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Solutions to the exam on Linear Algebra 1 (88-112), Fall 2006-7, "Moed" A

1. A) Let A, B be matrices s.t. $AB = 0$. We have then $(AB)^j = A(B^j) = 0$ for any $1 \leq j \leq k$. This means that each column of B is a solution to $AX = 0$. Hence $C(B) \subset Null(A)$. Reversing the argument we prove the opposite claim.

B) According to A) $AB = 0$ if and only if $C(B) \subset Null(A)$. We have $\dim C(B) = rank(B) = 2$ and $\dim Null(A) = n - rank(A) = 3 - 2 = 1$. Hence $\dim C(B) > \dim Null(A)$ and $C(B) \not\subset Null(A)$.

C) As in B) with $A = B$. $A^2 = 0$ implies that $C(A) \subset Null(A)$ and hence $\dim C(B) \leq \dim Null(A)$. We have $\dim Null(A) = n - rank(A)$ and $\dim C(A) = rank(A)$. This implies that $rank(A) \leq n - rank(A)$ or $rank(A) \leq n/2$.

2. A) 1. For an upper-triangular matrix A we have $A = -A^t$ implies $A = 0$. Hence $U \cap W = 0$.

2. We now prove that $V = U + W$. There are two ways. One is to compute the dimensions. An upper-triangular matrix is determined by its elements on and above the diagonal. There are $n(n+1)/2$ such elements and hence $\dim W = n(n+1)/2$ (for example standard matrices E_{ij} with $i \leq j$ form a basis of W.

On the other hand, antisymmetric matrices are determined by their elements above the diagonal. There are $n(n-1)/2$ such elements and hence $\dim U = n(n-1)/2$ (matrices $E_{ij} - E_{ji}$, $i \neq j$ for a basis of U). Since $U \cap W = 0$ form the theorem about the dimension of the sum we see that $\dim(U + W) = n(n-1)/2 + n(n+1)/2 = n = \dim V$. Hence $U + W = V$.

Another way to see that $V = U + W$ is to show that any matrix is a sum of an antisymmetric and of an upper-triangular matrices. For $A \in Mat_{n \times n}(F)$ denote by $L(A) \in Mat_{n \times n}(F)$ its lower-triangular part (i.e. A and $L(A)$ have the same elements below the diagonal and all elements of $L(A)$ on and above the diagonal are 0). Denote by $B = L(A) - L(A)^t$. We have then $A - B$ being upper-triangular and $B = -B^t$.

B) As $A = \{v_i\}$ spans V, for any $v \in V$ there are scalars a_i such that $\sum_i a_i v_i = v$. We obtain a nontrivial relation $\sum_i a_i v_i + (-1)v = 0$ for the set $A \cup v$. Hence it is linearly dependent.

3. A) Let P be a change of basis matrix from S to S' (i.e. $v'_i = \sum p_{ji} v_j$) and Q be a change of basis matrix from S' to S (i.e. $v_j = \sum q_{kj} v'_k$). Then the matrix QP gives an expression for vectors in the basis S' through itself (i.e. $v'_i = \sum_k (\sum_j q_{kj} p_{ji}) v'_k$). However, there is only one such an expression, namely $v'_i = v'_i$ since S' is linearly independent. Hence $QP = I$ and P is invertible.

B) $tr(AB) = \sum_i (AB)_{ii} = \sum_i (\sum_j a_{ij} b_{ji}) = \sum_i \sum_j a_{ij} b_{ji} = \sum_j (\sum_i b_{ji} a_{ij}) = \sum_j (BA)_{jj} = tr(BA)$.

4. Any vector space of dimension 2 is isomorphic (after a choice of basis) to $(\mathbb{Z}_2)^2$. If $U \subset V$ then $\dim U \leq 2$. If $\dim U = 0$ then $U = 0$, if $\dim U = 2$ then $U = V$. If $\dim U = 1$ the it is spanned by a non-zero vector (and consists of multiples of this vector). There are 3 different non-zero vectors in $(\mathbb{Z}_2)^2$: $(1,0),(0,1),(1,1)$. These vectors are not multiples of each other and hence there are 3 different subspaces of the dimension 1. Hence there are 5 different subspaces in $(\mathbb{Z}_2)^2$.

5. We have $A \cdot adj(A) = detA \cdot I_{n\times n}$ or $adj(A) = det(A)A^{-1}$. Also $det(\alpha A) = \alpha^n det(A)$. Hence $adj(adj(A)) = adj(det(A)A^{-1}) = det(det(A)A^{-1})(det(A)A^{-1})^{-1}$.

We have $det(det(A)A^{-1}) = det(A)^n det(A)^{-1}$ and $(det(A)A^{-1})^{-1} = det(A)^{-1}A$. Hence we arrive at $adj(adj(A)) = det(A)^{n-2}A$.