Character formulas and matrices

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$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 1 \end{pmatrix}$$

Abstract

We present a family of square matrices which are asymmetric variants of Walsh-Hadamard matrices. They originate in the study of character formulas, and provide a handy tool for translation of statements about permutation statistics to results in representation theory, and vice versa. They turn out to have many fascinating properties.

Outline

Character formulas

μ-unimodal permutations

- A sequence $(a_1, \ldots, a_n)$ of distinct positive integers is unimodal if there exists $1 \leq m \leq n$ such that

$$a_1 > a_2 > \ldots > a_m < a_{m+1} < \ldots < a_n.$$

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- Let $\mu = (\mu_1, \ldots, \mu_t)$ be a composition of n. A sequence of n positive integers is μ-unimodal if the first μ_1 integers form a unimodal sequence, the next μ_2 integers form a unimodal sequence, and so on.

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- A permutation $\pi \in S_n$ is μ-unimodal if the sequence $(\pi(1), \ldots, \pi(n))$ is μ-unimodal.

μ-unimodal permutations, descent set

- Let U_μ be the set of all μ-unimodal permutations in S_n.

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- Example: $n = 10$, $\mu = (3, 3, 4)$.

$$\pi = (\underbrace{4, 2, 10}_{\mu_1}, \underbrace{9, 7, 6}_{\mu_2}, \underbrace{5, 3, 1, 8}_{\mu_3}) \in U_\mu$$

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- The descent set of a permutation $\pi \in S_n$ is

$$\mathrm{Des}(\pi) := \{i : \pi(i) > \pi(i+1)\}.$$

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- Denote $I(\mu) := \{1, \ldots, n\} \setminus \{\mu_1, \mu_1 + \mu_2, \mu_1 + \mu_2 + \mu_3, \ldots\}$

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- Example: $I(\mu) = \{1, \ldots, 10\} \setminus \{3, 6, 10\} = \{1, 2, 4, 5, 7, 8, 9\}$
$$\mathrm{Des}(\pi) \cap I(\mu) = \{1, 4, 5, 7, 8\}$$

Formula 1: irreducible characters

Let λ and μ be partitions of n, let χ^λ be the character of the irreducible S_n-representation corresponding to λ, and let χ^λ_μ be its value on a conjugacy class of cycle type μ.

Theorem (Roichman '97)

$$\chi^\lambda_\mu = \sum_{\pi \in \mathcal{C} \cap U_\mu} (-1)^{|\operatorname{Des}(\pi) \cap I(\mu)|},$$

where $\mathcal{C}$ is any Knuth class of shape λ.

Formula 2: coinvariant algebra, homogeneous component

Let $\chi^{(k)}$ be the S_n-character corresponding to the symmetric group action on the k-th homogeneous component of its coinvariant algebra, and let $\chi^{(k)}_\mu$ be its value on a conjugacy class of cycle type μ.

Theorem (A-Postnikov-Roichman, '00)

$$\chi^{(k)}_\mu = \sum_{\pi \in L(k) \cap U_\mu} (-1)^{|\operatorname{Des}(\pi) \cap I(\mu)|},$$

where $L(k)$ is the set of all permutations of length k in S_n.

Formula 3: Gelfand model

A complex representation of a group or an algebra A is called a Gelfand model for A if it is equivalent to the multiplicity free direct sum of all irreducible A-representations. Let χ^G be the corresponding character, and let χ^G_μ be its value on a conjugacy class of cycle type μ.

Theorem (A-Postnikov-Roichman, '08)

The character of the Gelfand model of S_n at a conjugacy class of cycle type μ is equal to

$$\chi^G_\mu = \sum_{\pi \in Inv_n \cap U_\mu} (-1)^{|\operatorname{Des}(\pi) \cap I(\mu)|},$$

where $Inv_n := \{\sigma \in S_n : \sigma^2 = id\}$ is the set of all involutions in S_n.

Inverse formulas?

Question

Are these formulas invertible?

In other words: to what extent do the character values χ^*_μ $(\forall \mu)$ determine the distribution of descent sets?

Matrices

Subsets as indices

Definition

Let P_n be the power set (set of all subsets) of $\{1, \ldots, n\}$, with the anti-lexicographic linear order: for $I, J \in P_n$, $I \neq J$, let m be the largest element in the symmetric difference $I \triangle J := (I \cup J) \setminus (I \cap J)$, and define: $I < J \iff m \in J$.

Example

The linear order on P_3 is

$$\emptyset < \{1\} < \{2\} < \{1, 2\} < \{3\} < \{1, 3\} < \{2, 3\} < \{1, 2, 3\}.$$

P_n will index the rows and columns of our matrices.

Walsh-Hadamard matrices

The Walsh-Hadamard matrix H_n of order 2^n has entries

$$h_{I,J} := (-1)^{|I \cap J|} \qquad (\forall I, J \in P_n).$$

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$$H_n^t = H_n \qquad H_n H_n^t = 2^n I_{2^n}$$

Prefixes and runns

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Example

For $I = \{1,2,4,5,6,8,10\} \in P_{10}$:
$I_1 = \{1,2\}$, $I_2 = \{4,5,6\}$, $I_3 = \{8\}$, $I_4 = \{10\}$.

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Definition

For $I \in P_n$ let $I_1, \ldots, I_t$ be the runs in I. Define, for any $J \in P_n$:

$$a_{I,J} := \begin{cases} (-1)^{|I \cap J|}, & \text{if } I_k \cap J \text{ is a prefix of } I_k \text{ for each } k; \\ 0, & \text{otherwise.} \end{cases}$$

$A_n := (a_{I,J})_{I,J \in P_n}$, with P_n ordered as above.

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An auxiliary matrix:

$$b_{I,J} := \begin{cases} (-1)^{|I \cap J|}, & \text{if } I_k \cap J \text{ is a prefix of } I_k \text{ for each } k, \\ & \text{and } n \notin I \setminus J; \\ 0, & \text{otherwise.} \end{cases}$$

$B_n := (b_{I,J})_{I,J \in P_n}$.

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$I = \{1,2\}, \quad J = \{2\}, \quad I \cap J = \{2\}$ is not a prefix of I

$$A_n^t \neq A_n \qquad A_n A_n^t \neq 2^n I_{2^n} \qquad (n \geq 2)$$

Recursion

Lemma

$$A_n = \begin{pmatrix} A_{n-1} & A_{n-1} \\ A_{n-1} & -B_{n-1} \end{pmatrix} \qquad (n \geq 1)$$

with $A_0 = (1)$, *and*

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with $B_0 = (1)$.

For comparison:

$$H_n = \begin{pmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{pmatrix} \qquad (n \geq 1)$$

with $H_0 = (1)$.

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$$\det(A_n) = (n+1) \cdot \prod_{k=1}^{n} k^{2^{n-1-k}(n+4-k)} \qquad (n \geq 2)$$

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For comparison,

$$\det(H_n) = 2^{2^{n-1}n} \qquad (n \geq 2)$$

with $\det(H_0) = 1$ and $\det(H_1) = -2$.

Möbius inversion

Let Z_n be the zeta matrix of the poset P_n with respect to set inclusion:

$$z_{I,J} := \begin{cases} 1, & \text{if } I \subseteq J; \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$Z_n = \begin{pmatrix} Z_{n-1} & Z_{n-1} \\ 0 & Z_{n-1} \end{pmatrix} \qquad (n \geq 1)$$

with $Z_0 = (1)$. Its inverse is the Möbius matrix $M_n = Z_n^{-1}$, with entries $m_{I,J}$ defined by

$$m_{I,J} := \begin{cases} (-1)^{|J \setminus I|}, & \text{if } I \subseteq J; \\ 0, & \text{otherwise.} \end{cases}$$

It satisfies

$$M_n = \begin{pmatrix} M_{n-1} & -M_{n-1} \\ 0 & M_{n-1} \end{pmatrix} \qquad (n \geq 1)$$

with $M_0 = (1)$.

AM and BM

Denote now $AM_n := A_nM_n$, $BM_n := B_nM_n$ and $HM_n := H_nM_n$. It follows that

$$AM_n = \begin{pmatrix} AM_{n-1} & 0 \\ AM_{n-1} & -(AM_{n-1} + BM_{n-1}) \end{pmatrix} \qquad (n \geq 1)$$

with $AM_0 = (1)$ and

$$BM_n = \begin{pmatrix} AM_{n-1} & 0 \\ 0 & -BM_{n-1} \end{pmatrix} \qquad (n \geq 1)$$

with $BM_0 = (1)$, as well as

$$HM_n = \begin{pmatrix} HM_{n-1} & 0 \\ HM_{n-1} & -2HM_{n-1} \end{pmatrix} \qquad (n \geq 1)$$

with $HM_0 = (1)$.

Determinant computation (1)

By the *BM* recursion,

$$\det(BM_n) = \det(AM_{n-1})\det(-BM_{n-1}) \qquad (n \geq 1).$$

Now M_n is an upper triangular matrix with 1-s on its diagonal, so that

$$\det(M_n) = 1.$$

We conclude that

$$\det(B_n) = \delta_{n-1}\det(A_{n-1})\det(B_{n-1}) \qquad (n \geq 1),$$

where

$$\delta_n = (-1)^{2^n} = \begin{cases} -1, & \text{if } n = 0; \\ 1, & \text{otherwise.} \end{cases}$$

Determinant computation (2)

Similarly, for any scalar t and $n \geq 1$,

$$AM_n + tBM_n = \begin{pmatrix} (t+1)AM_{n-1} & 0 \\ AM_{n-1} & -AM_{n-1} - (t+1)BM_{n-1} \end{pmatrix}$$

and a similar argument yields

$$\det(A_n + tB_n) = \delta_{n-1} \det((t+1)A_{n-1}) \det(A_{n-1} + (t+1)B_{n-1})$$

It follows that

$$\det(A_n) = \left(\prod_{k=1}^{n} \delta_{n-k} \det(kA_{n-k}) \right) \cdot \det(A_0 + nB_0) =$$

$$= -(n+1) \cdot \prod_{k=1}^{n} k^{2^{n-k}} \cdot \prod_{k=1}^{n} \det(A_{n-k}) \qquad (n \geq 1).$$

Since $A_0 = (1)$ it follows that $\det(A_n) \neq 0$ for any nonnegative integer n.

Determinant computation (3)

The solution to this recursion, with initial value $\det(A_1) = -2$, is

$$\det(A_n) = (n+1) \cdot \prod_{k=1}^{n} k^{2^{n-1-k}(n+4-k)} \qquad (n \geq 2).$$

The *BM* recursion, with initial value $\det(B_1) = -1$, now yields

$$\det(B_n) = \prod_{k=1}^{n} k^{2^{n-1-k}(n+2-k)} \qquad (n \geq 2).$$

□

For comparison,

$$\det(H_n) = 2^{2^{n-1}} \det(H_{n-1})^2 \qquad (n \geq 2)$$

with initial value $\det(H_1) = -2$, so that

$$\det(H_n) = 2^{2^{n-1} n} \qquad (n \geq 2).$$

HM entries

$$HM_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & -2 & 4 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & -2 & 4 & 0 & 0 \\ 1 & 0 & -2 & 0 & -2 & 0 & 4 & 0 \\ 1 & -2 & -2 & 4 & -2 & 4 & 4 & -8 \end{pmatrix}$$

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Lemma

- **Zero pattern:** $(HM_n)_{I,J} \neq 0 \iff J \subseteq I$

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- **Signs:** $(HM_n)_{I,J} \neq 0 \Longrightarrow \operatorname{sign}((HM_n)_{I,J}) = (-1)^{|J|}$
- **Absolute values:** $(HM_n)_{I,J} \neq 0 \implies |(HM_n)_{I,J}| = 2^{|J|}$

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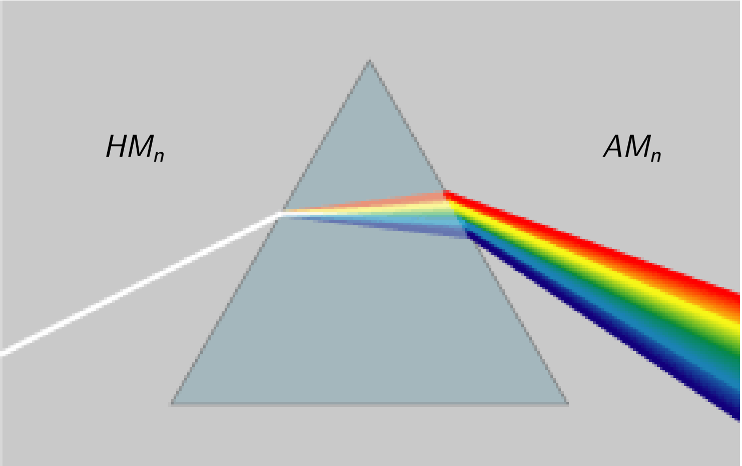
Theorem

- **Zero pattern:** $(AM_n)_{I,J} \neq 0 \iff J \subseteq I$
- **Signs:** $(AM_n)_{I,J} \neq 0 \Longrightarrow \text{sign}((AM_n)_{I,J}) = (-1)^{|J|}$
- **Absolute values:** *???*

Dispersion



AM entries (2)

Theorem

- **Zero pattern:** $(AM_n)_{I,J} \neq 0 \iff J \subseteq I$
- **Signs:** $(AM_n)_{I,J} \neq 0 \Longrightarrow \text{sign}((AM_n)_{I,J}) = (-1)^{|J|}$
- **Absolute values:**

$$(AM_n)_{I,J} \neq 0 \Longrightarrow |(AM_n)_{I,J}| = \prod_{k=1}^{t} (|J_k| + 1)^{\delta_k(I)}$$

where $J_1, \ldots, J_t$ *are the runs in* J *and, for* $J_k = \{m_k + 1, \ldots, m_k + \ell_k\}$ $(1 \leq k \leq t)$*:*

$$\delta_k(I) := \begin{cases} 0, & \text{if } m_k \in I; \\ 1, & \text{otherwise.} \end{cases}$$

Diagonal and last row

Corollary

- *All entries in the diagonal and last row of AM_n are non-zero.*
- *Diagonal:*

$$|(AM_n)_{J,J}| = \prod_{k=1}^{t} (|J_k| + 1)$$

- *Last row:*

$$|(AM_n)_{[n],J}| = \begin{cases} |J_1| + 1, & \text{if } 1 \in J; \\ 1, & \text{otherwise.} \end{cases}$$

- *Each nonzero entry $(AM_n)_{I,J}$ divides the corresponding diagonal entry $(AM_n)_{J,J}$ and is divisible by the corresponding last row entry $(AM_n)_{[n],J}$.*

Diagonal and last row (example)

$$AM_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & -1 & 3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & -2 & 4 & 0 & 0 \\ 1 & 0 & -2 & 0 & -1 & 0 & 3 & 0 \\ 1 & -2 & -1 & 3 & -1 & 2 & 1 & -4 \end{pmatrix}$$

$I = \{1, 2\}$

$I = \{1, 2, 3\}$

$\uparrow$

$J = \{1, 2\}$

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$$AM_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & -1 & 3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & -2 & 4 & 0 & 0 \\ 1 & 0 & -2 & 0 & -1 & 0 & 3 & 0 \\ 1 & -2 & -1 & 3 & -1 & 2 & 1 & -4 \end{pmatrix} \begin{matrix} \\ \\ \\ \\ \\ \\ I = \{2,3\} \\ I = \{1,2,3\} \end{matrix}$$

$\uparrow$

$J = \{2, 3\}$

Row sums

Lemma

- *The sum of all entries in row I of AM_n (or HM_n) is $(-1)^{|I|}$.*
- *The sum of absolute values of all entries in row I of AM_n is*

$$\prod_{k=1}^{t}(2^{|I_k|+1} - 1).$$

In HM_n the sum is $3^{|I|}$.

Column sums and square diagonal entries

Theorem

- *The sum of absolute values of all the entries in column J of AM_n is equal to the (J, J) diagonal entry of A_n^2, which in turn is equal to*

$$2^{n-t^*-|J^*|}\prod_{k=1}^{t^*}(|J_k^*|+2),$$

  where $J^* := J \setminus \{1\}$ and $J_1^*, \ldots, J_{t^*}^*$ are its runs.
- *For comparison, the sum of absolute values of all the entries in column J of HM_n is equal to the (J, J) diagonal entry of H_n^2, namely to the constant 2^n.*

Column sums and square diagonal entries

Example

$$AM_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & -1 & 3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & -2 & 4 & 0 & 0 \\ 1 & 0 & -2 & 0 & -1 & 0 & 3 & 0 \\ 1 & -2 & -1 & 3 & -1 & 2 & 1 & -4 \end{pmatrix}$$

column sums: 8 8 6 6 6 6 4 4

Column sums and square diagonal entries

Example

$$A_3^2 = \begin{pmatrix} 8 & 0 & 2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 8 & -2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 6 & 0 & -2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 6 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & -2 & 0 \\ 2 & 0 & 2 & 0 & 0 & 0 & 4 & 0 \\ 0 & 2 & 0 & 2 & 1 & 1 & 0 & 4 \end{pmatrix}$$

Inverse of AM

Theorem

- $$(AM_n^{-1})_{I,J} \neq 0 \iff J \subseteq I$$
- *For $J \subseteq I$,*
$$(AM_n^{-1})_{I,J} = (-1)^{|J|} \prod_{i \in I} \frac{d_{I,J}(i)}{e_{I,J}(i)},$$
where, for $i \in I_k$ (k-th run of I):
$$d_{I,J}(i) := \begin{cases} \max(I_k) - i + 1, & \text{if } i \in J; \\ 1, & \text{otherwise} \end{cases}$$
and
$$e_{I,J}(i) := \max(I_k) - i + 2.$$

Inverse of AM

Equivalently, for $J \subseteq I$,

$$(AM_n^{-1})_{I,J} = (-1)^{|J|} \prod_{k=1}^{t} \frac{1}{(|I_k|+1)!} \prod_{i \in I_k \cap J} (\max(I_k) - i + 1).$$

Note that the denominator $\prod_{k=1}^{t}(|I_k|+1)!$ is the cardinality of the parabolic subgroup $\langle I \rangle$ of S_{n+1} generated by the simple reflections $\{s_i : i \in I\}$.

Inverse of AM

Corollary

- *Each nonzero entry of AM_n^{-1} is the inverse of an integer.*
- *In each row of AM_n^{-1}, the sum of absolute values of all the entries is 1.*
- *In each row I of AM_n^{-1}, the first entry*

$$(AM_n^{-1})_{I,\emptyset} = \prod_{k=1}^{t} \frac{1}{(|I_k|+1)!}$$

divides all the other nonzero entries and the diagonal entry

$$(AM_n^{-1})_{I,I} = (-1)^{|I|} \prod_{k=1}^{t} \frac{1}{|I_k|+1}$$

is divisible by all the other nonzero entries.

Inverse of AM

Example

$$AM_3^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 \\ 1/6 & -1/3 & -1/6 & 1/3 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 \\ 1/4 & -1/4 & 0 & 0 & -1/4 & 1/4 & 0 & 0 \\ 1/6 & 0 & -1/3 & 0 & -1/6 & 0 & 1/3 & 0 \\ 1/24 & -1/8 & -1/12 & 1/4 & -1/24 & 1/8 & 1/12 & -1/4 \end{pmatrix}$$

Eigenvalues

$$A_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & \color{red}{0} & 1 \end{pmatrix}$$

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Question: What can be said about its eigenvalues?

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Answer: char. poly.$(A_2) = (x^2 - 4)(x^2 - 3)$

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$$A_2^2 = \begin{pmatrix} 4 & 0 & 1 & 0 \\ 0 & 4 & -1 & 0 \\ 0 & 0 & 3 & 0 \\ 1 & 1 & 0 & 3 \end{pmatrix}$$

Eigenvalues

$$A_3^2 = \begin{pmatrix} 8 & 0 & 2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 8 & -2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 6 & 0 & -2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 6 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & -2 & 0 \\ 2 & 0 & 2 & 0 & 0 & 0 & 4 & 0 \\ 0 & 2 & 0 & 2 & 1 & 1 & 0 & 4 \end{pmatrix}$$

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char. poly.$(A_3^2) = (x-8)^2(x-6)^4(x-4)^2$

Alas... A_3^2 is not diagonalizable !

Eigenvalues (conjecture)

Conjecture

The eigenvalues of A_n^2 (counted by algebraic multiplicity) are in $1 : 1$ correspondence with the diagonal entries of A_n^2.

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The eigenvalues of A_n^2 (counted by algebraic multiplicity) are in 1 : 1 correspondence with the diagonal entries of A_n^2.

The latter are explicitly known:

Theorem

The (J, J) diagonal entry of A_n^2 is equal to the sum of absolute values of all the entries in column J of AM_n, which in turn is equal to

$$2^{n-t^*-|J^*|}\prod_{k=1}^{t^*}(|J_k^*|+2)=\prod_k(\mu_k+1),$$

where μ is the composition of n corresponding to $J^* := J \setminus \{1\}$.

Back to characters

Fine sets

Definition

Let B be a set of combinatorial objects, and let $\mathrm{Des} : B \to P_{n-1}$ be a map which associates a "descent set" $\mathrm{Des}(b) \subseteq [n-1]$ to each element $b \in B$. Denote by B^μ the set of elements in B whose descent set $\mathrm{Des}(b)$ is μ-unimodal. Let ρ be a complex S_n-representation. Then B is called a fine set for ρ if, for each composition μ of n, the character value of ρ on a conjugacy class of cycle type μ satisfies

$$\chi^\rho_\mu = \sum_{b \in B^\mu} (-1)^{|\mathrm{Des}(b) \setminus S(\mu)|}.$$

Character values and descent sets

Theorem (Fine Set Theorem)

If B is a fine set for an S_n-representation ρ, then the character values of ρ uniquely determine the overall distribution of descent sets over B.

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Idea of proof

For a subset $J = \{j_1, \ldots, j_k\} \subseteq [n-1]$ let $s_J := s_{j_1} s_{j_2} \cdots s_{j_k} \in S_n$. Let χ^ρ be the vector with entries $\chi^\rho(s_J)$, for $J \in P_{n-1}$, and let v^B be the vector with entries

$$v_J^B := |\{b \in B \,:\, \mathrm{Des}(b) = J\}| \qquad (\forall J \in P_{n-1}).$$

Then, by definition, B is a fine set for ρ if and only if

$$\chi^\rho = A_{n-1} v^B.$$

The result follows since A_{n-1} is an invertible matrix.

Explicit inversion formula

Theorem

Let B be a fine set for an S_n-representation ρ. For every $D \subseteq [n-1]$, the number of elements in B with descent set D satisfies

$$|\{b \in B : \mathrm{Des}(b) = D\}| = \sum_J \chi^\rho(c_J) \sum_{I : D \cup J \subseteq I} (-1)^{|I \setminus D|} (AM_{n-1}^{-1})_{I,J}$$

where

$$(AM_{n-1}^{-1})_{I,J} = \frac{(-1)^{|J|}}{|\langle I \rangle|} \prod_{k=1}^{t} \prod_{i \in I_k \cap J} (\max(I_k) - i + 1),$$

$I_1, \ldots, I_t$ are the runs in I and $c_J := \prod_{j \in J} s_j$ is a Coxeter element in the parabolic subgroup $\langle J \rangle$.

Equivalence of classical theorems

For $0 \leq k \leq \binom{n}{2}$ let R_k be the k-th homogeneous component of the coinvariant algebra of the symmetric group S_n. For a partition λ, let $m_{k,\lambda}$ be the number of standard Young tableaux of shape λ with major index k.

Theorem (Lusztig-Stanley)

$$R_k \cong \bigoplus_{\lambda \vdash n} m_{k,\lambda} S^\lambda,$$

where the sum runs over all partitions of n and S^λ denotes the irreducible S_n-module indexed by λ.

Equivalence of classical theorems

The major index of a permutation π is $\mathrm{maj}(\pi) := \sum_{i \in \mathrm{Des}(\pi)} i$,
and its length $\ell(\pi)$ is the number of inversions in π.
For a subset $I \subseteq [n-1]$ denote $\mathbf{x}^I := \prod_{i \in I} x_i$.

Theorem (Foata-Schützenberger; Garsia-Gessel)

$$\sum_{\pi \in S_n} \mathbf{x}^{\mathrm{Des}(\pi)} q^{\ell(\pi)} = \sum_{\pi \in S_n} \mathbf{x}^{\mathrm{Des}(\pi)} q^{\mathrm{maj}(\pi^{-1})}.$$

The Fine Set Theorem implies

Corollary

The Foata-Schützenberger Theorem is equivalent to the Lusztig-Stanley Theorem.

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THANK YOU !