Tate's Thesis

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What follows is a slightly expanded version of the notes for two lectures on Tate's thesis given at the school on Automorphic Forms, L-functions and Number Theory at Hebrew University in Jerusalem in March of 2001.

Tate's thesis, *Fourier analysis in number fields and Hecke's zeta-functions*, Princeton, 1950, first appeared in print as Chapter XV of the conference proceedings *Algebraic Number Theory*, edited by Cassels and Frolich, published by the Thompson Book Co., Washington, D.C., in 1967. In it, Tate provides an elegant and unified treatment of the analytic continuation and functional equation of the L-functions attached by Hecke to his Größencharaktere in his pair of papers [heckeI] and [heckeII]. The power of the methods of (abelian) harmonic analysis in the setting of Chevalley's adèles/idèles provided a remarkable advance over the classical techniques used by Hecke[1]. In hindsight, Tate's work may be viewed as giving the theory of automorphic representations and L-functions of the simplest connected reductive group $G = GL(1)$, and so it remains a fundamental reference and starting point for anyone interested in the modern theory of automorphic representations.

These notes have two main goals. The first is to give a unified treatment, following Tate, of the analytic properties, analytic continuation, functional equation, etc., of the L-function $L(s, \chi)$ attached to any Hecke character χ for any number field. The second is to introduce the point of view and techniques of representation theory of adèle groups in the simplest case. For example, the progression from local results to global results and the role played by uniqueness theorems is rather typical in this business. These notes provide only a sketch and the reader should consult more extended sources for details.

References: Of course, the best reference is Tate's thesis itself! In addition, there are many other expositions, including Chapter VII of Weil [weilBNT], Chapter 3.1 of the book of Bump [bump], and the book of x and Ramakrishnan, [ramak]. In the present condensed survey, I more or less follow the approach sketched in Weil's 1966 Bourbaki talk *Fonction zeta et distributions*, [weilCP]. Weil's commentary on p.448 of [weilCP], volume III is also of interest.

§1. Adèles, idèles, and Dirichlet characters.

As a motivation for the role played by the idèles and quasi-characters, consider the following description of classical Dirichlet characters.

[1]Tate gives some interesting historical comments in the thesis and at the end of Chapter XV of [CF].

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For a positive integer N, a classical Dirichlet character $\underline{\chi}_N : \mathbb{Z} \to \mathbb{C}$ modulo N is a function obtained from a character $\chi_N : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ by first extending by 0 on $\mathbb{Z}/N\mathbb{Z}$ and then pulling back to $\mathbb{Z}$. We will call χ_N a Dirichlet character. If $N \mid M$, then χ_N defines a character $\chi_M = \chi_N \circ \mathrm{pr}_{M,N}$ of $(\mathbb{Z}/M\mathbb{Z})^\times$, by pulling back under the projection

$$\mathrm{pr}_{M,N} : \left(\mathbb{Z}/M\mathbb{Z}\right)^\times \longrightarrow \left(\mathbb{Z}/N\mathbb{Z}\right)^\times, \qquad N \mid M,$$

and there is an associated classical Dirichlet character $\underline{\chi}_M$ as well. The inverse limit of the system $\{\ \left(\mathbb{Z}/N\mathbb{Z}\right)^\times, \mathrm{pr}_{M,N}\ \}$, is the compact, totally disconnected, topological group[2]

$$\widehat{\mathbb{Z}}^\times := \varprojlim_N \left(\mathbb{Z}/N\mathbb{Z}\right)^\times,$$

and every Dirichlet character χ_N can be viewed as a *continuous* character χ of $\widehat{\mathbb{Z}}^\times$. The *conductor* of χ is the smallest N_0 for which χ is trivial on the kernel of the projection $\mathrm{pr}_{N_0} : \widehat{\mathbb{Z}}^\times \to (\mathbb{Z}/N_0\mathbb{Z})^\times$. Then $N_0 \mid N$ and χ is the pullback of a unique Dirichlet character χ_{N_0} of $(\mathbb{Z}/N_0\mathbb{Z})^\times$. The collection of classical Dirichlet characters $\underline{\chi}_M$ for $M \mid N_0$ all correspond to χ, where $\underline{\chi}_{N_0}$ is the unique *primitive* classical Dirichlet character, and the others are *imprimitive*. An analogous phenomena takes place in the dictionary between classical holomorphic modular forms and automorphic representations of $GL(2)$, where the normalized newforms play the role of the primitive classical Dirichlet characters.

Recall that, for a fixed prime p,

$$\begin{aligned}
\mathbb{Z}_p &:= \varprojlim_n \mathbb{Z}/p^n\mathbb{Z} \\
&= \{\alpha = a_0 + a_1 p + a_2 p^2 + \cdots \mid 0 \le a_i \le p-1\}, \\
\mathbb{Z}_p^\times &= \{\alpha \in \mathbb{Z}_p \mid a_0 \neq 0\}, \\
\mathbb{Q}_p &:= \{\alpha = \sum_i a_i p^i\}
\end{aligned}$$

etc.

Recall that the adèle ring $\mathbb{A}$ of $\mathbb{Q}$ is the restricted product[3]

$$\mathbb{A} = \mathbb{R} \times {\prod_p}' \mathbb{Q}_p$$

[2]Recall that this is, by definition,

$$\widehat{\mathbb{Z}}^\times = \{\ (x_N) \in \prod_N (\mathbb{Z}/N\mathbb{Z})^\times \ \mid\ \mathrm{pr}_{M,N}(x_M) = x_N\ \}.$$

The product of the finite groups $(\mathbb{Z}/N\mathbb{Z})^\times$ is compact and hence so it the closed subgroup $\widehat{\mathbb{Z}}^\times$. The projection maps

$$\mathrm{pr}_N : \widehat{\mathbb{Z}}^\times \to (\mathbb{Z}/N\mathbb{Z})^\times$$

are all surjective and continuous.

[3]Almost all components of an element (x_p) must lie in the subring $\mathbb{Z}_p$.

with respect to the subrings $\mathbb{Z}_p$. Also $\mathbb{Q} \hookrightarrow \mathbb{A}, \alpha \mapsto (\dots, \alpha, \alpha, \alpha, \dots)$, embedded diagonally, is a discrete subring. Similarly, the idèles group is the restricted product

$$\mathbb{A}^\times = \prod_v{}' \mathbb{Q}_v^\times, \qquad \mathbb{Q}^\times \hookrightarrow \mathbb{A}^\times,$$

and

$$\mathbb{A}^\times \simeq \mathbb{Q}^\times \times \mathbb{R}_+^\times \times \widehat{\mathbb{Z}}^\times, \qquad x = \alpha t u. \tag{1.1}$$

¿From this last decomposition, we see that any Dirichlet character χ defines a continuous character

$$\omega : \mathbb{Q}^\times \backslash \mathbb{A}^\times \longrightarrow \mathbb{C}^\times, \qquad \omega(x) = \omega(\alpha t u) = \chi(u).$$

The most general such *quasicharacter* has the form $\omega \cdot |\ |^s$, for $s \in \mathbb{C}$, where $|x| = |\alpha t u| = t$. Thus, the Dirichlet characters are precisely the quasicharacters of finite order.

Note that since

$$\left(\mathbb{Z}/N\mathbb{Z}\right)^\times = \prod_p \left(\mathbb{Z}/p^n\mathbb{Z}\right)^\times, \qquad n = n_p = \mathrm{ord}_p(N),$$

any Dirichlet character χ_N has a factorization

$$\chi_N = \otimes_p \, (\chi_N)_p.$$

This gives rise to a *factorization*

$$\chi = \otimes_p \, \chi_p$$

of the corresponding character χ of $\hat{\mathbb{Z}}^\times$, where each χ_p is the character of $\widehat{\mathbb{Z}}_p^\times$ associated to $(\chi_N)_p$. Note that for each place v, i.e., for each $\mathbb{Q}_p$ and for $\mathbb{Q}_\infty = \mathbb{R}$, we have an inclusion

$$\mathbb{Q}_v^\times \hookrightarrow \mathbb{Q}^\times \backslash \mathbb{A}^\times \qquad x_v \mapsto (\dots, 1, 1, x_v, 1, 1, \dots).$$

Thus, any quasicharacter ω determines quasicharacters ω_v of each $\mathbb{Q}_v^\times$, and there is a factorization

$$\omega = \otimes_v \, \omega_v \tag{1.2}$$

compatible with the previous one.

Exercise: (i) Check that an *odd* classical Dirichlet character $\underline{\chi}$ mod N (i.e., $\underline{\chi}(-a) = -\underline{\chi}(a)$) yields a character ω such that $\omega_\infty(x) = \mathrm{sgn}(x)$.
(ii) For ω associated to a classical Dirichlet character $\underline{\chi}_N$, modulo N, determine $\omega_p(p)$ for primes $p \nmid N$ and for primes $p \mid N$.

§2. L-functions for quasicharacters.

For $\boldsymbol{k}$ a general number field (or function field over a finite field[4]), there is a similar picture. We will use the following notation:

$$\begin{aligned}
\mathcal{O} &= \text{ ring of integers in } \boldsymbol{k},\\
\mathcal{O}_v &= \text{ it completion at a nonarchimedean place } v \ (= \text{a dvr}),\\
\mathcal{P}_v &= \text{ the maximal ideal of } \mathcal{O}_v,\\
\mathbb{F}_v &= \mathcal{O}_v/\mathcal{P}_v = \text{ the residue field at } v,\\
q_v &= |\mathcal{O}_v/\mathcal{P}_v| = \text{ its order}\\
\mathrm{ord}_v : \boldsymbol{k}_v &\longrightarrow \mathbb{Z}\cup\{\infty\} \qquad \text{the valuation}\\
|x|_v &: = q_v^{-\mathrm{ord}_v(x)} \qquad \text{the normalized absolute value.}\\
\mathcal{O}_v^\times &= \text{ the group of units,}
\end{aligned}$$

and so, $\mathcal{O}_v^\times \times \mathbb{Z} \xrightarrow{\sim} \boldsymbol{k}_v^\times$, $(u,n)\mapsto u\varpi_v^n$, where $\varpi_v \in \mathcal{O}_v$ is an element with $\mathrm{ord}_v(\varpi_v)=1$ (local uniformizer). For an archimedean place v with $\boldsymbol{k}_v \simeq \mathbb{R}$ (resp. $\boldsymbol{k}_v\simeq\mathbb{C}$), let $|x|_v$ denote the usual absolute value (resp. the square of the usual absolute value).

Then we have restricted products, the adèles,

$$\mathbb{A} = \prod_v{}' \boldsymbol{k}_v \qquad \text{and the idèles,} \qquad \mathbb{A}^\times = \prod_v{}' \boldsymbol{k}_v^\times,$$

with diagonal embeddings $\boldsymbol{k}\hookrightarrow\mathbb{A}$ and $\boldsymbol{k}^\times\hookrightarrow\mathbb{A}^\times$. Here restricted mean that for $x = (\dots,x_v,\dots)$ in $\mathbb{A}$ (resp. $\mathbb{A}^\times$), almost all components x_v lie in $\mathcal{O}_v$ (resp. $\mathcal{O}_v^\times$). There is an absolute value

$$|\ |_{\mathbb{A}} : \mathbb{A}^\times \longrightarrow \mathbb{R}_+^\times, \qquad |x| = \prod_v |x_v|_v.$$

which is trivial on $\boldsymbol{k}^\times$ (the product formula). The decomposition (1.1) becomes more complicated due to (i) global units and (ii) the nontriviality of the ideal class group. These are precisely the difficulties which Hecke had to work hard to overcome and which, by contrast, the adélic formalism handles so beautifully. In any case, there are inclusions

$$i_v : \boldsymbol{k}_v^\times \hookrightarrow \boldsymbol{k}^\times\backslash\mathbb{A}^\times, \qquad x_v \mapsto (\dots,1,1,x_v,1,1,\dots),$$

as before.

[4]We will exclude this case, simply to avoid the extra side comments it would require.

Definition 2.1: A *quasicharacter* or Hecke character is a continuous complex character $\omega : \mathbb{A}^\times \to \mathbb{C}^\times$, trivial on $\boldsymbol{k}^\times$.

Thus, one can think of ω as a character of $\mathbb{A}^\times$ occuring in the natural representation of this group by translations on the space of continuous functions

$$f : \boldsymbol{k}^\times \backslash \mathbb{A}^\times \longrightarrow \mathbb{C}.$$

There is a factorization

$$\omega = \otimes_v \, \omega_v$$

where $\omega_v = \omega \circ i_v$. It is possible to describe such characters in a more classical language, as in Kowalski's lecture, and one eventually arrives at the classical definition of the Größencharaktere given by Hecke.

Definition 2.2: (i) For a nonarchimedean place v, a quasicharacter $\omega_v : \boldsymbol{k}_v^\times \to \mathbb{C}^\times$ is *unramified* if it is trivial on $\mathcal{O}_v^\times$, and hence can be written in the form

$$\omega_v(x) = t_v^{\mathrm{ord}_v(x)}$$

for $t_v = t_v(\omega_v) \in \mathbb{C}^\times$.
(ii) A quasicharacter ω of $\boldsymbol{k}^\times \backslash \mathbb{A}^\times$ is unramified at v if its local component ω_v at v is unramified.

For any ω, there is a finite set of places $S = S(\omega)$, including all archimedean places[5] such that ω_v is unramified for all $v \notin S$. This is clear in the case $\boldsymbol{k} = \mathbb{Q}$ discussed in the previous section, where, if ω is associated to a primitive classical Dirichlet character $\underline{\chi}_N$ modulo N, then $S(\omega) = \{\infty\} \cup \{p \mid p \mid N\}$. Thus, associated to ω is a collection of complex numbers $\{t_v(\omega)\}_{v \notin S}$. For example, for $s \in \mathbb{C}$, the quasicharacter $\omega_s(x) = |x|_{\mathbb{A}}^s$ is everywhere unramified, and determines the set $\{q_v^{-s}\}$.

Definition 2.3: The (partial) **L-function** associated to a quasicharacter ω is the Euler product

$$\text{(EP)} \qquad L^S(s, \omega) = \prod_{v \notin S} (1 - t_v(\omega) \, q_v^{-s})^{-1},$$

where $S \supset S(\omega)$. The factors

$$L_v(s, \omega_v) := (1 - t_v(\omega) \, q_v^{-s})^{-1}$$

[5] where the notion of 'unramified' is not defined

are the *local L-factors* associated to the ω_v's.

Of course,

$$L^S(s,\omega) = L^S(0,\omega\omega_s),$$

so we could dispense with s in the notation. Usually, we will prefer to keep s and to assume that ω is a character, instead. In this case, the Euler product is absolutely convergent in the half plane $\mathrm{Re}(s) > 1$.

Now, the main goals are to prove the analytic properties of these functions, i.e.,

(1) to 'complete' the partial L-function $L^S(s,\omega)$ by including additional local L-factors for the primes $v \in S$, for example, for the archimedean places, and
(2) to prove the meromorphic analytic continuation and functional equation of the completed L–function.

These goal will be achieved by interpreting the local L-factors (resp. the global Euler product) as constants of proportionality between two naturally constructed basis elements of a one dimensional complex vector space of *distributions*!

§3. Local theory.

Eigendistributions. We fix a place v of $\boldsymbol{k}$ and, to streamline notation, we write $F = \boldsymbol{k}_v$. Let dx (resp. $d^\times x$) be a Haar measure on F (resp. $F^\times$). Observe that for $a \in F^\times$,

$$d(ax) = |a|\, dx$$

for the normalized absolute value, as defined above. Note that one then has $d^\times x = \mu\, |x|^{-1}\, dx$ for some positive constant μ. Also, for $s \in \mathbb{C}$, let

$$\omega_s(x) = |x|^s$$

be the associated character. If v is nonarchimedean, we write $\mathcal{O} = \mathcal{O}_v$, $\mathcal{P} = \mathcal{P}_v$, $q = q_v$, etc.. In this case, the character ω_s is unramified and only depends on the coset

$$s + \frac{2\pi i}{\log(q)}\mathbb{Z}.$$

Let

$$S(F) = \text{the space of Schwartz-Bruhat functions on } F.$$

$$S(F)' = \text{the space of tempered distributions on } F\text{, i.e.,}$$

the space of continuous linear functionals $\lambda : S(F) \to \mathbb{C}$.

Examples: (i) For F nonarchimedean, a complex valued function f on F is in $S(F)$ if and only if there is an integer $r \geq 0$ such that

$$\mathrm{supp}(f) \subset \mathcal{P}^{-r}$$

and

$$f = \text{ constant on cosets of } \mathcal{P}^r.$$

The space $S(F)$ is simply the complex vector space of all such functions and the tempered distributions are arbitrary $\mathbb{C}$–linear functionals on $S(F)$.
(ii) For $F = \mathbb{R}$ (resp. $\mathbb{C}$), $S(\mathbb{R})$ is the usual Schwartz space consisting of complex valued functions which, together with all of their derivatives, are rapidly decreasing, e.g.,

$$f(x) = h(x)\, e^{-\pi x^2}, \qquad \big(\text{ resp. } f(x) = h(x,\bar{x})\, e^{-2\pi x\bar{x}}\ \big)$$

for any polynomial $h \in \mathbb{C}[X]$ (resp. $h \in \mathbb{C}[X,Y]$). This is a Frechét space, etc. and the tempered distributions are the continuous linear functionals on it.

The multiplicative group $F^\times$ acts on $S(F)$ by

$$r(a)f(x) = f(xa),$$

for $x \in F$ and $a \in F^\times$, and on $S(F)'$ by

$$< r'(a)\lambda, f > = < \lambda, r(a^{-1})f >,$$

where $<\ ,\ >$ denotes the pairing of $S(F)'$ and $S(F)$.

Definition 3.1: For a quasicharacter ω of $F^\times$, let

$$S'(\omega) = \{\ \lambda \in S(F)' \mid r'(a)\lambda = \omega(a)\lambda\ \}$$

be the space of ω*–eigendistributions.*

The space $S'(\omega)$ can be analyzed 'geometrically'. Note that there are two orbits for the action of $F^\times$ on the additive group

$$F = \{0\} \cup F^\times.$$

Associated to this we have an inclusion

$$C_c^\infty(F^\times) \hookrightarrow S(F),$$

and, by duality, an exact sequence of distributions

$$0 \longrightarrow S(F)'_0 \longrightarrow S(F)' \longrightarrow C_c^\infty(F^\times)' \longrightarrow 0, \tag{3.1}$$

where $S(F)'_0$ is the subspace of distributions supported at 0. This sequence is compatible with the action of $F^\times$. Taking the ω–eigenspaces, we have the sequence

$$0 \longrightarrow S'(\omega)_0 \longrightarrow S'(\omega) \longrightarrow C_c^\infty(F^\times)'(\omega), \tag{3.2}$$

where $S'(\omega)_0$ is the space of ω-eigendistributions supported at 0. First, one has the following simple uniqueness result:

Lemma 3.2. *The space $C_c^\infty(F^\times)'(\omega)$ is one dimensional and is spanned by the distribution $\omega(x)\,d^\times x$. In particular, for any $\lambda \in S'(\omega)$, there is a complex number c such that*

$$\lambda\big|_{C_c^\infty(F^\times)} = c \cdot \omega(x)\,d^\times x,$$

i.e., if $\mathrm{supp}(f) \subset F^\times$ *is compact, then*

$$< \lambda, f > = c \cdot \int_{F^\times} f(x)\,\omega(x)\,d^\times x.$$

It remains to determine the space $S(F)'_0$ of tempered distributions supported at 0 and the subspace $S'(\omega)_0$. The delta distribution δ_0 defined by

$$< \delta_0, f > = f(0),$$

is obviously $F^\times$–invariant and supported at 0.

Lemma 3.3. *(i) If F is nonarchimedean, then*

$$S(F)'_0 = \mathbb{C} \cdot \delta_0 \subset S'(\omega_0),$$

and, for $\omega \neq \omega_0$, $S'(\omega)_0 = 0$.
(ii) If $F = \mathbb{R}$, then, setting $D = \frac{d}{dx}$,

$$S(F)'_0 = \bigoplus_{k=0}^{\infty} \mathbb{C} \cdot D^k \delta_0,$$

and

$$D^k \delta_0 \in S'(\omega),$$

where

$$\omega(x) = x^{-k}.$$

(iii) If $F = \mathbb{C}$, then, setting $D = \frac{\partial}{\partial x}$ and $\bar{D} = \frac{\partial}{\partial \bar{x}}$,

$$S(F)'_0 = \bigoplus_{k,l=0}^{\infty} \mathbb{C} \cdot D^k \bar{D}^l \delta_0.$$

Moreover,

$$D^k \bar{D}^l \delta_0 \in S'(\omega),$$

where

$$\omega(x) = x^{-k}\, \bar{x}^{-l}.$$

The fundamental local uniqueness result, whose proof we will sketch,2 is then:

Theorem 3.4. *For any quasicharacter ω,*

$$\dim S'(\omega) = 1.$$

Remark. (i) When $S'(\omega)_0 = 0$, i.e., when there are no ω–eigendistributions supported at 0, the reasoning above shows that the dimension of $S'(\omega)$ is at most one, so that only existence is at stake. This amounts to showing that the distribution $\omega(x)\, d^\times x$ in Lemma 3.2 can be extended to an ω-eigendistribution on $S(F)$. This will be established via local zeta integrals below.

(ii) When $S'(\omega)_0 \neq 0$, the problem will be to show that the distribution $\omega(x)\, d^\times x$ does *not* extend to an ω–eigendistribution. This more delicate fact also comes out of the analysis of local zeta integrals.

Example: Suppose that F is nonarchimedean. If ω is a *ramified* character, then $S'(\omega\omega_s)_0 = 0$, so that

$$\dim S'(\omega\omega_s) \leq 1$$

for all s. If ω is unramified with $\omega(x) = t^{\mathrm{ord}(x)}$, as above, then, by Lemma 3.3,

$$\dim S'(\omega\omega_s) \leq \begin{cases} 2 & \text{if } t \cdot q^{-s} = 1, \text{ and} \\ 1 & \text{otherwise.} \end{cases}$$

Zeta integrals. For a character ω of $F^\times$, i.e., a unitary quasicharacter, the *local zeta integral*

$$z(s, \omega; f) := \int_{F^\times} f(x)\, \omega\omega_s(x)\, d^\times x = \int_F f(x)\, \omega(x)\, \mu\, |x|^{s-1}\, dx,$$

is absolutely convergent for all $f \in S(F)$ provided $\mathrm{Re}(s) > 0$. Here $d^\times x = \mu\,|x|^{-1}\,dx$, as before. In this range, the distribution $f \mapsto z(z,\omega;f)$ defines a nonzero element

$$z(s,\omega) \in S'(\omega\omega_s).$$

Example: The unramified local theory. Suppose that F is nonarchimedean and suppose that ω is unramified, with $\omega(\varpi) = t$. If f has compact support in $F^\times$, then the integral

$$\int_{F^\times} f(x)\,\omega\omega_s(x)\,d^\times x$$

is entire. The idea, then, is to kill the support of an arbitrary $f \in S(F)$ by applying a suitable element of the group algebra $\mathbb{Z}[F^\times]$, more precisely, the element

$$\tau = [1] - [\varpi^{-1}], \tag{3.4}$$

where ϖ is a generator of $\mathcal{P}$. Since any $f \in S(F)$ is constant in a sufficiently small neighborhood of 0, we have

$$\big(r(\tau)f\big)(0) = 0,$$

and thus there is an element

$$z_o(s,\omega) \in S'(\omega\omega_s)$$

defined by

$$< z_o(s,\omega), f > := \int_{F^\times} \big(r(\tau)f\big)(x)\,\omega\omega_s(x)\,d^\times x. \tag{3.5}$$

On the other hand, in the halfplane $\mathrm{Re}(s) > 0$, where it is not necessary to kill support,

$$\begin{aligned} < z_o(s,\omega), f > &= \int_{F^\times} \big(\,f(x) - f(\varpi^{-1}x)\,\big)\,\omega\omega_s(x)\,d^\times x \\ &= \int_{F^\times} f(x)\,\omega\omega_s(x)\,d^\times x - \int_{F^\times} f(\varpi^{-1}x)\,\omega\omega_s(x)\,d^\times x \\ &= (1 - \omega\omega_s(\varpi))\cdot z(s,\omega;f) \\ &= (1 - t\,q^{-s})\cdot z(s,\omega;f) \\ &= L(s,\omega)^{-1}\cdot < z(s,\omega), f > . \end{aligned}$$

Strictly in terms of distributions, this says that, for $\mathrm{Re}(s) > 0$,

$$z(s,\omega) = L(s,\omega)\cdot z_o(s,\omega), \tag{3.6}$$

and hence gives the meromorphic analytic continuation of $z(s,\omega)$ to the whole s plane. Moreover (3.6) provides an interpretation of the local L-factor as a constant of proportionality between natural bases for the one dimensional space $S'(\omega\omega_s)$, away from the poles of $L(s,\omega)$.

Finally, to see why the distribution $z_o(s,\omega)$ is 'natural', we normalize the multiplicative Haar measure $d^\times x$ so that the units $\mathcal{O}^\times$ have volume 1. Then, we let

$$f^o = \text{ the characteristic function of } \mathcal{O}, \tag{3.7}$$

the ring of integers in F, and compute:

$$\begin{aligned} < z_o(s,\omega), f^o > &= \int_{F^\times} \big(f^o(x) - f^o(\varpi^{-1}x) \big)\, \omega\omega_s(x)\, d^\times x \\ &= \int_{\mathcal{O}^\times} d^\times x \\ &= 1, \end{aligned} \tag{3.8}$$

for all s. The point is that $r(\tau)f^o$ is the characteristic function of $\mathcal{O}^\times$. Thus, $z_o(s,\omega)$ is never zero and gives a basis vector for $S'(\omega\omega_s)$ for all s.

Remark. The L-factor as gcd. Relations (3.6) and (3.8) together show that $L(s,\omega)$ gives precisely the poles of the family of zeta integrals (3.3), i.e., that for any $f \in S(F)$, the function

$$\frac{z(s,\omega;f)}{L(s,\omega)} = < z_o(s,\omega), f >$$

is entire and that, for any given s, there is an f (specifically f^o) for which the value at s is nonzero.

Example: To complete the proof of Theorem 3.4 in the unramified case, there is still one point to check, namely that the space $S'(\omega_0)$ of invariant distributions, which contains $z_o(0,\omega_0)$, is actually only one dimensional. Here is the argument. There is an exact sequence of distributions

$$0 \longrightarrow \mathbb{C}\cdot\delta_0 \longrightarrow S(F)' \longrightarrow C_c^\infty(F^\times)' \longrightarrow 0 \tag{3.9}$$

and, passing to ω_0–eigendistributions, i.e., $F^\times$–invariant distributions, we have

$$0 \longrightarrow \mathbb{C}\cdot\delta_0 \longrightarrow S'(\omega_0) \longrightarrow \mathbb{C}\cdot d^\times x. \tag{3.10}$$

The point now is to show that, although $d^\times x$ is in the image of the quotient map in (3.9), it is not in the image of the quotient map in (3.10), i.e., the preimage of $d^\times x$ in $S(F)'$ is *not* invariant, and hence there is a nonzero map to $H^1(F^\times,\mathbb{C})$, where $F^\times$ acts trivially on $\mathbb{C}$.

Explicitly, the distribution $\lambda_0 \in S(F)'$ defined by

$$< \lambda_0, f > := < d^\times x, f - f(0) f^o > \tag{3.11}$$

gives a preimage of $d^\times x$ in $S(F)'$, which is clearly invariant under $\mathcal{O}^\times$ (since f^o is) and satisfies $< \lambda_0, f^o >= 0$. The distributions $r'(\varpi^{-1})\lambda_0$ and λ_0 agree on $C_c^\infty(F^\times)$ and hence differ by a distribution supported at 0, i.e.,

$$r'(\varpi^{-1})\lambda_0 = \lambda_0 + c\,\delta_0,$$

for some constant c. To determine c (which would be 0 if λ_0 were actually $F^\times$-invariant), we evaluate on f^o, using the group algebra element τ, as above:

$$\begin{aligned} -c &= < r'(\tau)\lambda_0, f^o > \\ &= < \lambda_0, r(\tau) f^o > \\ &= < d^\times x, r(\tau) f^o > \\ &= < z_o(0, \omega_0), f^o > \\ &= 1. \end{aligned}$$

Thus, on the two dimensional subspace of $S(F)'$ spanned by δ_0 and λ_0, $F^\times$ acts by the representation

$$\rho(x) = \begin{pmatrix} 1 & \operatorname{ord}(x) \\ & 1 \end{pmatrix}. \tag{3.12}$$

This finishes the proof of Theorem 3.4 in the unramified case.

Example: The ramified local theory. Suppose that F is nonarchimedean and ω is ramified, hence nontrivial on $\mathcal{O}^\times$. The conductor $c(\omega)$ of ω is the smallest integer c such that ω is trivial on $1 + \mathcal{P}^c$. Since any $f \in S(F)$ is constant in a neighborhood of 0, the integral

$$\int_{F^\times - \mathcal{P}^n} f(x)\, \omega\omega_s(x)\, d^\times x$$

is independent of n for n sufficiently large. This gives the analytic continuation of $z(s, \omega; f)$ to the whole s plane and hence gives a basis vector

$$z_o(s, \omega) := z(s, \omega),$$

for the one dimensional space $S'(\omega\omega_s)$ for all s. This finishes the proof of Theorem 3.4 in the ramified case. We set

$$L_v(s, \omega) = 1$$

in this case, and we note that, for

$$f^o(x) = \begin{cases} \omega(x)^{-1} & \text{if } x \in \mathcal{O}^\times, \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$< z_0(s,\omega), f^o > = \ 1.$$

Example: The archimedean case. Now it is the whole Taylor series at 0 of the function f which accounts for the poles of the zeta integral $z(s,\omega;f)$, rather than just the value $f(0)$ which arose in the non-archimedean case, cf. [bump], Proposition 3.1.7, p.271. It is easy to obtain the analytic continuation of the zeta integral $z(s,\omega;f)$ by integration by parts.

Here we just summarize the results.

If $F = \mathbb{R}$, then any quasicharacter has the form $\omega\omega_s$ where $s \in \mathbb{C}$ and

$$\omega(x) = x^{-a}, \qquad a = 0,\ 1.$$

Let

$$L(s,\omega) = \pi^{-\frac{s}{2}}\,\Gamma(\frac{s}{2}).$$

If $F = \mathbb{C}$, then any quasicharacter has the form $\omega\omega_s$ where $s \in \mathbb{C}$ and

$$\omega(x) = x^{-a}\bar{x}^{-b}, \qquad a,\ b \in \mathbb{Z}, \qquad \min(a,b) = 0.$$

Let

$$L(s,\omega) = (2\pi)^{1-s}\Gamma(s).$$

Proposition 3.5. *(i) The distribution*

$$z_o(s,\omega) := L(s,\omega)^{-1}\, z(s,\omega)$$

has an entire analytic continuation to the whole s plane and, for all s, defines a basis vector for the space $S'(\omega\omega_s)$.
(ii) If

$$f^o(x) = \begin{cases} x^a e^{-\pi x^2} & \textit{if } F = \mathbb{R}, \textit{ and} \\ x^a\,\bar{x}^b\, e^{-2\pi x\bar{x}} & \textit{if } F = \mathbb{C}, \end{cases}$$

then

$$< z_o(s,\omega), f^o > = \ 1.$$

Note that we are using the measure $d^\times x = |x|^{-1}\,dx$, where $|x|$ is the usual absolute value (resp. $|x| = x\bar{x}$) and dx is Lebesque measure (resp. twice Lebesque measure) when $F = \mathbb{R}$ (resp. $F = \mathbb{C}$). For (ii), see Lemma 8, p. 127 of [weilBNT].

Remark. In effect, the zeta distribution $z(s,\omega)$ has a meromorphic analytic continuation in s with simple poles at certain nonpositive integers. For example, when $F = \mathbb{R}$, the poles occur at the points $-r \in \mathbb{Z}_{\leq 0}$ with $r \equiv a \mod (2)$, and the residues are

$$\operatorname*{Res}_{s=-r}\ z(s,\omega) = c_r\, D^r \delta_0,$$

with a nonzero constant c_r. Here D is as in Lemma 3.3. The constant term in the Laurent expansion of $z(s,\omega)$ at $s = -r$ gives an extension to $S(F)$ of the distribution $x^{-r}\,d^\times x$ on $C_c^\infty(F^\times)$. A simple argument, as before, shows that this extension does *not* lie in the space $S'(\omega\omega_{-r})$. In the case $F = \mathbb{C}$, then the residue at $s = -r$ lies in $S'(\omega\omega_{-r})$, where $\omega\omega_{-r}(x) = x^{-a-r}\bar{x}^{-b-r}$ and is a nonzero multiple of $D^{a+r}\bar{D}^{b+r}\delta_0$.

Fourier Transforms. Fix a nontrivial character ψ of the additive group $F = F^+$, and identify F with its topological dual character group

$$\hat{F} := \mathrm{Hom}_{\mathrm{cont}}(F, \mathbb{C}^1),$$

by

$$F \overset{\sim}{\longrightarrow} \hat{F}, \qquad y \mapsto \big(\ x \mapsto \psi(xy)\ \big).$$

For F nonarchimedean, the conductor $\nu(\psi)$ of ψ is the largest integer ν such that ψ is trivial on $\mathcal{P}^{-\nu}$.

The Fourier transform

$$\hat{f}(x) = \int_F f(y)\,\psi(xy)\,dy$$

of a function $f \in S(F)$ is well defined and again lies in $S(F)$. (This is a fundamental property of the space of Schwartz–Bruhat functions.) The map $f \mapsto \hat{f}$ is an isomorphism $\hat{}: S(F) \overset{\sim}{\longrightarrow} S(F)$. There is a unique choice of the Haar measure, the self–dual measure with respect to ψ, such that Fourier inversion gives

$$\hat{\hat{f}}(x) = f(-x).$$

Supposing ψ to be given, we fix this choice of dx from now on. The Fourier transform of a distribution is defined by

$$<\hat{\lambda}, f> = <\lambda, \hat{f}> .$$

Lemma 3.6. *If $\lambda \in S'(\omega)$ is an ω–eigendistribution, then*

$$\hat{\lambda} \in S'(\omega^{-1}\omega_1),$$

i.e., $\hat{\lambda}$ is an $\omega^{-1}\omega_1$–eigendistribution.

Proof. An exercise!

Key point: By the uniqueness result Theorem 3.4, it follows that $\widehat{z_o(s,\omega)}$ is a constant multiple (depending on s and on ψ) of $z_o(1-s,\omega^{-1})$, i.e.,

$$\widehat{z_o(s,\omega)} = \epsilon(s,\omega,\psi)\, z_o(1-s,\omega^{-1}) \tag{3.13}$$

for a nonzero constant $\epsilon(s,\omega,\psi)$, the *local epsilon factor*, depending on s, ω and ψ. The following result gives explicit values of these factors.

Recall that, in the nonarchimedean case, $\nu = \nu(\psi)$ is the conductor of ψ. In the archimedean case, write

$$\psi(x) = \begin{cases} e(\beta x) & \text{for } \beta \in \mathbb{R}^\times, \text{ if } F = \mathbb{R}, \\ e(\beta x + \bar{\beta}\bar{x}) & \text{for } \beta \in \mathbb{C}^\times, \text{ if } F = \mathbb{C}, \end{cases}$$

where, for a real number t, $e(t) = e^{2\pi i t}$.

Proposition 3.7. (Local functional equations.)
(i) For F nonarchimedean and ω unramified:

$$\epsilon(s,\omega,\psi) = \omega(\varpi)^{-\nu}\, q^{(s-\frac{1}{2})\nu}.$$

(ii) For F nonarchimedean and ω ramified, with conductor c,

$$\epsilon(s,\omega,\psi) = \omega(\varpi)^{-\nu-c}\, \mathfrak{g}(\omega),$$

where $\mathfrak{g}(\omega)$ is the Gauss sum:

$$\mathfrak{g}(\omega) = .$$

(iii) For $F = \mathbb{R}$, write $\omega(x) = x^{-a}$ with $a = 0$, or 1. Then

$$\epsilon(s,\omega,\psi) = (\pi i)^a\, \pi^{\frac{1}{2}-s}.$$

(iv) For $F = \mathbb{C}$, write $\omega(x) = x^{-a}\bar{x}^{-b}$, for $a,\ b \in Z$ with $\min(a,b) = 0$. Then

$$\epsilon(s,\omega,\psi) = .$$

Proof. One simply evaluates both sides of (3.13) on the test function f^o, so that

$$\begin{aligned}\epsilon(s,\omega,\psi) &= <\widehat{z_o(s,\omega)}, f^o> \\ &= <z_o(s,\omega), \widehat{f^o}>.\end{aligned}$$

Thus one has to compute the Fourier transform of the standard function f^o. For example, in case (i), f^o is the characteristic function of $\mathcal{O}$, as above. Recall that $r(\tau)f^o$ is the characteristic function of $\mathcal{O}^\times$, and note that $r(\varpi^r)f^o$ is the characteristic function of $\mathcal{P}^{-r}$. Then,

$$\widehat{f^o} = \mathrm{vol}(\mathcal{O})\cdot r(\varpi^\nu)f^o$$

so Fourier inversion gives $\mathrm{vol}(\mathcal{O}) = q^{-\nu/2}$. Thus,

$$\begin{aligned}<\widehat{z_o(s,\omega)}, f^o> &=< z_o(s,\omega), q^{-\nu/2}\, r(\varpi^\nu)f^o> \\ &= q^{-\nu/2}\,\omega\omega_s(\varpi^\nu)^{-1}\cdot <z_o(s,\omega), f^o> \\ &= \omega(\varpi)^{-\nu}\, q^{(s-\frac{1}{2})\nu}\end{aligned}$$

whereas, $< z_o(1-s,\omega^{-1}), f^o >= 1$. This gives (i). The necessary calculations in the other cases can be found in Tate's thesis or in [weilBNT].

§4. Global theory.

Distributions. Returning to the number field $\boldsymbol{k}$ with its adèle ring $\mathbb{A}$, we define the space $S(\mathbb{A})$ of Schwartz–Bruhat functions on $\mathbb{A}$. This space contains all functions of the form

$$f = \otimes_v f_v$$

where $f_v \in S(\boldsymbol{k}_v)$ for all v and $f_v = f_v^o$, the characteristic function of $\mathcal{O}_v$ for almost all v. Note that

$$f(x) = \prod_v f_v(x_v)$$

is then well defined, as almost all factors are 1. The finite linear combinations of such factorizable functions are dense in $S(\mathbb{A})$. Again, $S(\mathbb{A})'$ is the space of tempered distributions, i.e., continuous linear functionals on $S(\mathbb{A})$.

Lemma 4.1. *Suppose that* $\{\lambda_v\}$ *is a collection of local tempered distributions* $\lambda_v \in S(\boldsymbol{k}_v)'$ *such that, for almost all* v, $<\lambda_v, f_v^o> = 1$. *Then there is a unique tempered distribution* $\lambda = \otimes_v \lambda_v$ *on* $S(\mathbb{A})$, *defined on factorizable functions by*

$$<\lambda, f> = \prod_v <\lambda_v, f_v>.$$

Conversely, if $\lambda \in S(\mathbb{A})'$ is nonzero, choose a factorizable f such that $\lambda(f) = 1$. Then, for any place v, we write

$$f = f_v \otimes f^v, \qquad f^v = \otimes_{w \neq v} f_w,$$

and we define a map

$$S(\boldsymbol{k}_v) \longrightarrow S(\mathbb{A}), \qquad g_v \mapsto g_v \otimes f^v,$$

by just varying the v component. Thus, we obtain local distributions

$$< \lambda_v, g_v > := < \lambda, g_v \otimes f^v >,$$

with $< \lambda_v, f_v^o > = 1$ for almost all v, and $\lambda = \otimes_v \lambda_v$.

The idèle group $\mathbb{A}^\times$ acts on $S(\mathbb{A})$ and $S(\mathbb{A})'$, and, for a quasicharacter ω of $\mathbb{A}^\times$, trivial on $\boldsymbol{k}^\times$, we let $S'(\omega)$ be the space of ω-eigendistributions. The factorization $\omega = \otimes_v \omega_v$ of the global quasicharacter gives rise to the decomposition

$$S'(\omega) = \otimes_v S_v'(\omega_v),$$

where we write $S_v' = S(\boldsymbol{k}_v)'$. The local uniqueness Theorem 3.4 then yields the global uniqueness result:

Theorem 4.2. *For all quasicharacter ω of $\mathbb{A}^\times$, the space of global ω-eigendistributions $S'(\omega)$ has dimension 1. For any $s \in \mathbb{C}$, the space $S'(\omega\omega_s)$ is spanned by the* **standard $\omega\omega_s$–eigendistribution**

$$z_o(s, \omega) := \otimes_v z_o(s, \omega_v).$$

Global zeta integrals and the completed L-function. For a character ω of $\boldsymbol{k}^\times \backslash \mathbb{A}^\times$ and a function $f \in S(\mathbb{A})$, we can define a global zeta integral by

$$z(s, \omega; f) = \int_{\mathbb{A}^\times} f(x)\, \omega\omega_s(x)\, d^\times x. \tag{4.1}$$

If f is factorizable, this is a product of the corresponding local integrals

$$z(s, \omega; f) = \prod_v z(s, \omega_v; f_v). \tag{4.2}$$

A key point here is that this integral *converges* precisely when the product of the local integrals converges. But since there is a finite set of places S such that for $v \notin S$, ω_v is unramified and $f_v = f_v^o$, we obtain convergence whenever the product

$$L^S(s, \omega) = \prod_{v \notin S} L_v(s, \omega_v)$$

is absolutely convergent, i.e., in the half plane $\mathrm{Re}(s) > 1$.

In terms of distributions, in the half plane $\mathrm{Re}(s) > 1$, we have $z(s,\omega) = \otimes_v z(s,\omega_v)$, and

$$z(s,\omega) = \Lambda(s,\omega) \cdot z_o(s,\omega), \tag{4.3}$$

where we define the **complete L-function**

$$\Lambda(s,\omega) := \prod_v L_v(s,\omega), \tag{4.4}$$

now including all local L-factors, and

$$z_o(s,\omega) = \otimes_v z_o(s,\omega_v), \tag{4.5}$$

is the standard $\omega\omega_s$–eigendistribution defined above. Thus, in the half-plane of convergence, the complete L-function can be viewed as a factor of proportionality between two 'natural' global eigendistributions.

Fourier transforms and the global functional equation. We fix a global additive character ψ of $\mathbb{A}$, trivial on $\boldsymbol{k}$. By resticting to local components, we can write

$$\psi(x) = \prod_v \psi_v(x_v).$$

For each nonarchimedean place v, the conductor $\nu_v := \nu(\psi_v)$ of ψ_v is defined. We then get an identification of $\mathbb{A}$ with its continuous character group, compatible with those defined by the components ψ_v in the local cases. We define a global Fourier transform

$$\widehat{\ } : S(\mathbb{A}) \xrightarrow{\sim} S(\mathbb{A}),$$

compatible with the local ones, i.e., if f is factorizable, then

$$\hat{f} = \otimes_v \hat{f}_v.$$

The ψ–self dual measure dx on $\mathbb{A}$ can be written as a product of the ψ_v–self dual measures on the $\boldsymbol{k}_v$'s. We define the Fourier transform of a distribution as before.

The desired analytic continuation and functional equation of the complete L-function $\Lambda(s,\omega)$ will follow from the conjunction of (i) the proportionality (4.3), (ii) the local functional equations of the $z_o(s,\omega_v)$'s and (iii) the following global identity:

Theorem 4.3. *The distribution $z(s, \omega)$, defined by the global zeta integral (4.1) for* $\mathrm{Re}(s) > 1$ *has a meromorphic analytic continuation to the whole s plane and satisfies the functional equation*

$$\widehat{z(s, \omega)} = z(1 - s, \omega^{-1}).$$

Assuming Theorem 4.3 for a moment, we immediately obtain the corresponding analytic continuation of $\Lambda(s, \omega)$ from (4.3), since the 'normalized' global distribution $z_o(s, \omega)$ is entire. Recall that the local functional equations have the form (3.13):

$$\widehat{z_o(s, \omega_v)} = \epsilon_v(s, \omega_v, \psi_v)\, z_o(1 - s, \omega_v^{-1}),$$

where $\epsilon_v(s, \omega_v, \psi_v)$ is given explicitly in Proposition 3.7. Note that almost all of the $\epsilon_v(s, \omega_v, \psi_v)$'s are 1, so we can define the **global epsilon factor**

$$\epsilon(s, \omega) = \prod_v \epsilon_v(s, \omega_v, \psi_v). \tag{4.6}$$

Here the additive character ψ has been omitted from the notation since, in fact, the product no longer depends on the choice made! Then,

$$\begin{aligned} \Lambda(s, \omega) \cdot \widehat{z_o(s, \omega)} &= \widehat{z(s, \omega)} \\ &= z(1 - s, \omega^{-1}) \\ &= \Lambda(1 - s, \omega^{-1}) \cdot z_o(1 - s, \omega^{-1}). \end{aligned} \tag{4.7}$$

Comparing the first and last expressions and using the local functional equations:

$$\Lambda(s, \omega) \cdot \epsilon(s, \omega, \psi) \cdot z_o(1 - s, \omega^{-1}) = \Lambda(1 - s, \omega^{-1}) \cdot z_o(1 - s, \omega^{-1}). \tag{4.8}$$

But, finally, the distribution $z_o(1-s, \omega^{-1})$ is nowhere zero by construction, and so we obtain **the functional equation**

Corollary 4.4.

$$\Lambda(1 - s, \omega^{-1}) = \epsilon(s, \omega) \cdot \Lambda(s, \omega).$$

Finally, the proof of Theorem 4.3 comes down to Riemann's classic argument based on Poisson summation! For this very standard calculation, we refer the reader to Tate's thesis, pp. 339–341, Weil [weilBNT], pp. 121–124, or Bump [bump], pp. 267–270.