Matrices with Integer Eigenvalues

$$\begin{pmatrix} 4 & 1 & 1 & 0 & -1 \\ 1 & 4 & 0 & 1 & -1 \\ 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ -1 & -1 & 0 & 0 & 3 \end{pmatrix}$$

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A Recent Monthly Paper

Almost All Integer Matrices Have No Integer Eigenvalues, G. Martin and E. B. Wong, Amer. Math. Monthly 116 (2009), 588-597.

We want to deal with (specific) exceptions.

Highlights

- Certain matrices with integer entries have, conjecturally, integer eigenvalues only.
- Partial results are known.
- The multiplicity of the zero eigenvalue has an algebraic interpretation.

A Signed Graph: Vertices

- $[0,r] := \{0,1,\ldots,r\}$
- For $k \geq 0$ and $0 \leq a,b \leq k+1$ let
  $\Omega_k(a,b) :=$ set of all pairs (U,V) s.t.
  $$U,V \subseteq [0,k], \quad |U| = a, \quad |V| = b.$$
- Weight: $wt(U,V) := \sum_{u \in U} u + \sum_{v \in V} v$
- $\Omega_k(a,b,w) :=$ set of all $(U,V) \in \Omega_k(a,b)$ with weight w.

A Signed Graph: Vertices (cont.)

- Example: $a = b = 2, \quad w = 4.$
- $k = 0,1: \quad \Omega_k(2,2,4) = \varnothing$
- $k = 2: \quad \Omega_k(2,2,4) = \{(12,01),(02,02),(01,12)\}$
- $k = 3: \quad \Omega_k(2,2,4) = \{(03,01),(12,01),(02,02),$
  $(01,12),(01,03)\}$

- Note: $w_{\min} = \binom{a}{2} + \binom{b}{2}, \quad w_{\max} = k(a+b) - \binom{a}{2} - \binom{b}{2}$

A Signed Graph: Edges

- For $(U,V),(\tilde{U},\tilde{V}) \in \Omega_k(a,b,w)$ and $u \in U, v \in V, z \in \mathbb{Z},$ write $(U,V) \underset{(u,v,z)}{\sim} (\tilde{U},\tilde{V})$ if

1. $\tilde{U} = (U \setminus \{u\}) \cup \{u+z\}$
2. $\tilde{V} = (V \setminus \{v\}) \cup \{v-z\}$
3. $u+v \leq k$

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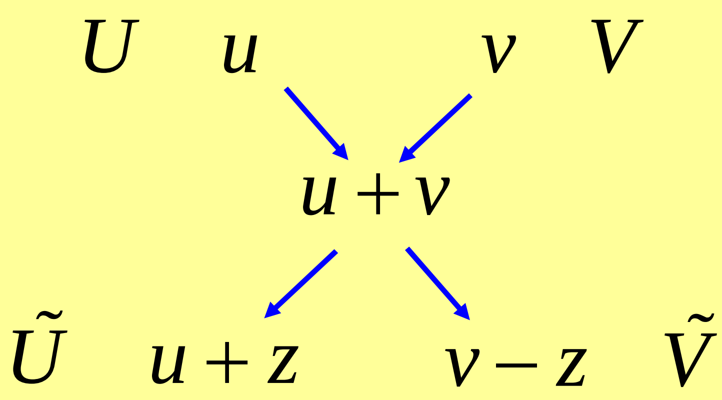

A Signed Graph

- $G_k(a,b,w)$ = graph with vertex set $\Omega_k(a,b,w)$ and edges corresponding to the various $(U,V) \underset{(u,v,z)}{\sim} (\tilde{U},\tilde{V})$.
- Loops and multiple edges may occur.
- Attach signs to edges:

Signs

- For $u = (u_1, \ldots, u_a), \quad \tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_a)$ define:
  $\varepsilon(u, \tilde{u}) = 0$ if u or $\tilde{u}$ has repeated elements;
  $\varepsilon(u, \tilde{u}) = sign(\sigma) sign(\tau)$ if
  $$u_{\sigma(1)} < \ldots < u_{\sigma(a)}, \quad \tilde{u}_{\tau(1)} < \ldots < \tilde{u}_{\tau(a)}.$$
- For $U = \{u_1, \ldots, u_a\}, \quad \tilde{U} = \{u_1, \ldots, u_i + z, \ldots, u_a\}$
  where $u_1 < \ldots < u_a$, define
  $$\varepsilon(U, \tilde{U}) := \varepsilon((u_1, \ldots, u_a), (u_1, \ldots, u_i + z, \ldots, u_a)).$$
- $\varepsilon((U, V), (\tilde{U}, \tilde{V})) := \varepsilon(U, \tilde{U}) \cdot \varepsilon(V, \tilde{V})$

Signed Adjacency Matrix

- $T_k(a,b,w) =$ the (signed) adjacency matrix of the graph $G_k(a,b,w)$.

  <u>Note:</u>

- Diagonal elements are nonnegative integers.

  Off-diagonal elements are $0, 1$ or -1.

Signed Adjacency Matrix

- Example: $k = 3, \quad a = 2, \quad b = 3, \quad w = 8$

$$T_k(a,b,w) = \begin{pmatrix} 4 & 1 & 1 & 0 & -1 \\ 1 & 4 & 0 & 1 & -1 \\ 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ -1 & -1 & 0 & 0 & 3 \end{pmatrix}$$

Conjectures

- Conjecture: [Hanlon, '92]

  All the eigenvalues of $T_k(a,b,w)$ are nonnegative integers.

Conjectures (cont.)

- Let $M_k(x,y,\lambda) := \sum_{a,b,r} m_k(a,b,r) x^a y^b \lambda^r$

  where $m_k(a,b,r) =$ multiplicity of r as an e.v. of $T_k(a,b) = \bigoplus_w T_k(a,b,w)$.

- Conjecture: [Hanlon, '92]

  $$M_k(x,y,\lambda) = \prod_{i=0}^{k} \left(1 + x + y + \lambda^{i+1} xy\right)$$

  Still open (in general)!

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where the Lie algebra L is either
* semisimple, or
* upper triangular (nilpotent), or
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* Heisenberg (nilpotent) — ?

Background (cont.)

- For the 3-dim Heisenberg $L_3 = \left\{ \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \right\}$, with basis $\{e, f, x\}$, the Laplacian $\partial\partial^* + \partial^*\partial$ for $L_3 \otimes \mathbb{C}[t]/(t^{k+1})$ is $\bigoplus_{a,b,w} T_k(a,b,w)$ (in a suitable basis).
- The property M conjecture is then equivalent to (an extension of)

$$M_k(x, y, 0) = (1 + x + y)^{k+1}$$

Partial Results

- [Hanlon, '92]
  Explicit eigenvalues in the stable case:

  * $a \le b$
  * $w \le \binom{a}{2} + \binom{b}{2}$
  * $k \ge (a-1) + (b-1) + w - \binom{a}{2} - \binom{b}{2}$

Partial Results (cont.)

- <u>Theorem</u>: [Hanlon, '92]
  In the stable case, $\Omega_k(a,b) \overset{1:1}{\leftrightarrow}$ pairs (λ, μ) of partitions with $|\lambda| + |\mu| = w - w_{\min}$.
  The eigenvalues of $T_k(a,b)$ are $|\lambda| - |\mu| + ab$.

  The proof uses Schur functions in two sets of variables.

Partial Results (cont.)

- [A.-Athanasiadis, '96]
  1. $a = 1, b = 2, \forall k:$ The nonzero eigenvalues of $\oplus_w T_k(a,b,w)$ are $\lambda = 1, \ldots, k+1,$ each with multiplicity k (and explicit distribution over w).
  2. $a = 1, \forall b, k:$ The multiplicity of the zero e.v.:

$$m_k(1,b,0) = \binom{k+1}{1 \quad b \quad k-b} = (k+1)\binom{k}{b}$$

Partial Results (cont.)

- [Hanlon-Wachs, '02]

  Extend result 2 above to $\forall a, b, k:$

$$m_k(a,b,0) = \begin{pmatrix} & & k+1 \\ a & b & k+1-a-b \end{pmatrix}$$

Partial Results (cont.)

- [Kuflik, '06]
  Distribution over w $(\forall a, b, k)$:
  $m_k(a,b,w,0) =$ coefficient of $q^{w-w_{\min}}$ in

$$\begin{bmatrix} & & k+1 \\ a & b & k+1-a-b \end{bmatrix}_q$$

  where

$$[m]!_q := [1]_q \cdot [2]_q \cdots [m]_q,$$

$$[m]_q := 1 + q + \ldots + q^{m-1}.$$

תודה על ההקשבה!