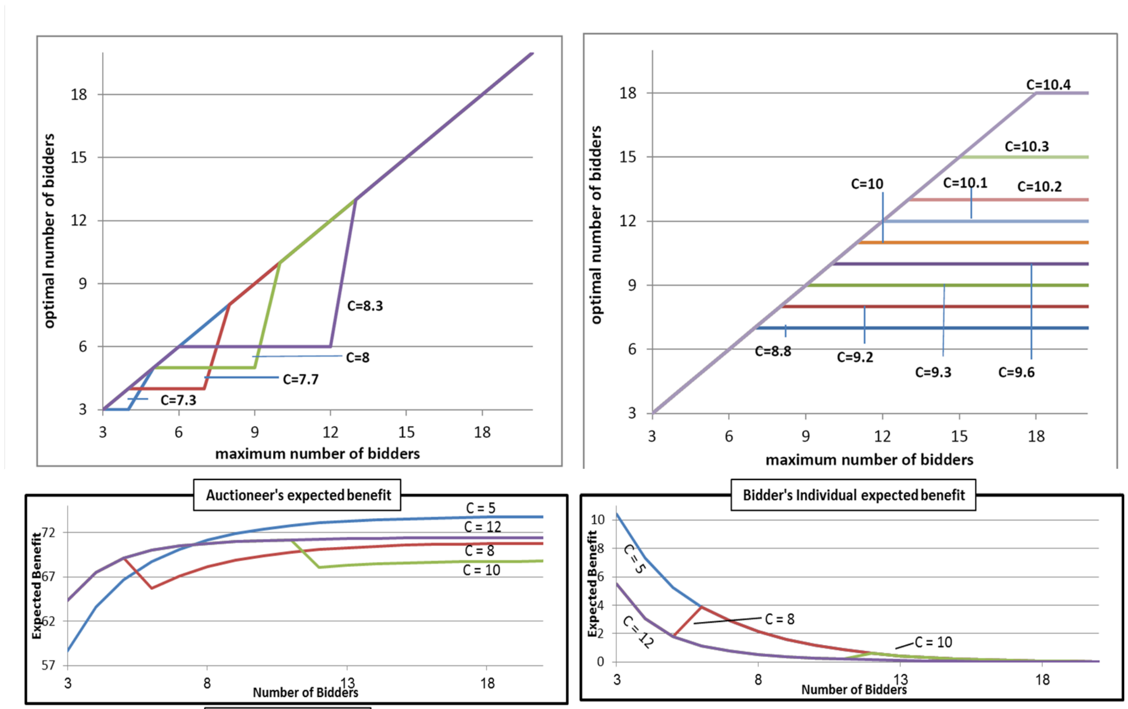
Figure 6 offers a similar analysis, emphasizing the effect of the level of heterogeneity between the bidders in the environment over the expected benefit of the different players in equilibrium. The setting used for this figure is based on 5 agents ($n = 5$) and its description of types and private values is given in the right bottom part of the figure. In this figure, the value of α controls the level of similarity between the different types of bidders (similar to the use in the setting on which Figure 4 was based). For $\alpha = 0$ the bidders are completely homogeneous, and as the value of α increases, the types become increasingly heterogeneous. As depicted in Figure 6, as the level of the heterogeneity increases, the cost at which the auctioneer switches to not purchasing the information in equilibrium increases and the expected auctioneer's benefit increases (for the same cost of purchasing the information, for those segments where the auctioneer either purchases or does not purchase the information in both cases). This is explained by the fact that the more heterogeneous the bidders, the more probable it is for each of the possible common values to result in a greater expected second-best bid if disclosed. A similar behavior is observed with the social welfare and the bidders' expected benefit. The first is explained by the fact that when the level of heterogeneity between the agents increases, the valuation of the bidder that values the item most also increases. The second is explained by the fact that the increase in the valuation of the bidder that values the item most is greater than the increase in the valuation of the one with the second highest valuation.

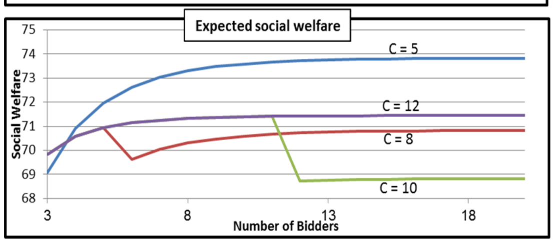
5.2. The Optimal Number of Bidders

The fact that the cost threshold by which the auctioneer no longer purchases the information in equilibrium increases as the number of bidders grows results in some non-intuitive model behavior, which relates to the auctioneer's preferred choice of the number of bidders participating in the auction. Indeed, if a similar decision is made for both n_1 and n_2, where $n_2 > n_1$, regarding purchasing or not purchasing the information then the auctioneer's expected benefit is necessarily greater with n_2. This is because any increase in the number of bidders increases the probability of having the second-best bid be of a type with a greater valuation of the auctioned item or service (hence a higher second-best bid as reflected in Figure 5). Nevertheless, since the cost at which the transition between purchasing the information to not purchasing it also increases as n increases, it is possible that for some costs, with n_1 bidders the information is not purchased in equilibrium and with $n_2 > n_1$ the information is purchased in equilibrium, yielding to the auctioneer an expected benefit lower than the one obtained for n_1. This situation is demonstrated in Figure 5 for cost $C = 23$. In this case the expected benefit of the auctioneer with 5 bidders is greater than her expected benefit with $n = 6$.

In fact, in the example given in Figure 5 if the number of bidders that will take part in the auction is between 3 and 6, then the auctioneer will favor having 6 bidders when the cost of obtaining the information is above 25 or below 18.5, only 5 bidders when within the range $22.3 - 25$, and only 4 bidders when within the range $18.5 - 22.2$. The expected social welfare is maximized when there are 6 bidders in the case where the cost of obtaining the information is below 17 or above 25, 5 bidders when the cost is within the range $22.3 - 25$, 4 bidders within the range $18.6 - 22.3$ and 3 bidders within the range $17 - 18.6$.

The above suggests that given the option to limit the number of bidders allowed to participate in the auction, the auctioneer should seriously consider using such a limit. Figure 7 depicts the auctioneer's expected-benefit-maximizing number of bidders, denoted N^*, given the maximum number of bidders interested in participating in the auction, N, for different C values, based on the setting described in the table at the bottom right part of the figure. It is noted that in this case, for any value $C < 6.8$ or $C > 10.5$, the optimal number of bidders is the maximum allowed number of bidders, i.e., $N^* = N$. For values $6.8 < C < 8.4$, for some intervals of N the auctioneer would prefer to use $N^* < N$ and for $8.4 < C < 10.5$, starting from some threshold value of N the auctioneer will always prefer to use some fixed value $N^* < N$. The figure also depicts the auctioneer's expected benefit, the bidders' expected benefit and the social welfare for C values that are within the different ranges specified above ($C < 6.8$, $6.8 < C < 8.4$, $8.4 < C < 10.5$ and $C > 10.5$). These help explain the patterns observed for N^*. When the cost of purchasing the information is relatively low ($C < 6.8$ in our case, represented by the $C = 5$ curve) the information is always purchased, regardless of the number of bidders in the auction (this is the equivalent to the interval $C < 13$ in Figure 5). Therefore an increase in the number of bidders is always favorable for the auctioneer. Similarly, when the information is very expensive ($C > 10.5$ in our case, represented by the $C = 12$ curve) the information is never purchased, regardless of the number of bidders in the auction (e.g., when the cost is greater than the highest possible private value of any of the bidders). For $6.8 < C < 8.4$ (represented by the $C = 8$ curve) the auctioneer will switch to not purchasing the information for some N, where the N' value below which the information is not purchased increases as C increases. Therefore, for any $N < N'$ the expected benefit of the auctioneer increases as N increases, however once reaching N' it decreases. This is because with N' the expected benefit is greater than with some greater N values, as it is better to have N' bidders and not purchase the information, than to have more than N' bidders however purchase the information. Still, since the expected benefit, even when purchasing the information, increases as the number of bidders increases, for some N'' it becomes once





		private values		
		x1	x2	x3
Type	p(type) \ p(value)	0.438	0.246	0.316
1	0.333	62	47	35
2	0.333	83	59	55
3	0.334	92	29	76

Figure 7: The auctioneer's expected-benefit-maximizing number of bidders as a function of the maximum number of bidders from whom she can choose, for different information purchasing costs (two upper graphs: left is for selected C values in the range 7.3-8.3, right is for selected C values in the range 8.8-10.4). The two middle graphs and the bottom left one depict the auctioneer's and bidders' expected benefit and the expected social welfare as a function of the number of bidders actually used (the horizontal axis), for different C values. The setting used is given in the bottom right table.

again more beneficial to increase the number of bidders in the auction, hence the auctioneer's expected benefit increases, once again, as a function of N. Finally, in the range $8.4 < C < 10.5$ (represented by the $C = 10$ curve), the auctioneer's expected benefit when not purchasing the information is greater than the expected benefit when purchasing the information, regardless of the number of agents used in the latter case. Therefore the value of N^* increases as N increases as long as the resulting equilibrium is not to purchase the information. However, starting from N'' for which the equilibrium is to purchase the information, the auctioneer is better off with having $N'' - 1$ bidders as the expected benefit with this number of bidders (and not purchasing the information) is greater than with more bidders (however having to purchase the information).

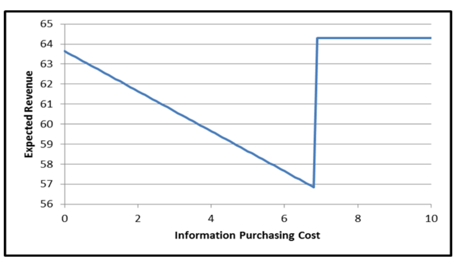
It is notable that a similar non-standard phenomena is observed with the bidders in our hybrid model. In traditional auction models having less bidders participate in the auction is known to be favorable to the bidders [7, 64, 52] — having less bidders means less competition which translates to a lower expected second-best price. In our case, however, the bidders may prefer having more competition, depending on the setting used. For example, from Figure 7 we observe that for $C = 8$ the expected benefit of the bidders for $N = 5$ is 1.8 (an average of 0.36 per bidder) and for $N = 6$ it is 3.88 (an average of 0.65 per bidder). A similar anomaly in nature is observed also with the social welfare. For example, from the social welfare graph in Figure 7 we observe a decrease in the social welfare from 71 to 69.6 in the transition from 5 bidders to 6 bidders when $C = 8$. These are once again explained by the change in the equilibrium structure due to the increase in the number of bidders used.

5.3. The Benefit of Having the Information-Provider and of Bidder-Awareness

Next, we investigate the benefit of having the option for the auctioneer to purchase the information in the first place, and the benefit to the different players from having the bidders become aware of that option. We begin with the question of whether or not the existence of an information-provider that is willing to sell the information to the auctioneer is always beneficial to any of the parties. Seemingly, since the auctioneer can choose between purchasing and not purchasing the information, one may assume that having such an option is always beneficial for the auctioneer. Nevertheless, this is not generally the case, and as illustrated in Figure 5 there are many settings where the opposite is true. Taking the case of $N = 6$ in Figure 5, if the cost of purchasing the information is $C < 6$ then the auctioneer's expected benefit is greater than the one resulting from not having the option to purchase the information in the first place. Still, even with that setting, for a cost $6 \leq C < 25$ the auctioneer is better off not having the option to purchase the information.

The explanation for this interesting phenomena according to which the existence of the option to purchase the information results in a substantial degradation in the auctioneer's benefit, as she could have benefited far more if such an option did not even exist, derives from the instability added to the model due to the availability of the information: the self-interested information-provider sets its price low enough such that a solution according to which the auctioneer does not obtain the information becomes unstable (because of the strong incentive to the auctioneer to deviate to obtaining the information and selectively disclosing it). The only stable applicable solution becomes the one where the information is purchased, and since the cost of obtaining the information is an inherent component in the auctioneer's expected benefit, the auctioneer ends up doing worse (compared to a similar setting without the option to obtain the information).

One interesting related observation from Figure 5 is that the expected benefit of the auctioneer, given a significantly high cost of purchasing the information, can potentially be greater than in the case where the information is relatively cheap or even free (e.g., in Figure 5, for $N = 6$ the auctioneer's expected benefit is greater when the cost of purchasing the information is 40 compared to when $C = 10$). In this kind of setting it could be beneficial for the auctioneer to pay the information-provider in order to make her leave the market completely or, alternatively, to convince her to charge more for the information provided (and publicize the new pricing). Figure 8 illustrates such a scenario, depicting the auctioneer's expected benefit as a function of the cost of purchasing the information. Here, for any cost $C \leq 6.9$ set by the information-provider, the auctioneer is better off paying her up to $(7.39 - 6.9) + C$ to leave the market (or charge $C > 6.9$), as in the latter case the auctioneer's expected benefit increases by that amount (compared to when charged $C \leq 6.9$).

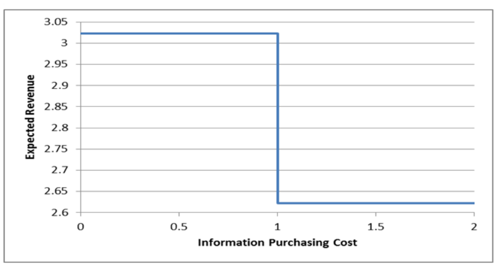


		private values		
		x1	x2	x3
Type	p(type) \ p(value)	0.438	0.246	0.316
1	0.333	62	47	35
2	0.333	83	59	55
3	0.334	92	29	76

Figure 8: An example in which the auctioneer is better off paying the information provider in order to make her leave the market completely or charge more for the information provided.

The benefit for the auctioneer when paying the information-provider to publish (and charge) a different price is not limited just to pushing the information-provider off the market as illustrated above. For example, from Figure 2 we conclude that given any cost $0.93 < C < 1.15$, the auctioneer can benefit form paying the information-provider $C - C'$ in exchange for setting a price $C' < 0.93$. We emphasize that despite the fact that the total payment that the auctioneer eventually pays the information-provider is C (for changing the price and purchasing the information), the resulting solution is different than the one obtained when the price is set to C'. This is because here the auctioneer first pays $C - C'$, and by the time the auction takes place this is already a sunk cost. Therefore, the auctioneer's considerations derive from the cost C', hence the equilibrium is different.

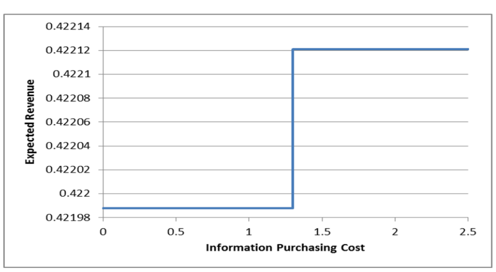
As for the bidders, in Section 3.5 it was established that their expected benefit does not depend on the cost of the information but rather on the transition between the different equilibria. If the information-provider is self-interested, then if there is a cost at which the information is purchased, then the information will necessarily be purchased (as the information-provider sets its price accordingly). We show that the determination of whether having the information-provider present is beneficial to bidders is setting-dependent. Figures 9 and 10 demonstrate the inconclusiveness of the effect: in Figure 9, the absence of the information-provider (which is equivalent to an information-provider charging a substantial price C for the information, hence the information is not purchased by the auctioneer) results in an equilibrium with an inferior individual bidder's benefit, whereas in Figure 10 it is her absence that improves the bidder's individual benefit.



n=3

		private values		
		x1	x2	x3
Type	p(type) \ p(value)	0.2	0.4	0.4
1	0.333	8	2	0.5
2	0.333	9	2.5	1
3	0.334	12	1	15

Figure 9: An example of a scenario in which the bidders are better off having an option for the auctioneer to purchase the external information.

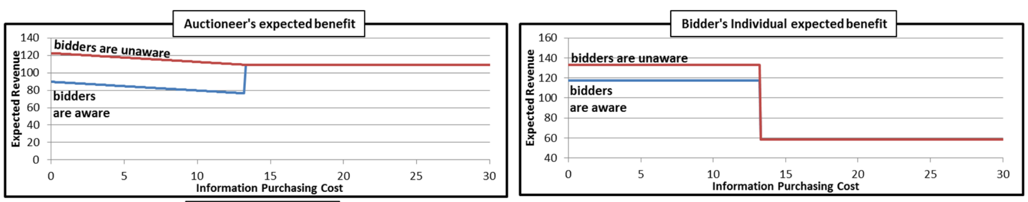


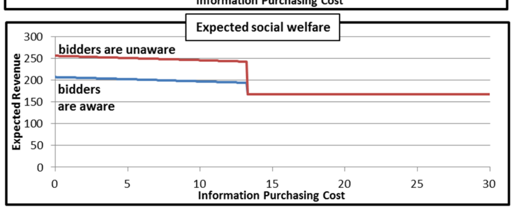
n=3

		private values		
		x1	x2	x3
Type	p(type) \ p(value)	0.2	0.4	0.4
1	0.333	8	2	0.5
2	0.333	9	0.25	1
3	0.334	12	1	1.5

Figure 10: An example for a scenario in which the bidders are better off not having an option for the auctioneer to purchase the external information.

We now move on to investigating the effect of bidders' awareness of the option to purchase the information over the different players' expected benefit. Interestingly, we can find examples for cases where bidders are better off being unaware than aware of the availability of the information to the auctioneer. Figure 11 illustrates such a setting. In this figure we can see that for any cost $C \leq 13.3$, the benefit for the bidders in the case of awareness is 117.2 and the benefit in the case of unawareness is 133, hence the bidders actually gained from their ignorance. The explanation for this behavior is, once again, originated in the stability considerations. When the bidders are unaware of the auctioneer's access to the information, the auctioneer purchases the information (if $C \leq 13.3$), disclosing the common value only if it is x_1 or x_2. This benefits the auctioneer (comparable to the case where bidders are aware of the information purchase option) however is also beneficial for the bidders, as it enables them to better distinguish between the different values whenever disclosed, hence avoiding high bids when the actual value for them is low. When the bidders are aware of the availability of information, however, the strategy $R^{auc} = \{x_1, x_2\}$ is precluded for stability considerations, resulting in an equilibrium $R^{auc} = \{x_1, x_3\}$, thus the bidders end up with a worse expected benefit. A counter-example, in which the bidders are better off being aware of the information's availability to the auctioneer can be found in Figure 12.



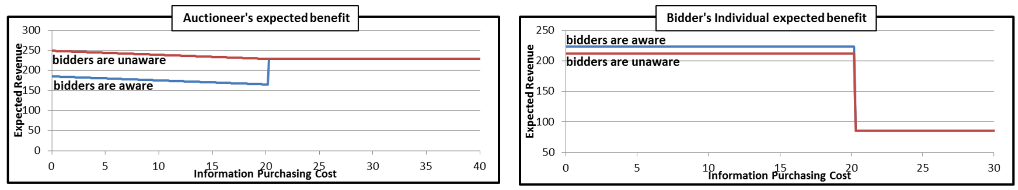


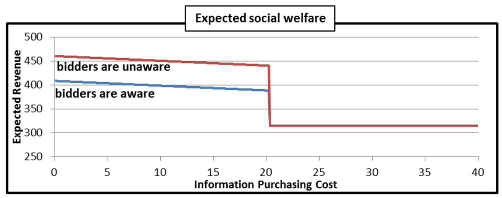
n=3

		private values		
		x1	x2	x3
Type	p(type) \ p(value)	0.333	0.333	0.334
1	0.333	5	20	15
2	0.333	300	23	20
3	0.334	35	30	520

Figure 11: The expected benefit of the auctioneer and bidders and the social welfare, as a function of the information cost, when bidders are aware and when unaware of the availability of such external information to the auctioneer. Here, bidders are better off being unaware of the availability of the information for $C \leq 13.3$.

From Figures 11 and 12 we observe that, as one may expect, the auctioneer benefits from having the bidders unaware of the information availability. Yet, a counter-example according to which the auctioneer prefers that the bidders become aware of her option to purchase the information can also be provided. This is demonstrated in Figure 13. Here, for any cost $0 \leq C \leq 4.05$ the equilibrium in the bidders' awareness case is $(\{x_1, x_2, x_3\}, \{x_1, x_2, x_3\})$ and its expected benefit





n=3		private values		
		x1	x2	x3
Type	p(type) \ p(value)	0.333	0.333	0.334
1	0.333	63	20	28
2	0.333	30	912	96
3	0.334	650	100	56

Figure 12: The expected benefit of the auctioneer and bidders and the social welfare, as a function of the information cost, when bidders are aware and when unaware of the availability of such external information to the auctioneer. Here, bidders are better off being aware of the availability of the information for $C \leq 20$.

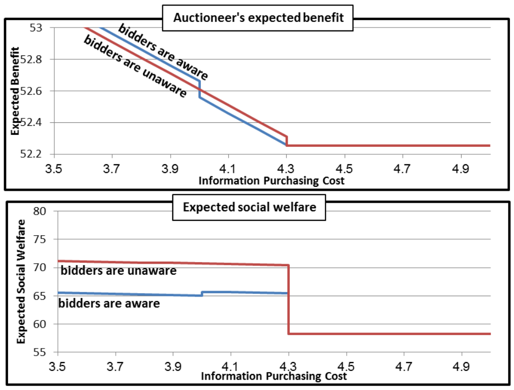
to the auctioneer is greater than in the case where bidders are unaware of the option of purchasing the information. Interestingly, for $4.05 < C \leq 4.3$ the equilibrium in the awareness-based variant becomes $(\{x_1, x_2, x_3, x_5\}, \{x_1, x_2, x_3, x_5\})$, resulting in a change in the auctioneer's preference towards preferring bidder's unawareness. For any cost $C > 4.3$ the auctioneer does not purchase the information and gains the same expected benefit regardless of the bidders' awareness or unawareness.

Finally, we note that Figures 11-13 all demonstrate the phenomena proved in Proposition 4, according to which the transition to not purchasing the information occurs at the same cost C both when the bidders are aware and when unaware of the auctioneer's option to purchase the information.

5.4. Social Planner's Interventions

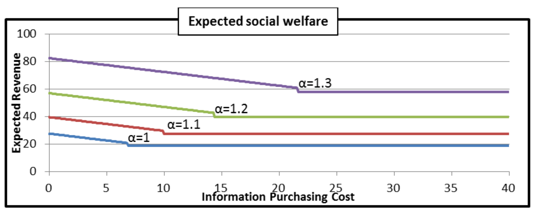
Finally, we demonstrate the effectiveness of market intervention in our case. The two mechanisms we use are, as discussed in Section 2, taxation and subsidy.

We begin with subsidy. A subsidy in our model is useful only if it drives the system to a different equilibrium, such that the change in the social welfare is greater than the change in the cost that the information-provider charges. An example of such case can be found in Figure 14. Here, it is possible that the price C set by the information-provider precludes the purchase of the information in equilibrium, e.g., in the case where the cost of producing the information is $C_\emptyset + \epsilon$ (where $\epsilon \to 0$). In such a case, a market planner may find it beneficial to subsidize the production cost so that the information-provider will offer the information at a cost $C < C_\emptyset$ (hence it will actually be purchased by the auctioneer). The increase



n = 4		private values				
		x1	x2	x3	x4	x5
Type	p(type) \ p(value)	0.09	0.35	0.25	0.06	0.25
1	0.42	61	45	31	50	56
2	0.23	89	18	38	80	51
3	0.35	21	79	86	4	40

Figure 13: The expected benefit of the auctioneer and bidders and the social welfare, as a function of the information cost, when bidders are aware and when unaware of the availability of such external information to the auctioneer. Here, the auctioneer prefers bidders' awareness of the availability of the information for $C \leq 4.05$ and unawareness otherwise.



n=5		private values		
		x1	x2	x3
Type	p(type) \ p(value)	0.307	0.446	0.247
1	0.333	2^{α}	$3^{3\alpha}$	4^{α}
2	0.333	$2^{2\alpha}$	$3^{2\alpha}$	$4^{2\alpha}$
3	0.334	$2^{3\alpha}$	3^{α}	$4^{3\alpha}$

Figure 14: The expected benefit and the social welfare as a function of the information cost, for different levels of the bidders' heterogeneity.

in social welfare in this case will be greater than the subsidy paid.

Next, we consider taxation. We note that the social-planner's proceeds from the tax (i.e., the social planner's "gains") are not added to the social-welfare calculations (however the tax is subtracted from the auctioneer's expected benefit, if the information is indeed purchased). Figure 15 depicts the expected social welfare for the setting used in Figure 2. As can be seen from the figure for any cost $C < 0.93$ (e.g., because the information-provider is active in other markets and hence set a fixed-price that is not the strategic-price for our specific market), setting a tax such that the cost of purchasing the information becomes within the range $0.93 - 1.15$ results in an increased expected social welfare.

A similar benefit in taxation can be demonstrated for the case where the information-provider is acting strategically. Consider Figure 16, which depicts the social welfare as a function of the information cost for the setting described in the table to the right of the figure. In this case, the social welfare with $C = 0$ is equal to the social welfare with $C_\emptyset$ (in which case the information is not purchased). Therefore, regardless of the price C set by the information-provider (including the case where the information-provider is acting strategically, setting a price $C_\emptyset - \epsilon$ ($\epsilon \rightarrow 0$)), a tax that is high enough to push the auctioneer to not purchase the information will necessarily increase social welfare. We note that unlike with all the other examples given, here if we include the information-provider's benefit in the social welfare measure then the described effect will not be obtained. This is due to the fact that the social welfare according to the latter measure will be a flat line regardless of the price charged for the information.

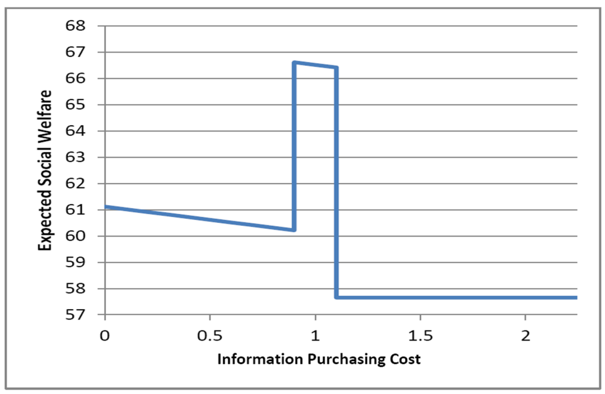
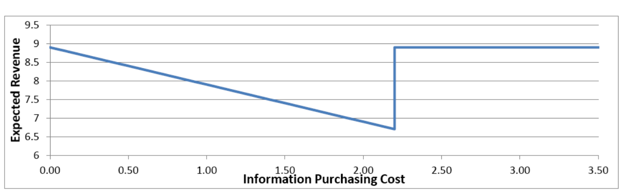


Figure 15: The expected social welfare, as a function of the information cost for the setting used in Figure 2. If the information cost is $C < 0.93$ then the taxation that will result in an effective cost in the range $0.93 - 1.15$ will increase social welfare.

6. Related work

In this section we review related work, divided according to the different aspects of the work presented in the paper, ranging from model classification through information disclosure issues and to the downsides of the availability of information in relevant domains.



(a) Advantage for taxation.

n=3		private values		
		x1	x2	x3
Type	p(value) / p(type)	0.438	0.246	0.316
1	0.333	1	2	3
2	0.333	1	4	9
3	0.334	1	8	27

(b) Setting used for Figure(a)

Figure 16: Example of a case where the social-planner can increase the social welfare by imposing a tax on the information cost.

6.1. Auctions as Means of Trading and Model Classification

Auctions are an effective means of trading and allocating goods whenever the seller is unsure about buyers' (bidders') exact valuations of the sold item [32, 33]. If the seller knew the bidders' valuations she could have allocated the item to the one that valuates it most. The advantage of many auction mechanism variants in this context is in the ability to effectively extract the bidders' valuations (e.g., English auction [40, 25, 28, 31], second-price sealed-bid auction [31, 46, 40] and VCG [49, 10, 24]), resulting in the most efficient allocation [32, 33]. Based on its many advantages, the mechanism is commonly used and researched [43, 5]. Over the years it has evolved to support various settings and applications such as on-line auctions where buyers and sellers arrive over time and the mechanism is required to make decisions about each bid as it is received [30, 37, 26], matching agents in dynamic two-sided markets [8], resource allocation [45, 44] and even for task allocation and joint exploration [20, 35]. In this context a great emphasis is placed on study of bidding strategies [59, 56, 4], the use of software agents to represent humans in auctions [12] combinatorial auctions [57] and the development of auction protocols that are truthful [8, 44, 13] and robust (e.g., against false-name bids when considering combinatorial auctions [65]).

Bidders' valuations are not the only source of uncertainty in auctions. Similar to the case of other forms of trading, there might be some uncertainty associated with the value of the sold (auctioned) item. This uncertainty may apply both to sellers and bidders, and is an inherent feature of many auction settings. The simplest setting in terms of the bidders' valuation, often termed *private value model* [32, 33, 21], is where all bidders know the value of the auctioned item to themselves. This model, however, implicitly assumes that no bidder knows the valuations of the other bidders with complete certainty. The more complex scenario is the case where the value of the auctioned item is unknown to the bidders at the time of the auction. Here, bidders may only have an estimate or some privately known signal, such as an expert's estimate, that is correlated with the true value [58, 23, 32, 33]. Moreover, commonly in this case, other bidders may have some information that, if known, would affect the valuation that each particular bidder associates with the item. Namely, values are unknown at the time of the auction and information known to other bidders may affect one's valuation. Such specification is termed *the interdependent values* [3, 33]. A special case of the interdependent values model, often termed *pure common value model*, is the one where the value, while unknown at the time of bidding, is the same for all bidders [33].

We illustrate the differences among the three models (private, interdependent and common value) using the "piece of art" example commonly found in auction theories textbooks. If a bidder wishes to buy a piece of art for her sole usage, her valuation will be based on the utility she derives from possessing it. This scenario typically aligns with the private value model. On the other hand, if the bidder is interested in buying this piece of art for her own usage but considers selling it sometime in the future, then the valuation model will be considered to be the interdependent model, as it combines private valuation (modeled as the privately known signal) and some uncertain resale value (modeled as the others' interdependent signal). In contrast, if the bidder is buying the piece of art only for the purpose of reselling it, then this case will be considered the common value model as the value derives directly and solely from the price which the public associates with this piece of art, which is both independent of the bidder's private considerations and unknown to her.

Many results known for the private value model do not hold for the more general case. For example, the known strategic equivalence between Vikcrey (second-price sealed-bid) and English auctions for the private model case does not necessarily hold for the other models [33]. In the English auction, bids are publicly announced and therefore new information is observed by bidders (concerning the valuations of the other bidders), resulting in a dynamic update of each bidder's beliefs and sequentially of her bidding strategy. In contrast, in the Vickrey auction no such information is observed as it is a closed form of auction. Moreover, in non-private models the dominant truth-telling strategy is no longer the optimal one [33]. To conclude, non-private models must be researched apart from the private value case.

The interdependent value model where the item's value is a combination of private and common values is sometimes referred to as a *correlated value model* [61]. However, this term is somehow ambiguous and often refers to a different model setting. For example, Eso [18] studies an auction model with risk-averse bidders where the correlated value notion stems from the correlation coefficient among the bidders' valuations. A similar model of correlated values was considered by Wang [62], aiming to determine the preferred selling mechanism (fixed price or auction) based on the distribution of the potential bidders' valuations.

Many works have attempted to deal with the problem of *uncertainty* in auctions. Most of these works refer to the uncertainty associated with bidders' information. For example, the bidders may be uncertain regarding the number of bidders participating in the auction, which is often the case in online auctions that apply English-type protocols (e.g., Dyer et al [15]). Alternatively, bidders are often assumed to be ignorant of their own private value and need to expend some computational efforts in order to reveal it (e.g., see Parkes [48] and Larson & Sandholm [36], who consider the problem in the context of bounded-rational and computationally-bounded agents). Finally, there are situations where the agents are uncertain regarding the tasks they need to execute (and consequently their bids) and have only partial control over their resources (e.g., Hosam and Kholdoun [29]). None of these works, however, deal with a setting similar to ours, and in particular do not address the question of information acquisition and disclosure, as well as the influence of a self-interested information-provider.

Another aspect according to which the auction literature classifies auction models is the symmetry among the different players, distinguishing between symmetric and asymmetric cases. In the context of the interdependent model, the symmetry has two aspects. The first concerns the symmetry of the bidders' valuation function, and the second concerns the symmetry in the knowledge available to the bidders regarding the auctioned item (e.g., knowing the signals' distribution). When considering asymmetric models, further interpretations of the concept may be found, e.g., models where the sellers are more informative than bidders [2, 17], and even variance in bidders' information (e.g., some bidders can be more informed than others). In the latter case, it has been shown that the advantageous bidder may benefit by signaling the less informative bidders to prevent them from bidding [54, 55]. Our model is neither purely symmetric nor purely asymmetric but rather a hybrid one. It assumes symmetry between bidders and sellers in terms of the a priori public information regarding the valuation of the characteristic variable, and between bidders whenever the auctioneer chooses to disclose information. It assumes asymmetry in terms of the bidders' valuation function, as each bidder's valuation function may be a different function of the characteristic variable. Another asymmetry in our model is in the availability of the information: only the auctioneer may purchase the information, and it is her choice whether or not to disclose it to the bidders.

6.2. Information Disclosure

The role of information revelation in the design of auctions has been investigated ever since Milgrom and Weber's seminal paper in 1982 [40]. According to the "linkage principle", on average (over the seller's information) the auctioneer's benefits are enhanced by committing to the policy of always providing the bidders with as much information as possible about the value of the good. This latter result is valid for common value auctions, and under some specific assumptions regarding the correlation between bidders' valuations. Since then, it has been shown that once one or some of the model's assumptions of the "linkage principle" change the principle fails to hold. For example, Perry and Renny [50] have shown its failure when considering multi-unit auctions. As demonstrated throughout our paper, the principle also fails in our hybrid model, where bidders' valuations are not necessarily correlated, making the dominance of the full information disclosure strategy setting-dependent.

In some situations, such as the one discussed in this paper, the auctioneer herself does not possess the information but rather needs to purchase it from an external source or invest some resources in order to produce it. In such settings the auctioneer needs to decide both whether to obtain the information, and if so, what parts of it should be disclosed. Alternatively, the auctioneer may allow bidders to assess information about the auctioned item on their own. In the latter case the auctioneer cannot verify whether and how bidders exercise this option and therefore any information they reveal remains hidden. Such a situation leads to settings where bidders are asymmetrically informed [38]. Works that studied common-value settings where bidders are not symmetrically exposed to information reached conclusions for specific mechanisms, while for others it was shown that multi-equilbria is possible (with no distinct choice that emerges), resulting in a non-conclusive determination of the results. For example, while in the first-price sealed-bid auction it is well known that the presence of an informational advantaged bidder will substantially reduce the seller's benefits [41, 63, 27], in the second-price sealed-bid auction there are still many open questions. This is mainly due to Milgrom's finding of the multiplicity of equilibria [39], which makes it difficult to determine which equilibrium bidders will follow. For this multiplicity equilibrium problem in environments with asymmetric bidders Selten [53] has proposed the Nash Equilibrium refinement, termed the Trembling-Hand Perfect Nash Equilibrium. According to this refinement, in two players game, the Nash equilibrium in which one of the players is playing dominated strategy is ruled out. However, with more than two players trembling-hand perfectness may get rid of more equilibria in which no player is playing a weakly dominated strategy. Other Nash equilibrium refinements were proposed to different asymmetric environments including the techniques of introducing additional rational "uninformed" or "random" players [16, 1]. The model used in this paper differs from these works in the sense that it assumes a symmetric common value and that the asymmetry is between the auctioneer (which is the advantageous informed player) and the bidders. Indeed, in our case the multi-equilibria problem arises, however since the auctioneer is the first mover in a sequential game (even if she eventually chooses not to purchase or disclose the information), the multiplicity Nash Equilibrium problem can be resolved by following the principle of a sub-game perfect equilibrium, leading to the choice of the Nash equilibrium that is best for the auctioneer ([47, 33]).

Goeree and Offerman [22, 23] consider a symmetric-bidders environment, which is similar to ours in the sense that bidders are characterized by both private and (uncertain) common value aspects. The uncertainty associated with the common value in their model is attributed both to the bidders and the auctioneer. Their work examines the effect of reducing the uncertainty associated with the common value, mostly in terms of reducing its variance, based on equilibrium analysis. From the model's point of view, the main difference between their work and ours is that they consider the case where the private and the common value elements are independent and additive and therefore the bidder's valuation function is the sum of the two, whereas our model assumes the bidders' private valuations to be a general (bidder-dependent) function of the common value. Furthermore, their model does not consider the option to purchase external information and the question of whether or not bidders are aware of this option, nor the existence of an external self-interested information-provider. Therefore, the various aspects and equilibrium dynamics associated with these latter model elements are precluded.

Works that do consider the option to obtain external information that can reduce uncertainty usually relate to the option of bidders to obtain such information. For example, Bergman et al [6] consider a setting with independent bidders' valuation of a single unit using Vickrey auction where it is possible for bidders to acquire information that can facilitate the assessment of the auctioned item's worth, prior to the auction stage. For example, consider the scenario where a number of established companies bid for the takeover of a certain target company. The bidders are assumed to be symmetrically informed but before submitting a bid it is highly expected that each competing company will hire a consulting firm to better assess the value of the target company. In their work Bergman et al analyze the bidder's private incentives to acquire information in such independent value models and establish a comparison between the number of bidders that choose to become informed (by buying information) in equilibrium and the number of informed bidders in the socially optimal allocation. They find that the number of bidders that decide to buy information is substantially greater than the number of bidders that should have bought it in order to achieve the socially optimal allocation. Similar to our use of tax and subsidies to allow a social planner to control the social efficiency, Bergman et al adopt the entry fees as a way that the social planner may control the information acquisition by bidders in equilibrium to make it as close as possible to the level that is socially efficient. The main difference between this work and ours is in the identity of the player that can obtain the external information. In our model it is the auctioneer rather than the bidders that can obtain the additional information. Therefore, in our case, one key question is what information will be disclosed, if the information is eventually purchased by the auctioneer and both sides need to model one another's decisions and beliefs regarding that aspect. Moreover, the influence of a self-interested information-provider on the resulting equilibrium was never addressed in these works.

Another relevant work is that of Emek et al [17], which investigates the scenario of a publisher who sells ad space to advertisers using a second-price sealed-bid auction in online advertising markets. The model they use assumes that the auctioneer possesses more accurate information than the advertisers (bidders), and the main question is which part of the information to disclose to the bidders in order to maximize the publisher's benefit. The auctioneer in their case divides the possible values into segments (disjoint clusters). The division into clusters is given to the bidders a priori, and once the true value is known to the auctioneer it only discloses the cluster it belongs to (thus eliminating all others) and the bidders place their bids accordingly. Based on the above protocol, the auctioneer needs to decide on the division of values into disjoint clusters, i.e., find the expected-benefit-maximizing division. Our work is similar to Emek et al in the sense that both assume: (i) the use of a second-price sealed-bid auction; (ii) the bidders are provided with some a priori probability regarding the value of the auctioned item; and (iii) the dependency of the private value on the common value. Despite these many similarities, the models and the resulting analysis are very different for the following reasons. First, and most important, Emek et al assume the auctioneer necessarily obtains the information and that the division into clusters is always given to the bidders a priori. Furthermore, not disclosing any information (signal) is not allowed. This means that their setting can be considered a classic Stackelberg game, as the auctioneer can always calculate the resulting second-best bid given the division it uses. Our problem, on the other hand, does not restrict the auctioneer to purchasing the information in the first place, and allows not disclosing any value even if the information is purchased. This changes the game to a Stackelberg game version with imperfect information of the bidders regarding the leader's actions in cases where no information is disclosed. In particular, bidders cannot distinguish, whenever no information is disclosed, between not purchasing the information in the first place and purchasing the information however not disclosing a value. Also, since Emek et al assume that the auctioneer necessarily obtains the information they do not deal with one of the inherent questions raised and analyzed in our paper, in addition to what information to disclose, which is whether or not to obtain the information in the first place. Moreover, in our model the information source is modeled as an external self-interested entity, hence the dynamics resulting from its benefit-maximizing strategy are also taken into considerations resulting in issues such as the benefit in not having the option to purchase the information in the first place and the individual's awareness of this option. Second, Emek et al assume the bidders types are known to the auctioneer, whereas we assume these are derived from a distribution of types. Finally, while the work of Emek et al considers the tradeoff between benefit maximization and social welfare, it does not consider external interventions in the form of subsidy and taxation.

Two interesting extensions to the work of Emek et al are the ones given by Miltersen and Sheffet [42] and Dughmi et al [14]. The first extends Emek et al's model to the case of mixed signaling scheme, showing that the problem is strongly related to a problem of optimally bundling divisible goods for auctioning. The second examines signaling for revenue maximization where social, legal, or practical constraints are placed on the auctioneer's signaling policy (limiting the number or nature of signals). Both these works differ from ours in the same way Emek et al's model and analysis differ from ours (as discussed above).

Based on the review given above, and to the best of our knowledge, an analysis that addresses a model with all of the different aspects included in the model analyzed in the current paper, is not provided in existing literature. Many prior models do consider a subset of our model characteristics, however as explained above, their results cannot simply be carried over to our model in order to point to the equilibrium that is likely to hold.

7. Conclusions and Future Research

The benefits of selective disclosure of information in mixed auction settings have been well established in prior works, as discussed in the previous section. The current paper considers the problem in a richer setting, where the availability of the information to the auctioneer is not trivial, but rather the auctioneer needs to decide whether or not she is interested in purchasing it. Bidders' awareness of the situation complicates the analysis in this case, as it requires the solution to be stable. The analysis given in the paper unfolds the equilibrium structure for such a model and facilitates its calculation for any setting. In particular, it shows how the problem of the auctioneer or the social planner can be simplified whenever having to reason about the equilibrium, which is likely to hold for any potential price set by the information provider.

The equilibrium-based analysis also facilitates the illustration of several interesting and often somewhat surprising properties of the model, related to the effect of the different model parameters over the auctioneer's expected-benefit in equilibrium. Some of them are in contrast to those characterizing traditional auction models (e.g., having more bidders participate in the auction is not necessarily in the auctioneer's best interest, and having less bidders is not necessarily in the bidders' best interest; having the option to purchase the information (at all or at a reduced price) can actually result in a degradation in the auctioneer's expected benefit). These results are attributed to the stability requirement — despite the superiority of the situation for the auctioneer (e.g., with the reduced cost of information, the greater number of bidders), the auctioneer's preferable solution cannot hold since the bidders know that if they act according to it, there is an incentive for the auctioneer to deviate.

As for bidders' awareness of the auctioneer's option to purchase the information, the paper demonstrates that such knowledge is not always beneficial in our setting: the auctioneer can potentially benefit from bidders' awareness, and bidders can benefit from being unaware. This, once again, is attributed to the stability constraint — becoming aware or unaware may lead to a different equilibrium solution, which might turn out to be more beneficial to any of the parties despite the loss due to bidders' awareness (for the auctioneer) or unawareness (for the bidders). Similarly, the option to purchase the information can have a substantial downside, and indeed the paper demonstrates situations where the existence of such an option leads to a degradation in the auctioneer's expected benefit. The interest in such a result is because seemingly it is the auctioneer's choice of whether to obtain the information or not. The explanation of this result is that once the bidders become strategic players, the auctioneer does not really have the flexibility of choosing whether or not to obtain the information. Instead, the choice of obtaining the information derives from stability considerations. One important implication of the above is that the auctioneer may find it beneficial to pay the information-provider to leave the market or not disclose her existence. Another implication is that a self-interested information-provider may find it beneficial not to publish her existence but rather contact the auctioneer and offer her services discretely.

Finally, the paper shows that in our unique setting, it might be beneficial for the auctioneer to pay an external information-provider in order to change the price she sets for the information (without leaving the market). Similarly, we demonstrate how a social planner can improve social welfare through subsidy and taxation. Both means are effective. In particular, in the case of a tax, the paper uses a strict assumption whereby the proceeds of the taxes are not re-distributed and thus the tax is taken into account as part of the social welfare only in the form of the tax payment that the auctioneer needs to pay.

We note that our model allows only the auctioneer to purchase the external information related to the common value. While this is applicable to many settings, for the justifications given in Section 2, there are situations where the auctioneer and bidders are symmetric in their ability to obtain the common value. Therefore, an important direction for future research is the modeling and analysis of settings where both the auctioneer and the bidders can purchase the external information. The analysis of such environments is likely to involve several aspects of coalition formation and cooperation among self-interested agents, as the bidders may find it beneficial to cooperate in obtaining the information (either for the purpose of reducing the cost each incurs or as a means of placing pressure on the auctioneer to obtain and disclose it). The stability in this case will need to be tested also for the bidder coalitions that will be formed. Furthermore, the auctioneer may find it beneficial in some cases to split the cost of obtaining the information with the bidders in exchange for committing to disclose the information. Another natural extension of this work is the analysis of mixed settings where only some of the bidders are aware of the fact that the auctioneer may purchase accurate information on the common value.

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Appendix - Using Continuous Distribution Functions

The continuous case demands some minor changes in the formulation given in the analysis section. When the distribution of X is continuous, the set R^{bidder} is a collection of intervals rather than discrete values, and the bid of a bidder of type t when no value is revealed is calculated as:

$$B(t, \emptyset) = \int_{x \notin R^{bidder}} V_t(x) f_x^*(x) dx \tag{14}$$

where $f_x^*(x)$ is the posterior probability distribution function, calculated as:

$$f_x^*(x) = \begin{cases} 0 & \text{if } x \in R^{bidder} \\ \frac{f_x(x)}{\int_{y \notin R^{bidder}} f_x(y) dy} & \text{if } x \notin R^{bidder} \end{cases} \tag{15}$$

The calculation of the expected benefit of bidders of type t, weighing the expected benefit for each value that X may obtain according to its occurrence probability, is much simpler compared to the discrete case, as we do not need to concern cases where there is more than one bidder placing the same highest bid:

$$
\begin{aligned}
u_{bidder}(t) = \int_{x \in R^{auc}} \Bigg[f_x(x) \Big(\int_{B(t',x)<B(t,x)} (n-1) f_x(t') dt' \\
\cdot \int_{B(t'',x) \leq B(t',x)} f_t(t'')^{n-2} (V_t(x) - B(t',x)) dt'' \Big) \Bigg] dx \\
+ \int_{x \notin R^{auc}} \Bigg[f_x(x) \Big(\int_{B(t',\emptyset)<B(t,\emptyset)} (n-1) f_x(t') dt' \\
\cdot \int_{B(t'',\emptyset) \leq B(t',\emptyset)} f_t(t'')^{n-2} \Big(\frac{\int_{y \notin R^{auc}} V_t(y) Pr(X=y) dy}{\int_{y \notin R^{auc}} Pr(X=y) dy} - B(t',\emptyset)) dt'' \Big) \Bigg] dx
\end{aligned}
\tag{16}
$$

and the expected benefit of a random bidder is given by: $u_{bidders} = \int_{t \in T} f_t(t) u_{bidder}(t) dt$.

The auctioneer's expect benefit from the auction in the continuous case is thus:

$$
u_{auc}(X=x) = \int_y n(n-1)(G(y,x))^{n-2}(1-G(y,x))g(y,x) y dy \tag{17}
$$

where $g(w,x)$ and $G(w,x)$ are given by:

$$
G(w,x) = \int_{B(t,x) \leq w} f_t(t) dt \quad ; \quad g(w,x) = \frac{dG(w,x)}{dw} \tag{18}
$$

Similarly, $u(R^{auc}, R^{bidder})$ can be calculated in this case using:

$$
\begin{aligned}
u(R^{auc}, R^{bidder}) = \int_{x \in R^{auc}} f_x(x) \cdot u_{auc}(x) dx \\
+ \int_{x \notin R^{auc}} f_x(x) \cdot u_{auc}(\emptyset) dx
\end{aligned}
\tag{19}
$$