Non-Crossing Partitions and a Diameter Problem

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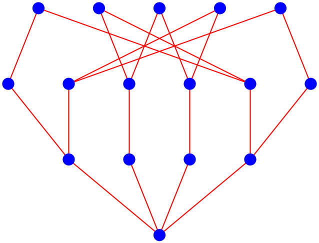
SaganFest: Gainesville, FL, March 2014



Outline

Non-crossing partitions

Maximal chains

EL labeling

Maximal elements

Radius and diameter

Open problems

To Bruce

Non-crossing partitions

$\{1,2\},\{3,5,6\},\{4\}$

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Non-crossing partitions

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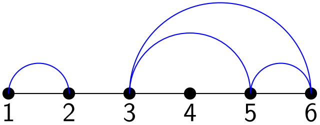

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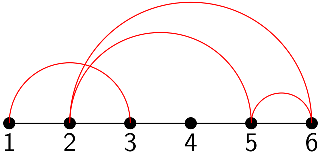

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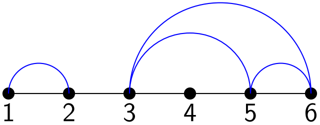

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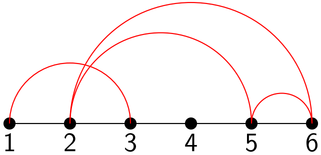
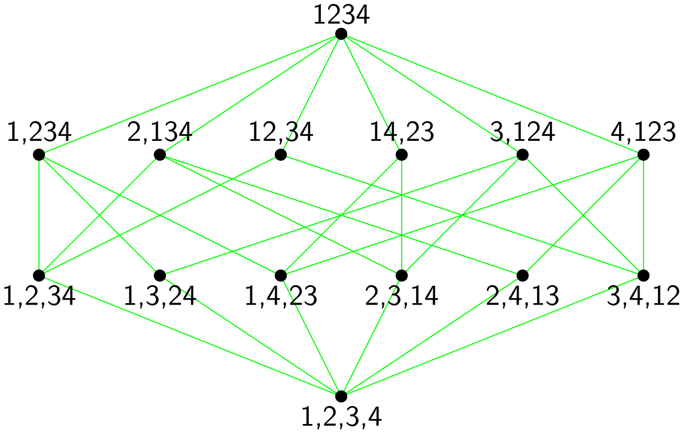


crossing partition

$NC_n :=$ lattice of all non-crossing partitions of $\{1,\ldots,n\}$,
ordered by refinement.

Non-crossing partitions



NC_4

Maximal chains in NC_n and the Hurwitz graph

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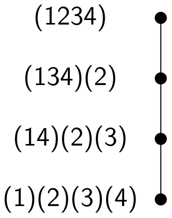

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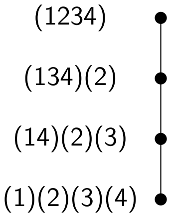
The Hurwitz graph $G_T(n)$: vertex set $= F_n$



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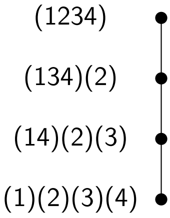

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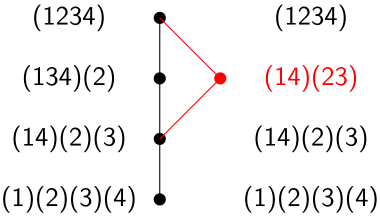
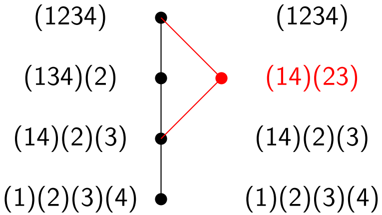


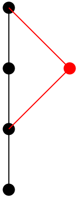
Maximal chains in NC_n and the Hurwitz graph

Alternative description:



Maximal chains in NC_n and the Hurwitz graph

Alternative description:

$(23)(34)(14) =$	(1234)		(1234)
$(34)(14) =$	$(134)(2)$		$(14)(23)$
$(14) =$	$(14)(2)(3)$		$(14)(2)(3)$
	$(1)(2)(3)(4)$		$(1)(2)(3)(4)$

Maximal chains in NC_n and the Hurwitz graph

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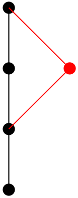

(1234) $= (24)(23)(14)$

(14)(23) $= (23)(14)$

(14)(2)(3) $= (14)$

(1)(2)(3)(4)

Maximal chains in NC_n and the Hurwitz graph

Alternative description: [Hurwitz 1891]

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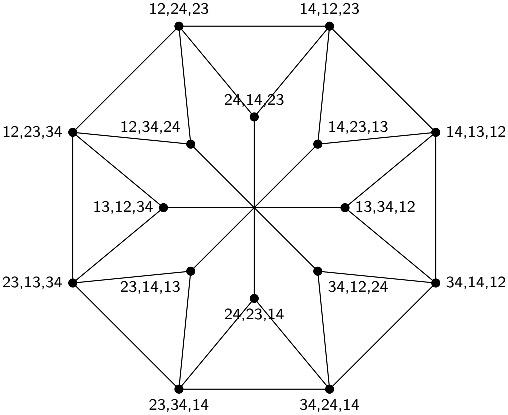
$w = (t_1, \ldots, t_i, t_{i+1}, \ldots, t_{n-1}) \in F_n$ is adjacent to its right shifts

$$R_i(w) := (t_1, \ldots, t_{i-1}, t_i t_{i+1} t_i, t_i, t_{i+2}, \ldots, t_{n-1})$$

and to its left shifts

$$L_i(w) := (t_1, \ldots, t_{i-1}, t_{i+1}, t_{i+1} t_i t_{i+1}, t_{i+2}, \ldots, t_{n-1}).$$

The Hurwitz graph $G_T(4)$



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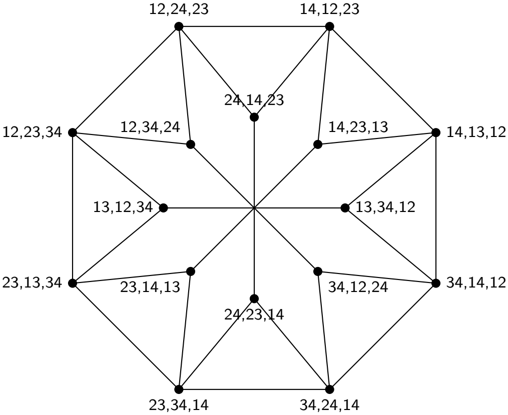
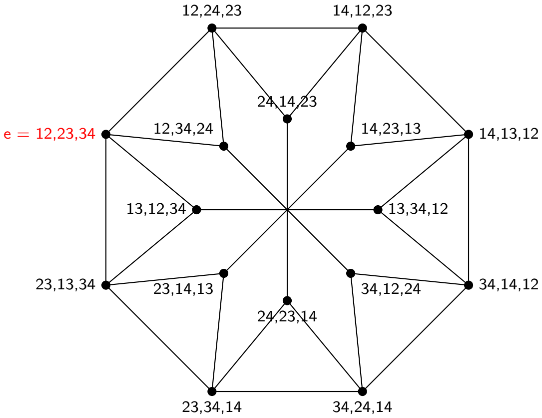


Question: Diameter of $G_T(n)$? Radius?

The poset $Weak(F_n)$



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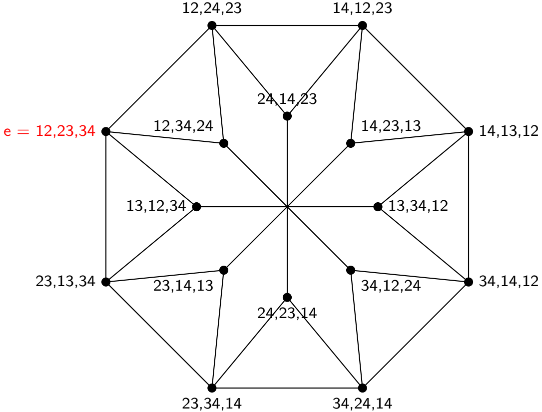
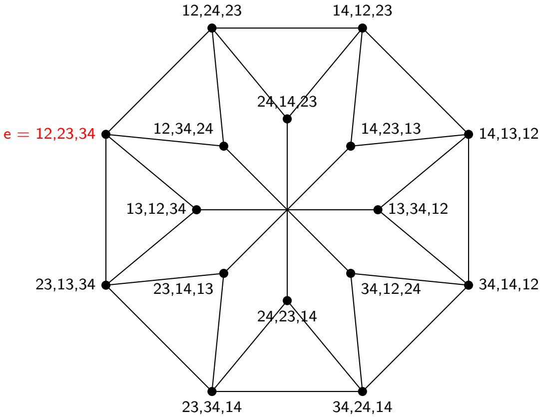


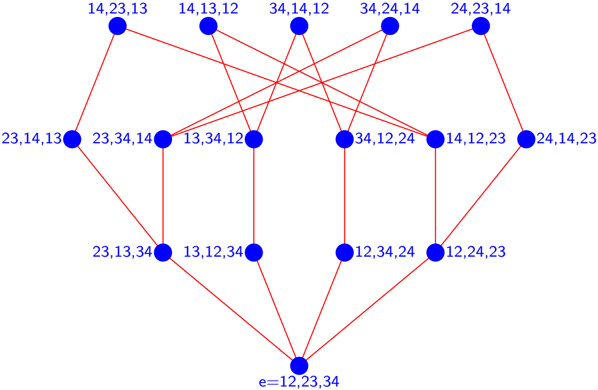
Choose a special vertex e

The poset $Weak(F_n)$



Choose a special vertex e and rank all vertices by their distance from e.

The poset $Weak(F_n)$



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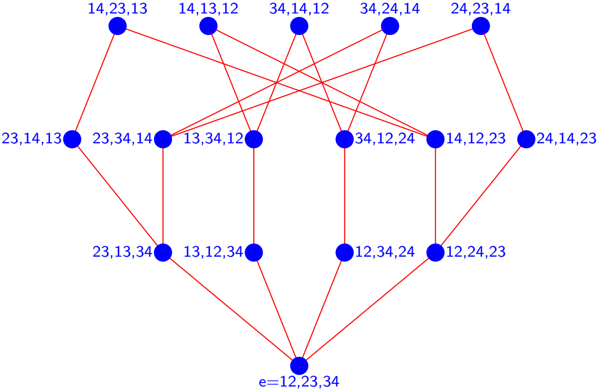

$|F_n| = n^{n-2}$

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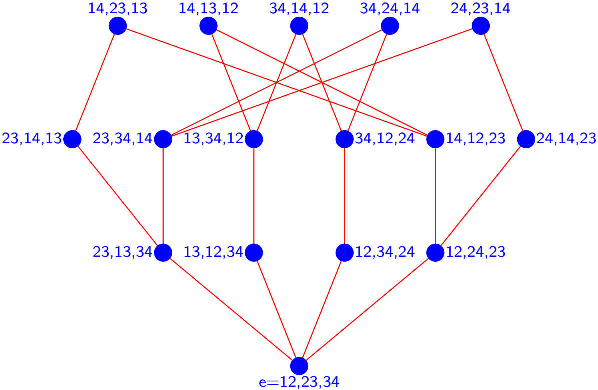
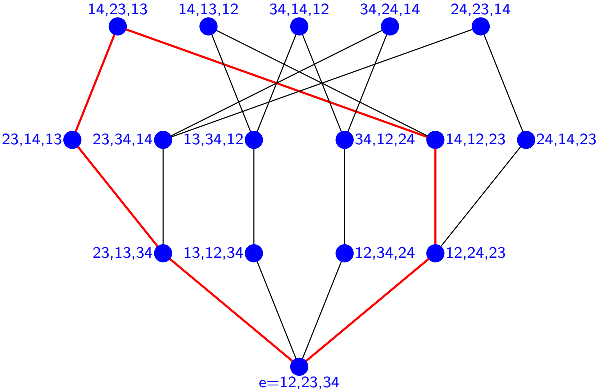


$$|F_n| = n^{n-2} \qquad |\max(F_n)| = C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$$

The map ϕ


14,23,13
14,13,12
34,14,12
34,24,14
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23,14,13
23,34,14
13,34,12
34,12,24
14,12,23
24,14,23
23,13,34
13,12,34
12,34,24
12,24,23
e=12,23,34

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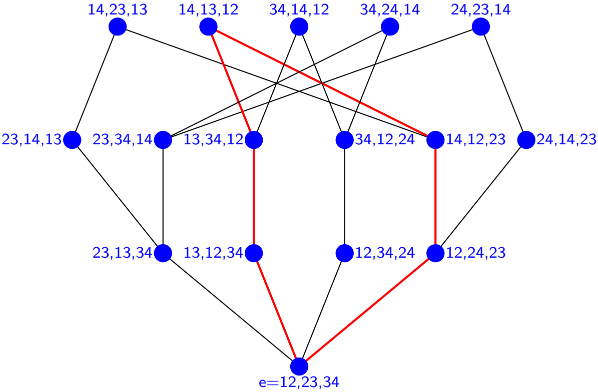
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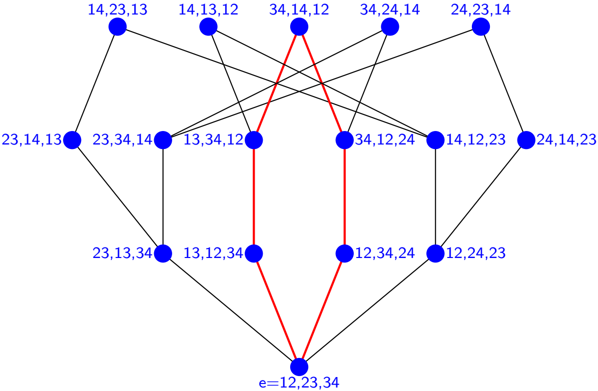
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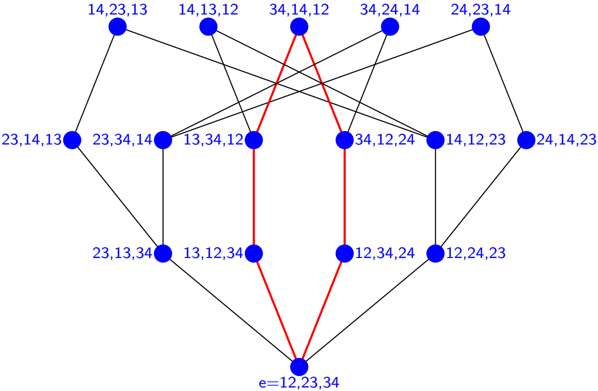

Theorem: $\exists\phi : F_n \to S_{n-1}$ whose restriction to any maximal interval $[e, w_0]$ is a poset isomorphism: $[e, w_0] \cong Weak(S_{n-1})$.

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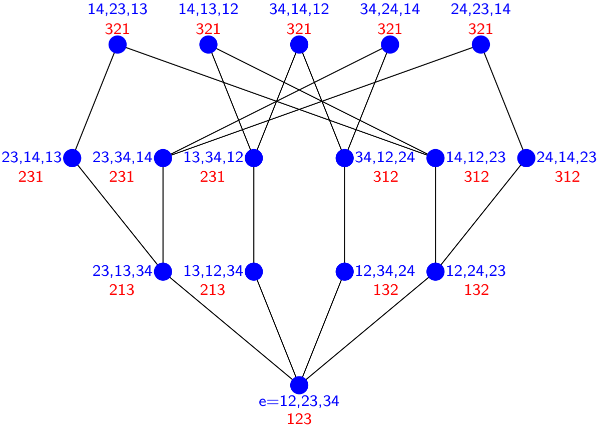

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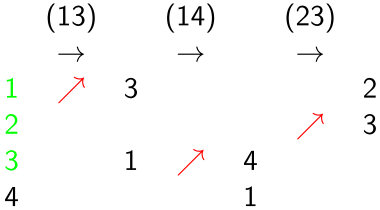

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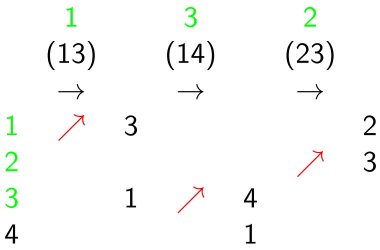

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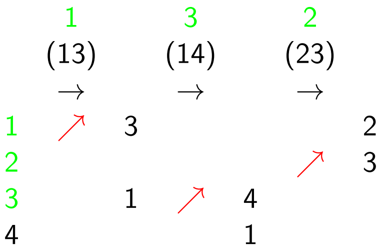
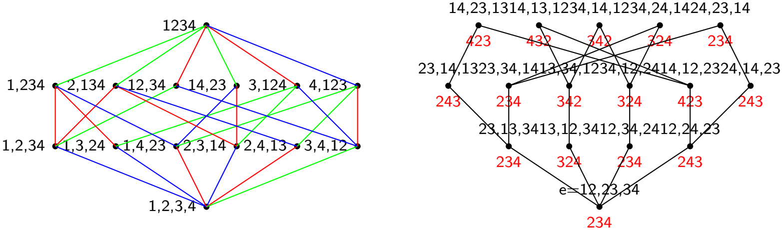
$$\underset{2}{(23)}\quad\underset{3}{(14)}\quad\underset{1}{(13)}\quad = (1234)$$



EL labeling of NC_n

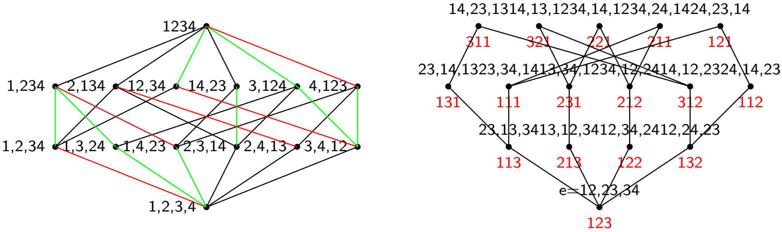
1 —— 2 —— 3 —— 4 ——

Color Legend



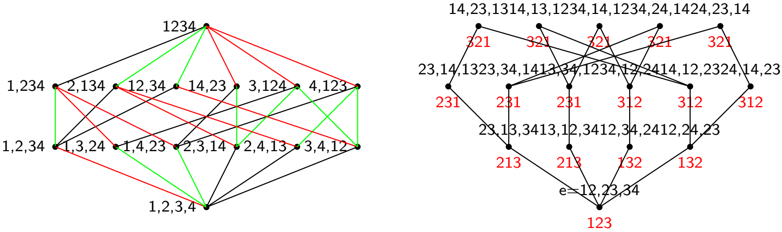
Björner's labeling of NC_4 and F_4

EL labeling of NC_n



Stanley's labeling of NC_4 and F_4

EL labeling of NC_n



ϕ labeling of NC_4 and F_4

EL labeling of NC_n

Merging a block $A = A_0 \cup A_1$ with a block B, where $\min A_0 < \min B < \min A_1$ (A_1 may be empty):

$$\text{Björner: label} = \min B$$

$$\text{Stanley: label} = \max A_0$$

$$\phi\text{: label} = \begin{cases} \max B, & \text{if } A_1 \neq \emptyset; \\ \max A, & \text{if } A_1 = \emptyset, \end{cases}$$

$$\text{since } (i,j) = \begin{cases} (\min B, \min A_1), & \text{if } A_1 \neq \emptyset; \\ (\min A, \min B), & \text{if } A_1 = \emptyset. \end{cases}$$

EL labeling of NC_n

Björner's labeling

Stanley's labeling

ϕ labeling

The 0-Hecke algebra

The 0-*Hecke algebra* $\mathcal{H}_{n-1}(0)$ is an associative algebra over $\mathbb{Q}$ generated by

$$\{T_i : 1 \le i \le n-2\}$$

with defining relations

$$T_i^2 = T_i \qquad (1 \le i \le n-2),$$

$$T_i T_j = T_j T_i \qquad (|i-j| > 1)$$

and

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \qquad (1 \le i \le n-3).$$

$\mathcal{H}_{n-1}(0)$ action on F_n

The descent set of a permutation π is

$$\mathrm{Des}(\pi) := \{i \mid \pi(i) > \pi(i+1)\}.$$

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Define an action ρ of $\mathcal{H}_{n-1}(0)$ on F_n by:

$$\rho(T_i)(w) := \begin{cases} R_i(w), & \text{if } i \in \mathrm{Des}(\phi(w)) \text{ and } R_i(w) < w; \\ L_i(w), & \text{if } i \in \mathrm{Des}(\phi(w)) \text{ and } L_i(w) < w; \\ w, & \text{if } i \notin \mathrm{Des}(\phi(w)). \end{cases}$$

for any $1 \leq i \leq n-2$ and $w \in F_n$.

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for any $1 \leq i \leq n-2$ and $w \in F_n$.

Theorem: ρ is a faithful $\mathcal{H}_{n-1}(0)$ action on F_n. Each maximal interval is ρ-invariant.

Alternating non-crossing trees

The number of maximal elements in F_n is the Catalan number C_{n-1}. There is a nice bijection with alternating non-crossing trees.

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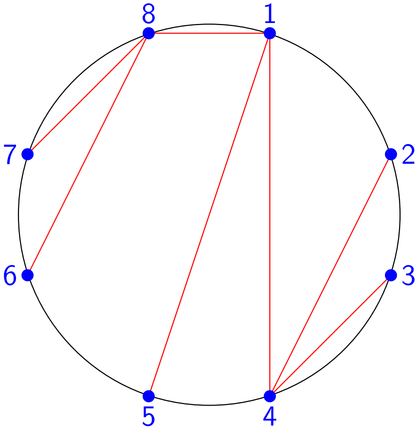
A tree with integer-labeled vertices is alternating if the vertices along any path in it form an alternating sequence:

$$\ldots > i < j > k < \ldots;$$

equivalently, if the neighbors of each vertex i are all larger, or all smaller, than i.

Alternating non-crossing trees



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Theorem: [Gelfand, Graev and Postnikov] The number of alternating non-crossing trees on n vertices is C_{n-1}.

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Theorem: The Goulden-Yong correspondence gives a bijection between the maximal elements in F_n and the alternating non-crossing trees on n vertices.

Inversions: right, left and neutral

For $w = (t_1, \ldots, t_{n-1}) \in F_n$, an inversion of $\pi := \phi(w)$ is a pair (i, j) such that $i < j$ but $\pi^{-1}(i) > \pi^{-1}(j)$.

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$$\mathrm{Inv}(w) = \mathrm{Inv}_R(w) \cup \mathrm{Inv}_L(w) \cup \mathrm{Inv}_N(w)$$

Let $\mathrm{inv}(w) := |\,\mathrm{Inv}(w)|$ etc.

q,t-enumeration

Carlitz and Riordan defined a q-Catalan number $C_n(q)$ using the recursion

$$C_{n+1}(q) := \sum_{k=0}^{n} q^{(k+1)(n-k)} C_k(q) C_{n-k}(q) \qquad (n \geq 0)$$

with $C_0(q) := 1$. For combinatorial interpretations of this number see, e.g., [Fürlinger and J. Hofbauer, Butler, Sagan-Savage] and sequence A138158 in [Sloane OEIS].

Define (q,t)-Catalan numbers by

$$\tilde{C}_{n+1}(q,t) := \sum_{k=0}^{n} q^k t^{n-k} \tilde{C}_k(q,t) \tilde{C}_{n-k}(q,t) \qquad (n \geq 0)$$

with $\tilde{C}_0(q,t) = 1$.

q,t-enumeration

Observation: For every $n \geq 0$,

$$\tilde{C}_n(q,t) = \tilde{C}_n(t,q)$$

and

$$C_n(q) = q^{\binom{n}{2}} \tilde{C}_n(q^{-1}, 1).$$

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Theorem:

$$\sum_{w \in \max(F_{n+1})} q^{\mathrm{inv}_R(w)} t^{\mathrm{inv}_L(w)} = \tilde{C}_n(q,t)$$

Radius and diameter

The radius of a graph G is

$$\text{rad}(G) := \min_{v \in V} \max_{w \in V} d(v, w)$$

and its diameter is

$$\text{diam}(G) := \max_{v,w \in V} d(v, w).$$

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Theorem: The radius of the Hurwitz graph

$$\mathrm{rad}(G_T(n)) = \binom{n-1}{2}.$$

Radius and diameter

Clearly, for any graph G,

$$\operatorname{rad}(G) \leq \operatorname{diam}(G) \leq 2\operatorname{rad}(G).$$

Corollary: The diameter of $G_T(n)$ satisfies

$$\binom{n-1}{2} \leq \operatorname{diam}(G_T(n)) \leq 2\binom{n-1}{2}.$$

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The proof uses a "bubble sort" algorithm to transform any two given words $v, w \in F_n$ to a common word by applying various R_j and L_j operators.

Open problems

- Find more accurately the diameter of the Hurwitz graph.

  Conjecture: The diameter of $G_T(n)$ is $\binom{n-1}{2} + O(n)$.

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- Find more accurately the diameter of the Hurwitz graph.

  Conjecture: The diameter of $G_T(n)$ is $\binom{n-1}{2} + O(n)$.

- Same for the Hurwitz graph of other Coxeter groups.

  Conjecture: The radius of the Hurwitz graph of type B_n is $\binom{n}{2} + 1$.

Dates

March

S	M	T	W	Th	F	Sat
23	24	25	26	27	28	29

SaganFest

Dates

March

S	M	T	W	Th	F	Sat
23	24	25	26	27	28	29

SaganFest Birthday

Dates

March

S	M	T	W	Th	F	Sat
23	24	25	26	27	28	29

SaganFest (24–25) Birthday (29)

	Gregorian
This year:	29 March 2014

Dates

March

S	M	T	W	Th	F	Sat
23	24	25	26	27	28	29

SaganFest Birthday

	Gregorian		Hebrew
This year:	29 March 2014	=	27 Adar B 5774 (Sat)

Dates

March

S	M	T	W	Th	F	Sat
23	24	25	26	27	28	29

SaganFest (24–25) Birthday (29)

	Gregorian		Hebrew
This year:	29 March 2014	=	27 Adar B 5774 (Sat)
60 years ago:	29 March 1954	=	24 Adar B 5714 (M)

Dates

March

S	M	T	W	Th	F	Sat
23	24	25	26	27	28	29

SaganFest (24–25) Birthday (29)

	Gregorian		Hebrew
This year:	29 March 2014	=	27 Adar B 5774 (Sat)
60 years ago:	29 March 1954	=	24 Adar B 5714 (M)
After sunset:			25 Adar B 5714 (T)

Dates

March

S	M	T	W	Th	F	Sat
23	24	25	26	27	28	29
	SaganFest	SaganFest		Birthday (H)		Birthday (G)

	Gregorian		Hebrew
This year:	29 March 2014	=	27 Adar B 5774 (Sat)
60 years ago:	29 March 1954	=	24 Adar B 5714 (M)
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Dates

March/Adar B

S	M	T	W	Th	F	Sat
23/21	24/22	25/23	26/24	27/25	28/26	29/27
	SaganFest			Birthday (H)		Birthday (G)

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To Bruce

To Bruce

$|A_5| = 60$

To Bruce

$$|A_5| = 60$$
$$|S_5| = 120$$

To Bruce

$$|A_5| = 60$$
$$|S_5| = 120$$

Have happy and fruitful next 60 years!

To Bruce

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