Cyclic descents, standard Young tableaux and Gromov-Witten invariants

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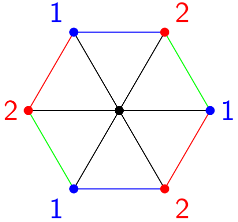

Based on joint works with

Ira Gessel (Brandeis)
Sergi Elizalde (Dartmouth)
Vic Reiner (Minnesota)
Yuval Roichman (Bar-Ilan)

Outline

Cyclic descents

Cyclic descents of permutations

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The descent set of a permutation $\pi = (\pi_1, \ldots, \pi_n)$ in the symmetric group $\mathfrak{S}_n$ is

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where $[m] := \{1, 2, \ldots, m\}$.

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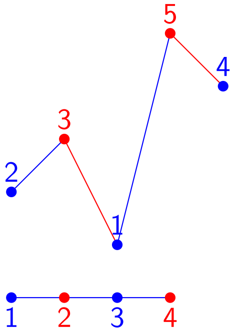

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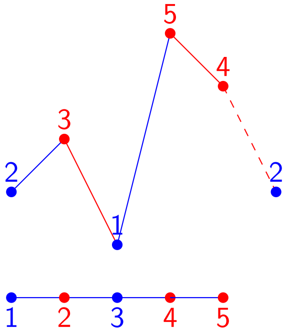

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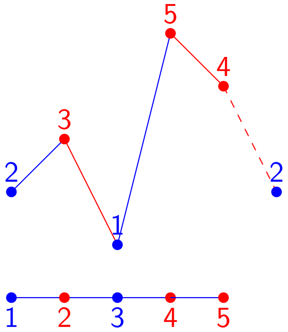

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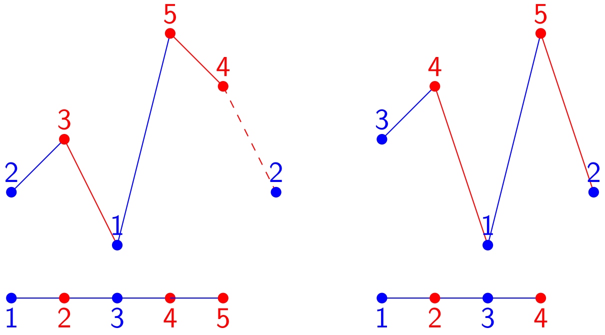

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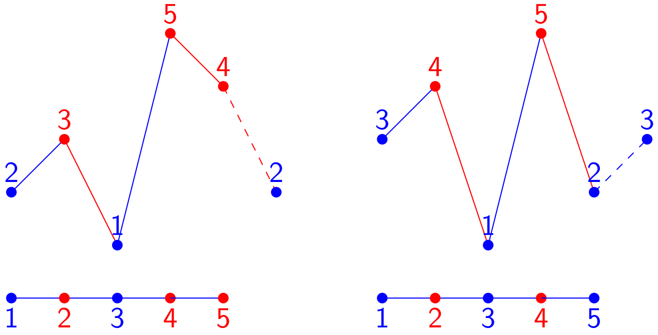


Question:

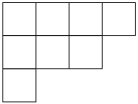
Can a similar concept be defined in other contexts?
E.g., for standard Young tableaux?

Standard Young Tableaux

A shape λ of size n is a partition $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$. It has a corresponding diagram.

Example

$$\lambda = (4, 3, 1)$$



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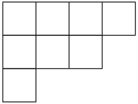

A standard Young tableau (SYT) T of shape λ is a filling of the diagram of λ by the numbers $1, \ldots, n$, each one appearing once, such that the entries increase along rows (from left to right) and along columns (from top to bottom).

Example

$$\lambda = (4,3,1)$$

1	2	4	8
3	5	7	
6			

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Denote the set of all standard Young tableaux of shape λ/μ by $\mathrm{SYT}(\lambda/\mu)$.

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Motivating Problem:

Define a cyclic descent set for SYT of any shape λ/μ.

SYT of rectangular shapes

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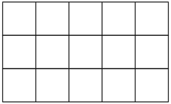

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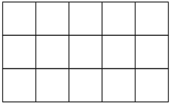

Theorem (Rhoades '10)

For $r|n$, *let* $\lambda = (r^{n/r}) = (r, \ldots, r) \vdash n$ *be a rectangular shape. Then there exists a cyclic descent map* $\mathrm{cDes} : \mathrm{SYT}(\lambda) \to 2^{[n]}$ *s.t., for all* $T \in \mathrm{SYT}(\lambda)$,

$$\mathrm{cDes}(T) \cap [n-1] \quad = \quad \mathrm{Des}(T)$$

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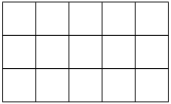

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$$\begin{aligned} \mathrm{cDes}(T) \cap [n-1] &= \mathrm{Des}(T) \\ \mathrm{cDes}(p(T)) &= p_n(\mathrm{cDes}(T)) \end{aligned}$$

where p_n acts on the set of integers $\mathrm{cDes}(T)$ *by adding* $1 \pmod{n}$ *to each element, and p acts on the SYT T by Schützenberger's jeu-de-taquin promotion.*

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Two orbits of SYT:

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$$\{1,4\} \quad \{2,5\} \quad \{3,\color{red}{6}\} \quad ; \quad \{1,3,5\} \quad \{2,4,\color{red}{6}\}$$

Formalization

Let us formalize the concept of a cyclic descent set. Recall the bijection $p_n : 2^{[n]} \longrightarrow 2^{[n]}$ induced by the cyclic shift $i \mapsto i + 1 \pmod{n}$.

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Definition

Let $\mathcal{T}$ be a finite set, and $\mathrm{Des} : \mathcal{T} \longrightarrow 2^{[n-1]}$ any map. A cyclic extension of Des is a pair (cDes, p), where $\mathrm{cDes} : \mathcal{T} \longrightarrow 2^{[n]}$ is a map and $p : \mathcal{T} \longrightarrow \mathcal{T}$ is a bijection, satisfying the following axioms: for all T in $\mathcal{T}$,

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$$\begin{array}{rl} \text{(extension)} & \mathrm{cDes}(T) \cap [n-1] = \mathrm{Des}(T), \\ \text{(equivariance)} & \mathrm{cDes}(p(T)) = p_n(\mathrm{cDes}(T)), \\ \text{(non-Escher)} & \varnothing \subsetneq \mathrm{cDes}(T) \subsetneq [n]. \end{array}$$

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Examples

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Motivating Problem:

Does Des on $\mathrm{SYT}(\lambda/\mu)$ have a cyclic extension ?

More examples

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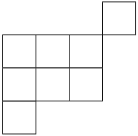
For $\lambda \vdash n-1$ let $\lambda^{\square}$ be the skew shape obtained from λ by placing a disconnected box at its upper right corner.

Example

$$(3,3,1)^{\square} =$$



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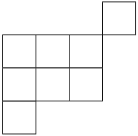
$(3,3,1)^{\square} =$ 

Theorem (Elizalde-Roichman '15)

For every partition $\lambda \vdash n-1$ there exists a cyclic descent extension on $\mathrm{SYT}(\lambda^{\square})$.

More examples

Theorem (A-Elizalde-Roichman '16)

Each of the following shapes carries a cyclic descent extension:

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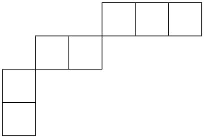

(*strip*)

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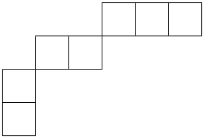

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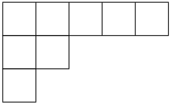

(*hook plus one box*)

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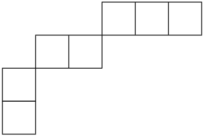

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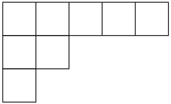

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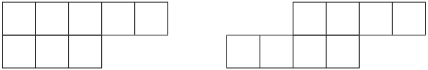

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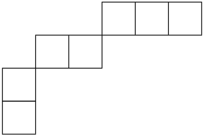

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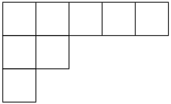

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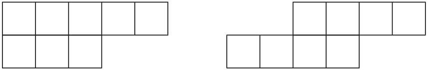

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The proofs are explicit and combinatorial.

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So far - so good!

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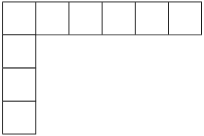

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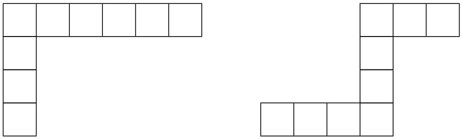

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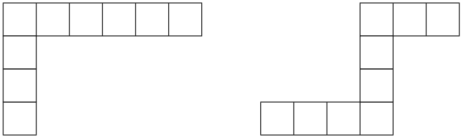
A connected skew shape λ/μ is a ribbon if it does not contain a 2×2 square.
In particular, for $\mu = \varnothing$, a non-skew ribbon is a hook $\lambda = (n-k, 1^k)$.

Example



Proposition *If λ/μ is a connected ribbon, then* $\mathsf{SYT}(\lambda/\mu)$ *does not have a cyclic descent extension.*

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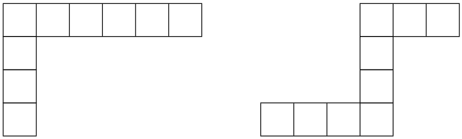

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Existence and uniqueness

Main theorem

Recall our

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Theorem (A-Reiner-Roichman '17)

1. *(existence) For every skew shape λ/μ of size n, which is not a connected ribbon, there exists a cyclic descent extension.*
2. *(uniqueness) For any such shape, all cyclic descent extensions $\mathrm{cDes} : \mathrm{SYT}(\lambda/\mu) \to 2^{[n]}$ have the same fiber sizes $|\,\mathrm{cDes}^{-1}(J)|$, uniquely determined by λ/μ and $J \subseteq [n]$.*

Near-hooks

Near-hooks

In general, the descent map cDes is not unique; only the fiber sizes are. However, in some special cases the map itself is unique.

Theorem

Let λ/μ be skew shape with $n \geq 2$ cells, and let $1 \leq k \leq n-1$ be an integer. Then TFAE:

1. *All the tableaux in* $\mathrm{SYT}(\lambda/\mu)$ *have the same cyclic descent number* k.
2. *The set of descent numbers of* $\mathrm{SYT}(\lambda/\mu)$ *is* $\{k-1, k\}$.
3. *Either λ/μ or its reverse is "one cell away from a hook", namely has one of the forms:*
   (a) Hook minus its corner cell: $(n-k+1, 1^k)/(1) = (1^k) \oplus (n-k)$.
   (b) Hook plus a disconnected cell: $(n-k, 1^{k-1}) \oplus (1)$ *or* $(1) \oplus (n-k, 1^{k-1})$.
   (c) Hook plus an internal cell: $(n-k, 2, 1^{k-2})$, *with* $2 \leq k \leq n-2$.

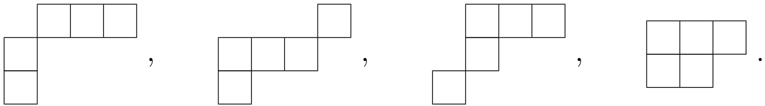
The shapes (a), (b) and (c) will be called near-hooks.

Near-hooks

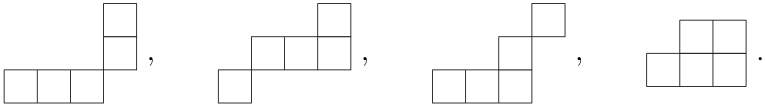
Near-hooks

Example

Near-hooks, for $n = 5$ and $k = 2$:



Their reverses:



Exceptional (Escher) cyclic descents

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What happens if we relax the non-Escher condition?

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Definition

Let $\mathcal{T}$ be a finite set, and $\mathrm{Des} : \mathcal{T} \longrightarrow 2^{[n-1]}$ any map. An exceptional (Escher) cyclic extension of Des is a pair (cDes_*, p), where $\mathrm{cDes}_* : \mathcal{T} \longrightarrow 2^{[n]}$ is a map and $p : \mathcal{T} \longrightarrow \mathcal{T}$ is a bijection, satisfying the following axioms:

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$$
\begin{array}{rl}
\text{(extension)} & \mathrm{cDes}_*(T) \cap [n-1] = \mathrm{Des}(T), \\
\text{(equivariance)} & \mathrm{cDes}_*(p(T)) = p_n(\mathrm{cDes}_*(T)), \\
\text{(Escher)} & (\exists T \in \mathcal{T})\, \mathrm{cDes}_*(T) \in \{\varnothing, [n]\}.
\end{array}
$$

Exceptional (Escher) cyclic descents

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Theorem

Let λ/μ be a skew shape of size $n \geq 2$. The usual descent map Des *on* $\mathrm{SYT}(\lambda/\mu)$ *has an exceptional cyclic extension* (cDes_*, p) *if and only if λ/μ has one of the following forms. In each case, all such extensions have the same fiber sizes* $|\,\mathrm{cDes}_*^{-1}(J)|$ $(\forall J \subseteq [n])$*.*

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1. $\lambda/\mu = (n)$*, a single row:* $\mathrm{cDes}_*(T) = \varnothing$ *for the unique SYT* T*.*
2. $\lambda/\mu = (1^n)$*, a single column:* $\mathrm{cDes}_*(T) = [n]$ *for the unique SYT* T*.*
3. $\lambda/\mu = (1)^{\oplus n}$ *has* n *connected components, each of size* 1*, with* n *even. In this case there is also a non-Escher cyclic extension, and the fiber sizes satisfy*

$$|\,\mathrm{cDes}_*^{-1}(J)| = |\,\mathrm{cDes}^{-1}(J)| + (-1)^{|J|} \qquad (\forall J \subseteq [n]).$$

In particular, $|\,\mathrm{cDes}_*^{-1}(\varnothing)| = |\,\mathrm{cDes}_*^{-1}([n])| = 1$*.*

Exceptional (Escher) cyclic descents

Exceptional (Escher) cyclic descents

Remarks

1. For $n = 1$, there are two distinct exceptional cyclic extensions, one with $\mathrm{cDes}_*(T) = \varnothing$ and the other with $\mathrm{cDes}_*(T) = [1]$, for the unique SYT T.
2. For $\lambda/\mu = (1)^{\oplus n}$ there is a natural descent-preserving bijection between $\mathrm{SYT}(\lambda/\mu)$ and the symmetric group $\mathfrak{S}_n$. It follows that, for even n, there is a definition for the cyclic descents of permutations whose distribution is slightly different from Cellini's!

Exceptional (Escher) cyclic descents

Exceptional (Escher) cyclic descents

Example

The symmetric group S_4.

$\mathrm{cDes}/\mathrm{cDes}_*$

4123	3412	2341	1234
$\{1\}/\{1,4\}$	$\{2\}/\{2,4\}$	$\{3\}/\{3,4\}$	$\{4\}/\varnothing$
1432	2143	3214	4321
$\{2,3,4\}/\{2,3\}$	$\{1,3,4\}/\{1,3\}$	$\{1,2,4\}/\{1,2\}$	$\{1,2,3\}/\{1,2,3,4\}$

Tools

Schur functions

For $\lambda \vdash n$ let the Schur function s_λ be

$$\sum_{T \in SSYT(\lambda)} \prod_i x_i^{\text{number of } i \text{ entries in } T},$$

where $SSYT(\lambda)$ is the set of semi-standard Young tableaux of shape λ (weakly increasing along rows, and strictly increasing along columns).

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Example $SSYT(2,1) =$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$$

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2	

1	2
2	

1	1
3	

1	2
3	

1	3
2	

1	3
3	

2	2
3	

2	3
3	

$\dots$

$$s_{2,1} = x_1^2 x_2 + x_1 x_2^2$$

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$$s_{2,1} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + 2x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + \ldots$$

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Schur functions are symmetric, and form a basis for the space of symmetric functions.

Complete homogeneous functions

$\lambda = (5)$ 

For the special case of a one-row shape $\lambda = (n)$, the Schur function $h_n = s_{(n)}$ is the complete homogeneous symmetric function:

$$h_n = \sum_{i_1 \leq \ldots \leq i_n} x_{i_1} \cdots x_{i_n}.$$

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Define also, for a sequence $N = (n_1, \ldots, n_k)$,

$$h_N = h_{n_1} \cdots h_{n_k}.$$

Schur positivity

A symmetric function is called Schur positive if all coefficients of its expansion in the Schur basis are nonnegative.

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where the Littlewood-Richardson coefficients $c^\lambda_{\mu,\nu} \geq 0$ have a combinatorial interpretation.

Ribbon Schur functions

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For a subset $J = \{j_1 < j_2 < \ldots < j_t\} \subseteq [n-1]$ define the associated composition

$$\mathrm{co}(J) := (j_1,\ j_2 - j_1,\ j_3 - j_2, \ldots,\ n - j_t)$$

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Theorem (Gessel '83)

For any skew shape λ/μ and $J \subseteq [n]$,

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The ribbon Schur functions $s_{\mathrm{co}(J)}$ are Schur positive.

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For a subset $\varnothing \neq J = \{j_1 < j_2 < \ldots < j_t\} \subseteq [n]$ define the associated cyclic composition

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and the corresponding affine ribbon Schur function

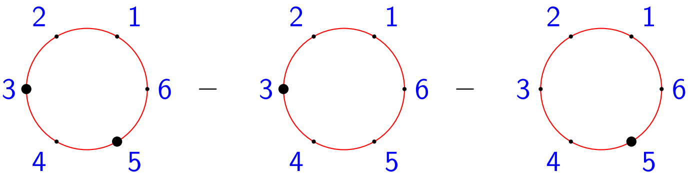
$$\tilde{s}_{\mathrm{cc}(J)} := \sum_{\varnothing \neq I \subseteq J} (-1)^{|J \setminus I|} h_{\mathrm{cc}(I)}.$$

Affine ribbon Schur functions

Example

Let $n = 6$ and $J = \{3, 5\}$. The affine ribbon Schur function is

$$\tilde{s}_{\mathrm{cc}(\{3,5\})} = h_{\mathrm{cc}(\{3,5\})} - h_{\mathrm{cc}(\{3\})} - h_{\mathrm{cc}(\{5\})}$$
$$= h_{(2,4)} - h_{(6)} - h_{(6)}.$$



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Example

For $n = 6$ and $J = \{3, 5\}$,

$$\tilde{s}_{cc(\{3,5\})} = s_{4,2} + s_{5,1} - s_6.$$

Gromov-Witten invariants

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- string theory: Free energy in type IIA superstring theory
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Specifically, let the Grassmannian $\mathrm{Gr}_{k,n}$ be the projective variety of all k-dimensional subspaces of $\mathbb{C}^n$.

Let $P_{k,n}$ be the set of all partitions λ whose shape fits in a $k \times (n-k)$ rectangle, namely $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $n-k \geq \lambda_1 \geq \ldots \geq \lambda_k \geq 0$.

Gromov-Witten invariants

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Fix a flag of subspaces $\{0\} = V_0 \subset V_1 \subset \ldots \subset V_n = \mathbb{C}^n$. For each $\lambda \in P_{k,n}$ Define the corresponding Schubert variety $\Omega_\lambda \subset \mathrm{Gr}_{k,n}$ as the set of all subspaces $X \in \mathrm{Gr}_{k,n}$ such that the dimensions of its intersections with the various subspaces V_i in the flag satisfy suitable bounds (depending on λ).

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For a nonnegative integer d and partitions $\lambda, \mu, \nu \in P_{k,n}$, the (3-point) Gromov-Witten invariant $C_{\mu,\nu}^{\lambda,d}$ is the number of rational curves of degree d in $\mathrm{Gr}_{k,n}$ that intersect fixed generic translates of the Schubert varieties $\Omega_{\lambda^\vee}$, Ω_μ and Ω_ν, provided that this number is finite. This happens exactly when $|\mu| + |\nu| = nd + |\lambda|$.

Gromov-Witten invariants

Fix a flag of subspaces $\{0\} = V_0 \subset V_1 \subset \ldots \subset V_n = \mathbb{C}^n$. For each $\lambda \in P_{k,n}$ Define the corresponding Schubert variety $\Omega_\lambda \subset \mathrm{Gr}_{k,n}$ as the set of all subspaces $X \in \mathrm{Gr}_{k,n}$ such that the dimensions of its intersections with the various subspaces V_i in the flag satisfy suitable bounds (depending on λ).

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For $d = 0$, $C_{\mu,\nu}^{\lambda,0} = c_{\mu,\nu}^{\lambda}$ are the Littlewood-Richardson coefficients.

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Important: The geometric description implies that

$$C_{\mu,\nu}^{\lambda,d} \geq 0 \qquad (\forall d, \lambda, \mu, \nu)$$

Existence

Recall that the affine ribbon Schur functions $\tilde{s}_{\mathrm{cc}(J)}$ are not always Schur positive. Can this be made more precise?

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Theorem (Postnikov '05, McNamara '06, A-Reiner-Roichman)

For all $\varnothing \neq J \subseteq [n]$ *of size* $k > 0$

$$\tilde{s}_{cc(J)} + \sum_{i=0}^{k-1} (-1)^{k-i} s_{(n-i,1^i)}$$

is Schur positive (and hook-free).

Existence

Proof idea:

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$$p_n = x_1^n + x_2^n + \ldots = \sum_{i=0}^{n-1} (-1)^i s_{(n-i,1^i)}$$

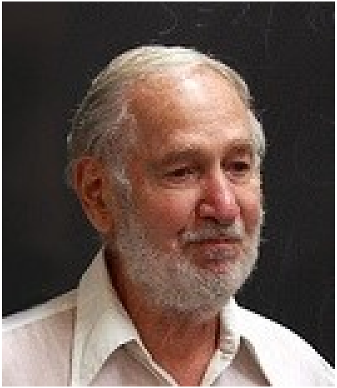
is the n-th power symmetric function.
Postnikov proved that, restricting to k variables only (namely letting $x_{k+1} = \ldots = 0$),

$$s_{\lambda/d/\mu}(x_1, \ldots, x_k) = \sum_{\nu \subseteq k \times (n-k)} C_{\mu,\nu}^{\lambda,d} s_\nu(x_1, \ldots, x_k),$$

where $C_{\mu,\nu}^{\lambda,d} \geq 0$ are the aforementioned Gromov-Witten invariants.

A topological interpretation

A topological interpretation



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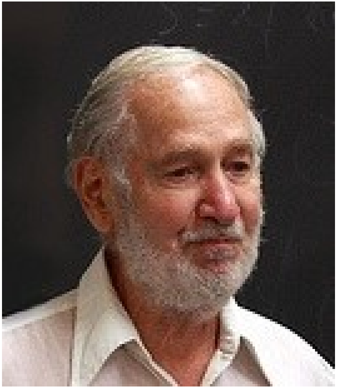
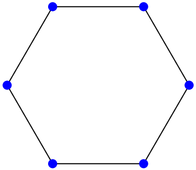


Robert Steinberg
May 25, 1922 - May 25, 2014

A topological interpretation

A topological interpretation

The Coxeter complex $\Sigma(W)$ of type A_2:



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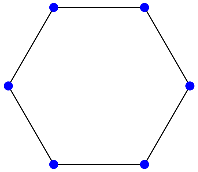
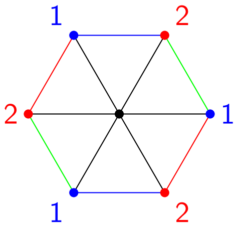


The Steinberg torus $\widetilde{\Delta} = \Sigma(\widetilde{W})/\mathbb{Z}\Phi^\vee$ of type $\widetilde{A}_2$:



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The Coxeter complex $\Delta = \Sigma(A_{n-1})$, and each of its type-selected subcomplexes Δ_J (for $J \subseteq [n-1]$), are Cohen-Macaulay. Their top cohomology groups carry $\mathfrak{S}_n$-representations corresponding to the ribbon Schur functions $s_{\mathrm{co}(J)}$.

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The Coxeter complex $\Delta = \Sigma(A_{n-1})$, and each of its type-selected subcomplexes Δ_J (for $J \subseteq [n-1]$), are Cohen-Macaulay. Their top cohomology groups carry $\mathfrak{S}_n$-representations corresponding to the ribbon Schur functions $s_{\mathrm{co}(J)}$.

The Steinberg torus $\widetilde{\Delta}$ is not Cohen-Macaulay. Its Euler characteristic carries the (virtual) $\mathfrak{S}_n$-representation

$$\sum_{i \geq 0} (-1)^i \operatorname{ch}(C^i(\widetilde{\Delta})) = \sum_{i \geq 0} (-1)^i \operatorname{ch}(H^i(\widetilde{\Delta}))$$

which corresponds to the symmetric function identity

$$\sum_{\varnothing \neq I \subseteq [n]} (-1)^{n-|I|} h_{\mathrm{cc}(I)} = \sum_{i=0}^{n-1} (-1)^{n-1-i} s_{(n-i,1^i)} = \tilde{s}_{\mathrm{cc}([n])}.$$

There are analogues for type-selected subcomplexes.

Cyclic quasi-symmetric functions

A quasi-symmetric function is a formal power series $f \in \mathbb{Z}[[x_1, x_2, \ldots]]$ of bounded degree such that, for any $t \geq 1$, any two increasing sequences $i_1 < \ldots < i_t$ and $i'_1 < \ldots < i'_t$ of positive integers, and any sequence $(m_1, \ldots, m_t)$ of positive integers, the coefficients of $x_{i_1}^{m_1} \cdots x_{i_t}^{m_t}$ and $x_{i'_1}^{m_1} \cdots x_{i'_t}^{m_t}$ in f are equal. The set $QSym$ of all quasi-symmetric functions is a graded ring, and its n-homogeneous part $QSym_n$ has as a basis Gessel's fundamental quasi-symmetric functions F_J, indexed by all subsets $J \subseteq [n-1]$. Its dimension is 2^{n-1}.

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Theorem (Gessel '84)

For any skew shape λ/μ,

$$\sum_{T \in \mathrm{SYT}(\lambda/\mu)} F_{\mathrm{Des}(T)} = s_{\lambda/\mu}.$$

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Theorem (A-Gessel-Reiner-Roichman '17)

The set cQSym of all cyclic quasi-symmetric functions is a graded ring, and its n-homogeneous part $QSym_n$ has as a basis suitable (normalized) fundamental cyclic quasi-symmetric functions $\widehat{F}_A$, indexed by the orbits A of the $\mathbb{Z}/n\mathbb{Z}$-action (by cyclic shifts) on the nonempty subsets $J \subseteq [n]$. Its dimension is

$$\frac{1}{n} \sum_{d|n} \varphi(d)(2^{n/d} - 1).$$

Cyclic quasi-symmetric functions

Cyclic quasi-symmetric functions

Theorem (A-Gessel-Reiner-Roichman '17)

For any skew shape λ/μ which is not a connected ribbon,

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For any skew shape λ/μ which is not a connected ribbon,

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Corollary

For any non-hook shape ν and set $J \subseteq [n]$, the Gromov-Witten invariant $C^{\lambda,1}_{\lambda,\nu}$ is equal to the coefficient of $\widehat{F}_{[J]}$ in the expansion of s_ν, where the partition λ corresponds to the cyclic composition $\mathrm{cc}(J)$.

Summary and open problems

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- For almost all skew shapes λ/μ, the fiber size distribution of this cyclic extension is unique.
- The proof (of existence) involves toric Schur functions and the nonnegativity of Gromov-Witten invariants.

Open Problems

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Find an explicit combinatorial description of a cyclic descent extension on SYT(λ/μ).

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Problem

For each non-hook partition $\lambda \vdash n$ *find a cyclically closed subset* $A \subseteq \mathfrak{S}_n$ *such that*

$$\sum_{\pi \in A} \mathbf{x}^{\mathrm{cDes}(\pi)} = \sum_{T \in \mathsf{SYT}(\lambda)} \mathbf{x}^{\mathrm{cDes}(T)}.$$

Thank You!