Characters, Descents and Matrices

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$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 1 \end{pmatrix}$$

Abstract

A certain family of square matrices plays a major role in character formulas for the symmetric group and related algebras. These matrices are asymmetric variants of Walsh-Hadamard matrices, and have some fascinating properties which may be explained by use of Möbius inversion. They provide a tool for translation of statements about permutation statistics to results in representation theory, and vice versa.

Outline

Character Formulas

μ-unimodal permutations

- A sequence $(a_1, \ldots, a_n)$ of distinct positive integers is unimodal if there exists $1 \leq m \leq n$ such that

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- Let $\mu = (\mu_1, \ldots, \mu_t)$ be a composition of n. A sequence of n positive integers is μ-unimodal if the first μ_1 integers form a unimodal sequence, the next μ_2 integers form a unimodal sequence, and so on.

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- A permutation $\pi \in S_n$ is μ-unimodal if the sequence $(\pi(1), \ldots, \pi(n))$ is μ-unimodal.

μ-unimodal permutations, descent set

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- Example: $n = 10$, $\mu = (3, 3, 4)$.

$$\pi = (4, 2, 10, 9, 7, 6, 5, 3, 1, 8) \in U_\mu$$

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- Example: $I(\mu) = \{1, \ldots, 10\} \setminus \{3, 6, 10\} = \{1, 2, 4, 5, 7, 8, 9\}$

$$\mathrm{Des}(\pi) \cap I(\mu) = \{1, 4, 5, 7, 8\}$$

Formula 1: irreducible characters

Let λ and μ be partitions of n, let χ^λ be the character of the irreducible S_n-representation corresponding to λ, and let χ^λ_μ be its value on a conjugacy class of cycle type μ.

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Theorem (Roichman '97)

$$\chi^\lambda_\mu = \sum_{\pi \in \mathcal{C} \cap U_\mu} (-1)^{|\operatorname{Des}(\pi) \cap I(\mu)|},$$

where $\mathcal{C}$ is any Knuth class of shape λ.

Formula 2: coinvariant algebra, homogeneous component

Let $\chi^{(k)}$ be the S_n-character corresponding to the symmetric group action on the k-th homogeneous component of its coinvariant algebra, and let $\chi^{(k)}_{\mu}$ be its value on a conjugacy class of cycle type μ.

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Theorem (A-Postnikov-Roichman '00)

$$\chi^{(k)}_\mu = \sum_{\pi \in L(k) \cap U_\mu} (-1)^{|\operatorname{Des}(\pi) \cap I(\mu)|},$$

where $L(k)$ is the set of all permutations of length k in S_n.

Formula 3: Gelfand model

A complex representation of a group or an algebra A is called a Gelfand model for A if it is equivalent to the multiplicity free direct sum of all irreducible A-representations. Let χ^G be the corresponding character, and let χ^G_μ be its value on a conjugacy class of cycle type μ.

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Theorem (A-Postnikov-Roichman '08)

The character of the Gelfand model of S_n at a conjugacy class of cycle type μ is equal to

$$\chi^G_\mu = \sum_{\pi \in Inv_n \cap U_\mu} (-1)^{|\operatorname{Des}(\pi) \cap I(\mu)|},$$

where $Inv_n := \{\sigma \in S_n : \sigma^2 = id\}$ is the set of all involutions in S_n.

Iwahori-Hecke algebra

Let $\mathcal{H}_n(q)$ be the algebra over $\mathbb{Q}$ generated by $T_1, \ldots, T_{n-1}$ subject to the relations

$$(T_i + q)(T_i - 1) = 0 \qquad (\forall i)$$

$$T_i T_j T_j T_i \qquad (|j - i| > 1)$$

and

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \qquad (1 \leq i < n - 1).$$

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Remark. All above formulas extend to $\mathcal{H}_n(q)$ when replacing (-1) by $(-q)$.

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Remark. All above formulas extend to $\mathcal{H}_n(q)$ when replacing (-1) by $(-q)$.

Example. The character of the Gelfand model of $\mathcal{H}_n(q)$ at the element T_μ is equal to

$$\sum_{\pi \in Inv_n \cap U_\mu} (-q)^{|\operatorname{Des}(\pi) \cap I(\mu)|},$$

Inverse formulas?

Question

Are these formulas invertible?

In other words: to what extent do the character values χ^*_μ $(\forall \mu)$ determine the distribution of descent sets?

Matrices

Walsh-Hadamard matrices

Recursive definition

$$H_n = \begin{pmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{pmatrix} \qquad (n \geq 1)$$

with $H_0 = (1)$.

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Subsets as indices

Definition

Let P_n be the power set (set of all subsets) of $\{1, \ldots, n\}$, with the anti-lexicographic linear order: for $I, J \in P_n$, $I \neq J$, let m be the largest element in the symmetric difference $I \triangle J := (I \cup J) \setminus (I \cap J)$, and define: $I < J \iff m \in J$.

Example

The linear order on P_3 is

$$\emptyset < \{1\} < \{2\} < \{1,2\} < \{3\} < \{1,3\} < \{2,3\} < \{1,2,3\}.$$

Fact (explicit description of H_n)

The Walsh-Hadamard matrix *H_n of order 2^n has entries*

$$h_{I,J} := (-1)^{|I \cap J|} \qquad (\forall I, J \in P_n).$$

where rows and columns of H_n are indexed by P_n ordered as above.

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Note that

$$H_n^t = H_n$$

and

$$H_n H_n^t = 2^n I_{2^n}$$

The matrices A and B

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$$A_n = \begin{pmatrix} A_{n-1} & A_{n-1} \\ A_{n-1} & -B_{n-1} \end{pmatrix} \qquad (n \geq 1)$$

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For comparison:

$$H_n = \begin{pmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{pmatrix} \qquad (n \geq 1)$$

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Prefixes and runs

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Example

For $I = \{1, 2, 4, 5, 6, 8, 10\} \in P_{10}$:
$I_1 = \{1, 2\}$, $I_2 = \{4, 5, 6\}$, $I_3 = \{8\}$, $I_4 = \{10\}$.

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Lemma (explicit description of A_n and B_n)

For $I \in P_n$ let $I_1, \ldots, I_t$ be the runs in I. Define for any $J \in P_n$:

$$a_{I,J} := \begin{cases} (-1)^{|I \cap J|}, & \text{if } I_k \cap J \text{ is a prefix of } I_k \text{ for each } k; \\ 0, & \text{otherwise.} \end{cases}$$

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Then

$$A_n = (a_{I,J})_{I,J \in P_n} \quad \text{and} \quad B_n = (b_{I,J})_{I,J \in P_n}$$

with P_n ordered as above.

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$$A_n^t \neq A_n \qquad (n \geq 2)$$

Determinant

Theorem

A_n and B_n are invertible for all $n \geq 0$.

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$$\det(A_n) = (n+1) \cdot \prod_{k=1}^{n} k^{2^{n-1-k}(n+4-k)} \qquad (n \geq 2)$$

while $\det(A_0) = 1$ *and* $\det(A_1) = -2$.

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while $\det(A_0) = 1$ and $\det(A_1) = -2$.

For comparison,

$$\det(H_n) = 2^{2^{n-1}n} \qquad (n \geq 2)$$

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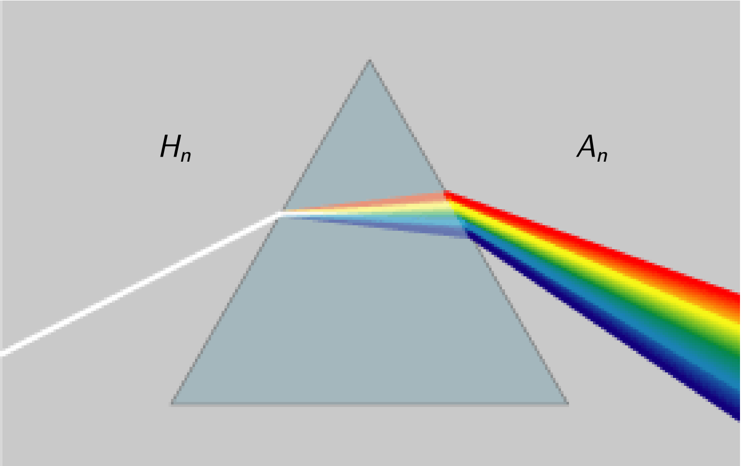
From white light to rainbow colors



Möbius inversion

Let Z_n be the zeta matrix of the poset P_n with respect to set inclusion:

$$z_{I,J} := \begin{cases} 1, & \text{if } I \subseteq J; \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$Z_n = \begin{pmatrix} Z_{n-1} & Z_{n-1} \\ 0 & Z_{n-1} \end{pmatrix} \qquad (n \geq 1)$$

with $Z_0 = (1)$. Its inverse is the Möbius matrix $M_n = Z_n^{-1}$, with entries $m_{I,J}$ defined by

$$m_{I,J} := \begin{cases} (-1)^{|J \setminus I|}, & \text{if } I \subseteq J; \\ 0, & \text{otherwise.} \end{cases}$$

It satisfies

$$M_n = \begin{pmatrix} M_{n-1} & -M_{n-1} \\ 0 & M_{n-1} \end{pmatrix} \qquad (n \geq 1)$$

with $M_0 = (1)$.

AM and BM

Denote now $AM_n := A_n M_n$, $BM_n := B_n M_n$ and $HM_n := H_n M_n$. It follows that

$$AM_n = \begin{pmatrix} AM_{n-1} & 0 \\ AM_{n-1} & -(AM_{n-1} + BM_{n-1}) \end{pmatrix} \qquad (n \geq 1)$$

with $AM_0 = (1)$ and

$$BM_n = \begin{pmatrix} AM_{n-1} & 0 \\ 0 & -BM_{n-1} \end{pmatrix} \qquad (n \geq 1)$$

with $BM_0 = (1)$, as well as

$$HM_n = \begin{pmatrix} HM_{n-1} & 0 \\ HM_{n-1} & -2HM_{n-1} \end{pmatrix} \qquad (n \geq 1)$$

with $HM_0 = (1)$.

HM entries

$$HM_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & -2 & 4 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & -2 & 4 & 0 & 0 \\ 1 & 0 & -2 & 0 & -2 & 0 & 4 & 0 \\ 1 & -2 & -2 & 4 & -2 & 4 & 4 & -8 \end{pmatrix}$$

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Lemma

- **Zero pattern:** $(HM_n)_{I,J} \neq 0 \iff J \subseteq I$

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- **Signs:** $(HM_n)_{I,J} \neq 0 \implies \text{sign}((HM_n)_{I,J}) = (-1)^{|J|}$

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Lemma

- **Zero pattern:** $(HM_n)_{I,J} \neq 0 \iff J \subseteq I$
- **Signs:** $(HM_n)_{I,J} \neq 0 \Longrightarrow \operatorname{sign}((HM_n)_{I,J}) = (-1)^{|J|}$
- **Absolute values:** $(HM_n)_{I,J} \neq 0 \Longrightarrow |(HM_n)_{I,J}| = 2^{|J|}$

AM entries

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- **Absolute values:**

$$(AM_n)_{I,J} \neq 0 \Longrightarrow |(AM_n)_{I,J}| = \prod_{k=1}^{t} (|J_k| + 1)^{\delta_k(I)}$$

where $J_1, \ldots, J_t$ *are the runs in* J *and, for* $J_k = \{m_k + 1, \ldots, m_k + \ell_k\}$ $(1 \leq k \leq t)$*:*

$$\delta_k(I) := \begin{cases} 0, & \text{if } m_k \in I; \\ 1, & \text{otherwise.} \end{cases}$$

Diagonal and last row

Corollary

- *All entries in the diagonal and last row of AM_n are non-zero.*
- *Diagonal:*

$$|(AM_n)_{J,J}| = \prod_{k=1}^{t}(|J_k| + 1)$$

- *Last row:*

$$|(AM_n)_{[n],J}| = \begin{cases} |J_1| + 1, & \text{if } 1 \in J; \\ 1, & \text{otherwise.} \end{cases}$$

- *Each nonzero entry $(AM_n)_{I,J}$ divides the corresponding diagonal entry $(AM_n)_{J,J}$ and is divisible by the corresponding last row entry $(AM_n)_{[n],J}$.*

Diagonal and last row (example)

$$AM_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & -1 & 3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & -2 & 4 & 0 & 0 \\ 1 & 0 & -2 & 0 & -1 & 0 & 3 & 0 \\ 1 & -2 & -1 & 3 & -1 & 2 & 1 & -4 \end{pmatrix}$$

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Eigenvalues

$$A_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 1 \end{pmatrix}$$

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$$A_2^2 = \begin{pmatrix} 4 & 0 & 1 & 0 \\ 0 & 4 & -1 & 0 \\ 0 & 0 & 3 & 0 \\ 1 & 1 & 0 & 3 \end{pmatrix}$$

Eigenvalues

$$A_3^2 = \begin{pmatrix} 8 & 0 & 2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 8 & -2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 6 & 0 & -2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 6 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & -2 & 0 \\ 2 & 0 & 2 & 0 & 0 & 0 & 4 & 0 \\ 0 & 2 & 0 & 2 & 1 & 1 & 0 & 4 \end{pmatrix}$$

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char. poly.$(A_3) = (x^2 - 8)(x^2 - 6)^2(x^2 - 4)$

Eigenvalues

Conjecture

The eigenvalues of A_n^2 (counted by algebraic multiplicity) are in $1 : 1$ correspondence with the diagonal entries of A_n^2 (which are explicitly known).

Eigenvalues

Theorem (G. Alon '13)

The eigenvalues of A_n^2 (counted by algebraic multiplicity) are in $1:1$ correspondence with the diagonal entries of A_n^2, and thus in 2:1 correspondence with the compositions $\mu = (\mu_1, \ldots, \mu_t)$ of n:

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Similarly, The eigenvalues of B_n^2 are in $1:1$ correspondence with the diagonal entries of B_n^2, and thus in 2:1 correspondence with the compositions of n:

$$\pi'_\mu = \prod_{i=1}^{t-1} (\mu_i + 1).$$

Eigenvalues

$$A_3 \sim \begin{pmatrix} 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Back to Characters

Fine sets

Definition

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Then $\mathcal{B}$ is called a fine set for a complex S_n-representation ρ if, for each composition μ of n, the character value of ρ on a conjugacy class of cycle type μ satisfies

$$\chi^{\rho}_{\mu} = \sum_{b \in \mathcal{B}^{\mu}} (-1)^{|\mathrm{Des}(b) \setminus S(\mu)|}.$$

Character values and descent sets

Theorem (Fine Set Theorem)

If $\mathcal{B}$ is a fine set for an S_n-representation ρ, then the character values of ρ uniquely determine the overall distribution of descent sets over $\mathcal{B}$.

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For a subset $J = \{j_1, \ldots, j_k\} \subseteq [n-1]$ let $s_J := s_{j_1} s_{j_2} \cdots s_{j_k} \in S_n$.

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$$v_J^{\mathcal{B}} := |\{b \in \mathcal{B} \,:\, \mathrm{Des}(b) = J\}| \qquad (\forall J \in P_{n-1}).$$

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The result follows since A_{n-1} is an invertible matrix.

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$$A_2 v^{S_2} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 1 \end{pmatrix} v^{S_2} = \begin{pmatrix} 6 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Explicit inversion formula

Theorem

Let B be a fine set for an S_n-representation ρ. For every $I \subseteq [n-1]$, the number of elements in B with descent set D satisfies

$$|\{b \in B : \mathrm{Des}(b) = D\}| = \sum_{J} \chi^{\rho}(c_J) \sum_{I : D \cup J \subseteq I} (-1)^{|I \setminus D|} (AM_{n-1}^{-1})_{I,J}$$

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where

$$(AM_{n-1}^{-1})_{I,J} = \frac{(-1)^{|J|}}{|\langle I \rangle|} \prod_{k=1}^{t} \prod_{i \in I_k \cap J} (\max(I_k) - i + 1),$$

$I_1, \ldots, I_t$ are the runs in I and $c_J := \prod_{j \in J} s_j$ is a Coxeter element in the parabolic subgroup $\langle J \rangle$.

Equivalence of classical theorems

Corollary

Given two symmetric group modules with fine sets, the isomorphism of these modules is equivalent to equi-distribution of the descent set on their fine sets.

Equivalence of classical theorems

The major index of a permutation π is $\mathrm{maj}(\pi) := \sum_{i \in \mathrm{Des}(\pi)} i$,
and its length $\ell(\pi)$ is the number of inversions in π.
For a subset $I \subseteq [n-1]$ denote $\mathbf{x}^I := \prod_{i \in I} x_i$.

Theorem (Foata-Schützenberger; Garsia-Gessel)

$$\sum_{\pi \in S_n} \mathbf{x}^{\mathrm{Des}(\pi)} q^{\ell(\pi)} = \sum_{\pi \in S_n} \mathbf{x}^{\mathrm{Des}(\pi)} q^{\mathrm{maj}(\pi^{-1})}.$$

Equivalence of classical theorems

For $0 \leq k \leq \binom{n}{2}$ let R_k be the k-th homogebeous component of the coinvariant algebra of the symmetric group S_n.

For a partition λ, let $m_{k,\lambda}$ be the number of standard Young tableaux of shape λ with major index k.

Theorem (Lusztig-Stanley)

$$R_k \cong \bigoplus_{\lambda \vdash n} m_{k,\lambda} S^{\lambda},$$

where the sum runs over all partitions of n and S^{λ} denotes the irreducible S_n-module indexed by λ.

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$B_k = \{\pi \in S_n : \operatorname{maj}(\pi^{-1}) = k\}$ is a fine set for the representation $\rho_k := \bigoplus_{\lambda \vdash n} m_{k,\lambda} S^{\lambda}$.

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Thus $\rho_k \cong R_k$ if and only if the distributions of the descent set over B_k and L_k are equal.

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- ... and offer many more riddles, awaiting (your) solution!

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THANK YOU !