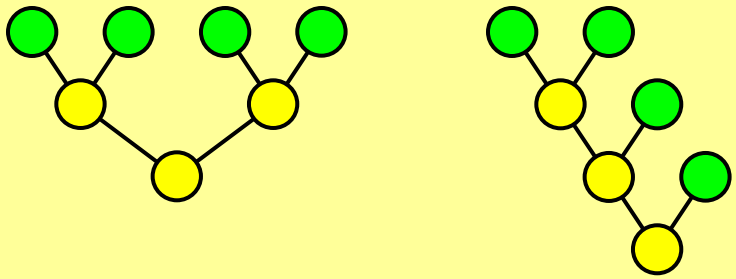
Testimony Combinations: Mathematical Aspects of a Talmudic Problem



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Some Basics in Jewish Law

- Two witnesses are needed to enforce payment of a claimed debt
- One witness suffices only to require an oath

Some Surprises...

- A witness on a loan on Sunday
  and a witness on a loan on Monday
  - can together enforce payment!
- A witness on a loan of 100 on Sunday
  and a witness on a loan of 200 on Monday
  - can together enforce payment of 100!

Notation

For testimony amounts

$$a, b$$

the *payment value* is

$$p := \min(a, b)$$

and the *oath value* is

$$q := \max(a, b) - \min(a, b) = |a - b|$$

How about more witnesses?

- *[Shulchan Aruch, Choshen Mishpat 30,3]*

  Alice claims that Bob owes her 1500. She brings 5 witnesses: one saying “I saw a loan of 100”, one saying “I saw a loan of 200”, one saying “300”, one “400” and one “500”. If, according to the witnesses, the loans took place on different times - then Bob must pay Alice 700 and take an oath on 100.

Why?

Why?

- *[Nachmanides = רמב"ן]*

  Combine the witness of 200 with that of 300, to make Bob pay 200 out of 300. Then combine the witness of 400 with that of 500, to make him pay 400 out of 500. Then combine the witness of 100 with that of 500 on the 100 remaining in his testimony... or with that of 300 on the 100 remaining in his testimony.

Why? (*Nachmanides*, cont.)

- There is another way: Combine the witness of 400 with that of 500 to make Bob pay 400. Then combine the 100 remaining from the testimony of 500 with the witness of 300 to make him pay 100. Then combine the witness of 200 with the 200 remaining from the witness of 300 to make him pay 200. Finally, the witness of 100, who is not combined, requires an oath on 100.

Nachmanides' Principle

- Increase the amount (*payment value*) as much as possible, by combining testimonies in an optimal way
  - "וכן עיקר - להעלות החשבון בכל מה שנוכל ולהצריף כל עדות העדים לתועלת התובע"

Why? (another way)

- *[Nimukey Yoseph]*

  Combine the witness of 200 with that of 300 (for an outcome of 200). Combine the witness of 400 with that of 500 (for an outcome of 400). Then combine the 100 remaining from the witness of 300 to the 100 remaining from the witness of 500 (for an outcome of 100).

Is there a difference?

- *[Bayit Chadash = R. Yoel Sirkis]*

  Perhaps Nachmanides cannot accept the combination suggested by Nimukey Yoseph, since he does not permit to combine a 100, which remained from a previous combination, with another 100, which also remained from a combination.
- Namely: each combination should involve at least one “original witness”.

Payment Value and Oath value

- Let $a_1, a_2, ..., a_n$ be testimony values, and fix a combination pattern. Let p be the resulting payment value, and let q be the oath value.

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- **Example:**

$$100 + 200 + 300 + 400 + 500 = 2 \cdot 700 + 100$$

Payment Value and Oath value

- **Proof:** Each penny can either combine with another penny, contributing 1 to p, or not combine – and contribute to q.
- **Corollary:** Maximizing the Payment Value is equivalent to minimizing the Oath Value.
- We shall concentrate on minimizing the Oath Value q.

Algebraic Structure

- Let S be the set of nonnegative real numbers (or nonnegative integers). For $a, b \in S$ denote

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(This is the Oath Value for a, b)

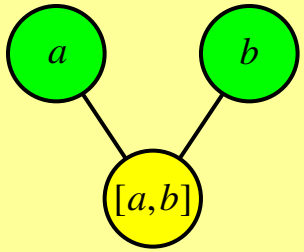
Algebraic Structure

- **Claim:**
- 1. $[a,b]=[b,a]$
- 2. $[a,0]=[0,a]=a$
- 3. $[a,a]=0$

- **Note:** $[\cdot,\cdot]$ is *not* associative!

$$[[100,200],300]=200\neq 0=[100,[200,300]]$$

Description by a Binary Tree



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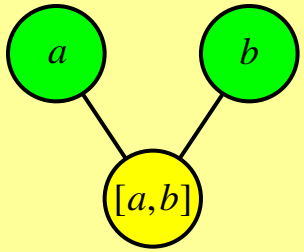

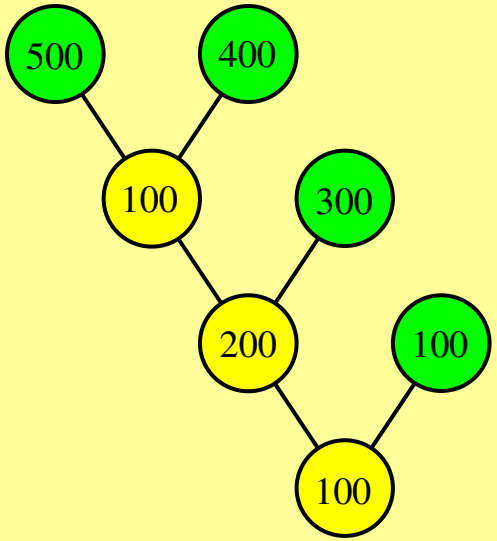

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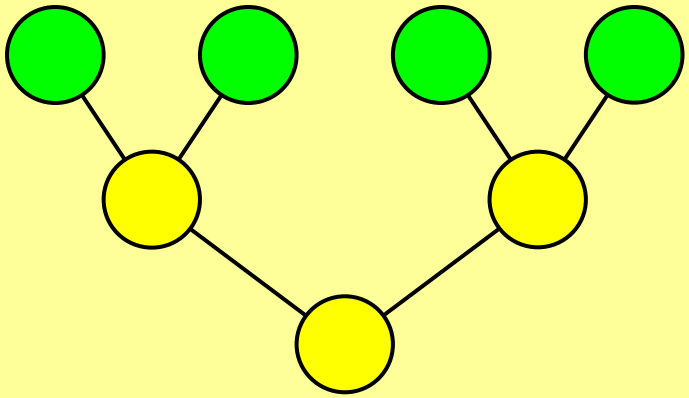
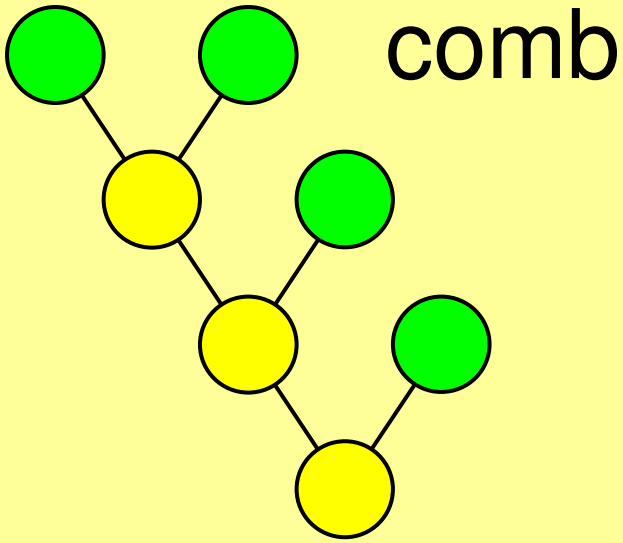
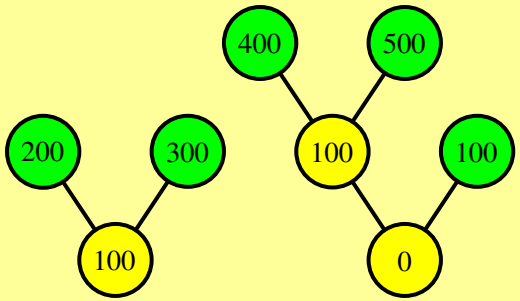
The *Bayit Chadash* explanaion of *Nachmanides*

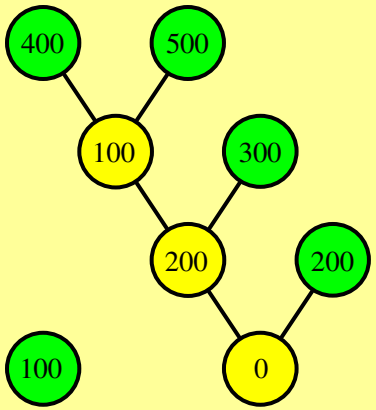




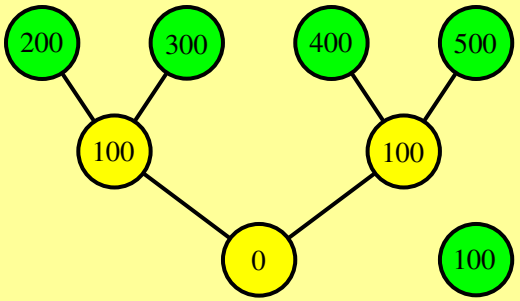
Binary Forests



Nachmanides 1



Nachmanides 2



Nimukey Yoseph

Forests and Trees

- **Claim:** The minimal Oath Value can always be obtained by a binary tree (i.e., a connected forest).

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- **Question:** Can *Nachmanides* (a la *Bayit Chadash*) restrict to a binary (connected) comb?

Binary Trees and Combs

- **Main Theorem:** Any Oath Value obtainable by a binary tree is actually obtainable by a binary comb. Thus *Nachmanides* = *Nimukey Yoseph*, *eventually*.

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- **Main Theorem:** Any Oath Value obtainable by a binary tree is actually obtainable by a binary comb. Thus *Nachmanides* = *Nimukey Yoseph*, *eventually*.
- **Definition:** A number is a *feasible Oath Value* if there exists a binary tree (comb) that produces it as an Oath Value.

Feasible Oath Values

- **Theorem:** Given testimonies $a_1, a_2, ..., a_n$, a signed sum

$$q = \varepsilon_1 a_1 + \varepsilon_2 a_2 + ... + \varepsilon_n a_n$$

where

$$\varepsilon_1, \varepsilon_2, ..., \varepsilon_n \in \{+1, -1\}$$

is a feasible Oath Value iff

$$0 \le q \le \max\{a_i \mid \varepsilon_i > 0\}$$

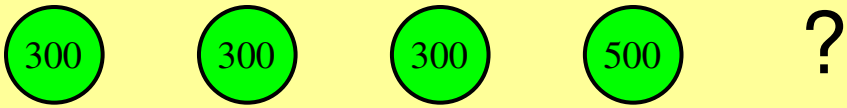
Feasible Oath Values

- **Example:**

$$q = 400 = 300 + 300 + 300 - 500$$

is not a feasible Oath Value, even though

$q < 500$



Related Issues

- The Partition hyperplane arrangement
- The Partition Problem (NP-complete)
- The Karmarkar-Karp “differencing method”
- A probabilistic “rationale”

Thank You!